Physics 305, Fall 2008
Problem Set 5

due Thursday, October 22

1. **1D molecules (30 points):** One could think of this problem as a warm-up for the hydrogen molecule ion presented in Griffiths 7.3. Consider the following model Hamiltonian of a one dimensional molecule:

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - g\delta(x) - g\delta(x-a) \ ,$$

where $g > 0$ and $a > 0$. In the case where the second delta function is set to zero, the (properly normalized) ground state wave function is

$$\psi_0(x) = \sqrt{\kappa}e^{-\kappa|x|}$$

where $\kappa = mg/\hbar^2$. In the case where the first delta function is set to zero, there is a corresponding ground state wave function we will label $\psi_a(x)$. Now H can be solved exactly, but we will instead consider the approximate ground state

$$\psi(x) = \mathcal{N}(\psi_0(x) + \psi_a(x)) \ .$$

a. Calculate the value of $\mathcal{N}$ that makes $\psi(x)$ properly normalized.

b. Calculate the expectation value of the Hamiltonian $\langle H \rangle$ in the state $|\psi\rangle$. Show that you can express the expectation value in the form

$$\langle H \rangle = -\frac{\hbar^2\kappa^2}{2m}\frac{1 + e^{-\kappa a}(3 + \kappa a) + 2e^{-2\kappa a}}{1 + e^{-\kappa a}(1 + \kappa a)} \ .$$

c. Imagine the nuclei, represented by the delta functions, experience a repulsive potential of the form

$$V(a) = \frac{1}{10}\frac{\hbar^2\kappa}{2m}\frac{1}{a} \ .$$

At what value of κa is the total energy minimized? What is the energy at the minimum, expressed in units of $\hbar^2\kappa^2/2m$. (You will only be able to determine these values numerically.) What can you conclude from the value of the energy at the minimum?

2. **Radioactive Decay (20 points):** The two nuclei, ^{226}Ra and ^{226}Th, with $Z = 88$ and $Z = 90$, both disintegrate by α emission with the emitted α particles having energies of 4.9 MeV and 6.5 MeV, respectively. Assuming that the nuclear radius is the same for both nuclei, $R = 7.3 \times 10^{-13}$ cm, estimate the ratio of their half-lives using the WKB approximation. The α particle is less strongly bound as Z increases (at fixed $A = 226$ in this case) because of the Coulomb repulsion among the protons. This suggests the nuclear radius R actually increases a little with increasing Z. What will be the sign of the effect of this radius change on your ratio of half-lives?

3. **WKB and central force potentials (30 points):** The radial wave functions and eigen energies for bound states of the hydrogen atom with large quantum number [more precisely, large $(n - l)$] are well approximated using the WKB method. Let's generalize the Coulomb potential to an attractive power-law central potential with general exponent:

$$V(r) = -\frac{A}{r^b} ,$$

with A positive and, for parts (a) and (b) below, $0 < b < 2$. As usual, a particle of mass m is moving in this potential.

a. Show that within the WKB approximation there are an infinite number of bound states for each value of the angular momentum quantum number l when $0 < b < 2$.

b. For the $l = 0$ states, what are the bound state eigenenergies at large n? Again, use WKB and its "matching rules." Express the answer in terms of a finite, dimensionless, unevaluated integral. Look up the integral in a table, or have a computer program such as Mathematica evaluate it for you.