Physics 305, Fall 2008
Problem Set 4

due Thursday, October 15

1. **Toy Model of a 1d Metal (25 points):** Consider a system of N electrons moving in a box of length na with periodic boundary conditions. The Hamiltonian for this system is

$$H = \sum_{i=1}^{N} h_i \; ; \quad h_i = -\frac{\hbar^2}{2m}\frac{d^2}{dx_i^2} - g\sum_{j=1}^{n}\delta(x_i - ja)$$

where $g > 0$. There exist both positive and negative energy single particle states for these electrons.

   a. What is the analog of equation (5.64) in Griffiths for the positive energy states? for the negative energy states?

   b. Assume N is very large and define the quantity $\beta \equiv mga/\hbar^2$. For what value(s) of β do half of the states in the first band have negative energy? For what value(s) of β do all of the states in the first band have negative energy. For $\beta = 5/2$, what is the size of the energy gap between the first band and the second band.

2. **BEC of Lithium-7 (40 points):** At about the same time that Ketterle, Wiemann, and Cornell formed Bose-Einstein condensates of rubidium-87 and sodium-23 atoms (and later won the Nobel prize), Randy Hulet's lab in Texas was trying to form a BEC of lithium-7 atoms in a harmonic trap. We will try to model Hulet's system with a mean-field approach, using the Gross-Pitaevskii equation — essentially a nonlinear generalization of Schrödinger's equation:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \frac{Ng}{2}|\psi|^2$$

where H acts on the wave function ψ. Here ψ is a single particle wave function. At low temperatures, all the N bosons should be in the same state, and the contribution of the interactions to the energy should scale as the local density of the wave function $|\psi|^2$, giving rise to the third term in our Hamiltonian. One very significant difference between these experiments is that lithium-7 atoms attract ($g < 0$) while rubidium-87 and sodium-23 repel ($g > 0$).

   a. Explain how lithium-7 is a boson.

   b. Using a Gaussian trial wave function $\psi = c\, e^{-r^2/2a^2}$, calculate the expectation value of the energy $E(a) = \langle H \rangle$ as a function of a. [Hint: The expectation value has the form $E(a) = A/a^2 + Ba^2 + C/a^3$ for some constants A, B, and C which you need to determine.]

   c. In the limit in which the interaction energy is large compared to the kinetic energy, minimize $E(a)$ as a function of a for the repulsive case $g > 0$. How do a_{min} and $\langle H \rangle_{\text{min}}$ scale with N?

d. Later in the course, I hope to be able to derive the formula

$$g = \frac{4\pi\hbar^2\ell}{m} \ ,$$

where ℓ is the "scattering length" for the bosons. Assume the trap has a frequency $\omega = 2\pi \times 145$ Hz and the "scattering length" $\ell = -1.5$ nm for lithium-7. What is the maximum number of lithium-7 atoms that can be placed in the trap.