Physics 305, Fall 2008
Problem Set 7

due Thursday, November 13

1. **Hydrogen in an electric field:** A hydrogen atom, initially in its ground state, is placed in a spatially-uniform electric field $\mathbf{E}(t)$ that has the time dependence

$$\mathbf{E}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \mathbf{E}_0 e^{-\gamma t} & \text{if } t > 0 \ . \end{cases}$$

   What is the probability, to lowest nonvanishing order in $|\mathbf{E}_0|$, that for $t \to \infty$ this hydrogen atom is in an excited state with principal quantum number $n = 2$? (Ignore the coupling to photons, so the excited states are assumed to be stable.)

2. **Time dependent harmonic oscillator:** Consider a harmonic oscillator with the Hamiltonian

$$H(t) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega(t)^2}{2}x^2 \ ,$$

   with

$$\omega(t) = \omega_0 + \delta\omega \ \sin \Omega t \ ,$$

   and the frequencies ω_0 and $\delta\omega$ are independent of time with $\delta\omega \ll \omega_0$. Assume that this oscillator is in its ground state at time $t = 0$. At first order in perturbation theory, what excited state(s) will be populated? Calculate the probability of the oscillator being excited at time t, again to lowest nonvanishing order in perturbation theory. Use the formulation of the harmonic oscillator (with $\omega = \omega_0$) in terms of raising and lowering operators in solving all parts of this problem. (You may assume in the last steps of this problem that $|\Omega/2 - \omega_0| \ll |\Omega/2 + \omega_0|$.)

   (This problem is a rough model of a process that is important in nonlinear optics, where a laser with frequency Ω alters the dielectric properties of a material and thereby produces two photons of frequency $\omega_0 \approx \Omega/2$.)

3. **Einstein A and B coefficients:** This problem is to make sure that you have read and understood Griffiths 9.3.1. Consider a system that consists of atoms with two energy levels E_1 and E_2 and a thermal gas of photons. There are N_1 atoms with energy E_1, N_2 atoms with energy E_2 and the energy density of photons with frequency $\omega = (E_2 - E_1)/\hbar$ is $W(\omega)$. In thermal equilbrium at temperature T, W is given by the Planck distribution:

$$W(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp(\hbar\omega/k_B T) - 1} \ .$$

   According to Einstein, this formula can be understood by assuming the following rules for the interaction between the atoms and the photons

   - Atoms with energy E_1 can absorb a photon and make a transition to the excited state with energy E_2; the probability per unit time for this transition to take place is proportional to $W(\omega)$, and therefore given by

$$P_{\text{abs}} = B_{12} W(\omega)$$

     for some constant B_{12}.

- Atoms with energy E_2 can make a transition to the lower energy state via stimulated emission of a photon. The probability per unit time for this to happen is

$$P_{\text{stim}} = B_{21} W(\omega)$$

for some constant B_{21}.
- Atoms with energy E_2 can also fall back into the lower energy state via spontaneous emission. The probability per unit time for spontaneous emission is independent of $W(\omega)$. Let's call this probability

$$P_{\text{spont}} A_{21} \ .$$

A_{21}, B_{21}, and B_{12} are known as Einstein coefficients.

a. Write a differential equation for the time dependence of the occupation numbers N_1 and N_2.

b. What is the lifetime of the excited energy level E_2 at very low temperature?

c. Determine the distribution $W(\omega)$ in thermal equilibrium as a function of the Einstein coefficients.

Assume that the ratio N_2/N_1 in thermal equilibrium is given by the Boltzmann factor

$$\frac{N_1}{N_2} = \exp(\hbar\omega/k_B T) \ .$$

d. By comparing the result of part (c) with the Planck distribution, show that

$$P_{\text{abs}} = P_{\text{stim}} = \langle n \rangle P_{\text{spont}}$$

where $\langle n \rangle = 1/(\exp(\hbar\omega/k_B T) - 1)$ is the average number of photons with frequency ω. Give an interpretation of this formula. When is spontaneous emission dominant? When is stimulated emission dominant?