Physics 305, Fall 2008
Problem Set 6
due Thursday, November 12

1. **A charged particle in a magnetic field (30 points):** Consider an electron of charge $-e$ and mass $m$ in a constant magnetic field $B = B\hat{z}$. For simplicity, we will assume that the electron is constrained to move in the $xy$-plane, and we will ignore its spin.

   a. Using a separation of variables ansatz,

   $$\Psi(x, y) = \chi(x)\psi(y),$$

   find the eigen-energies for the electron in a gauge where $A = By\hat{x}$.

   b. Now consider the case where the electron is moving inside a square with periodic boundary conditions and area $A = L^2$. Assume that

   $$\frac{AeB}{\hbar} \gg 1.$$ 

   How many quantum states are there with a given energy? What do you think is the reason for this assumption about the size of $A$?

   c. Increase the number of electrons inside the rectangle to $N$. Ignore electron-electron interaction. Describe qualitatively what happens to the Fermi sea as we increase the magnetic field from zero.

2. **Spin precession and “spin echo” (40 points):** The following problem is a warm-up for the time-dependent perturbation theory we will consider after fall break. The Hamiltonian of a $s = 1/2$ spin in a time-dependent magnetic field $\vec{B}(t)$ is

   $$H(t) = \mu\vec{B}(t) \cdot \vec{\sigma},$$

   where $\mu > 0$ is the spin’s magnetic moment and $\vec{\sigma}$ are the Pauli matrices for the spin. Let the spin be initially oriented along the positive $z$-axis at time $t = 0$. We will subject it to a certain sequence of fields that are a caricature of some things that can be done in MRI imaging.

   a. The $\pi/2$ pulse: First apply a field of magnitude $B_x$ along the positive $x$-axis from time $t = 0$ to $t = t_1$. What must the product $B_xt_1$ be in order for this application to cause a 90 degree rotation of the spin?

   b. What is the resulting orientation of the spin at time $t_1$?

   c. Next apply a field of magnitude $B_z$ along the positive $z$-axis, and leave it on for a longer time, so the spin may rotate many times. What is the resulting time dependence of $\mu\langle \vec{\sigma} \rangle$, the expectation value of the spin’s magnetic moment at time $t$ (it is a vector)? For a proton spin in a 1 Tesla field, what is the precession frequency?
In solids (e.g. bone, etc.), variation in the material’s magnetic properties cause the field $\vec{B}$ to vary slightly over the positions of the various nuclear spins. Variation in the magnetic field in turn causes the spins’ precession rates to differ, and the rotating magnetic moments quickly get out of phase, eliminating the coherent macroscopic precessing magnetization that would otherwise be present. The following “spin echo” technique can bring back the coherence:

After $B_z$ is on from time $t_1$ to $t_1 + t_2$, turn it off and apply $B_x$ as in part (a), but now for time $2t_1$, so the spins rotate by 180 degrees about the $x$-axis (a “$\pi$-pulse”). Then apply $B_z$ again for the same time duration $t_2$. Note $t_2$ is long, so each spin turns many times about the $z$-axis during the applications of $B_z$, and the product $B_zt_2$ is not chosen to give any particular number of rotations of the spin. However, we assume the two applications of field along the $z$-axis are identical in magnitude and duration.

d. At the end of these 4 applications of the field ($B_x$ for $t_1$, $B_z$ for $t_2$, $B_x$ for $2t_1$, and then $B_z$ for $t_2$ again), what is the orientation of the spin? Show that it does not depend on $B_z$, so all the spins will be aligned at this time, even though $B_z$ varies over the material. This reappearance of a coherent rotating magnetization is called the “spin echo”.

Have a happy fall break!