Physics 305, Fall 2008
Problem Set 8

due Thursday, December 3

1. **Einstein A and B coefficients (25 pts):** This problem is to make sure that you have read and understood Griffiths 9.3.1. Consider a system that consists of atoms with two energy levels E_1 and E_2 and a thermal gas of photons. There are N_1 atoms with energy E_1, N_2 atoms with energy E_2 and the energy density of photons with frequency $\omega = (E_2 - E_1)/\hbar$ is $W(\omega)$. In thermal equilbrium at temperature T, W is given by the Planck distribution:
$$W(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar\omega/k_B T) - 1} \ .$$
According to Einstein, this formula can be understood by assuming the following rules for the interaction between the atoms and the photons

   - Atoms with energy E_1 can absorb a photon and make a transition to the excited state with energy E_2; the probability per unit time for this transition to take place is proportional to $W(\omega)$, and therefore given by
   $$P_{\text{abs}} = B_{12} W(\omega)$$
   for some constant B_{12}.
   - Atoms with energy E_2 can make a transition to the lower energy state via stimulated emission of a photon. The probability per unit time for this to happen is
   $$P_{\text{stim}} = B_{21} W(\omega)$$
   for some constant B_{21}.
   - Atoms with energy E_2 can also fall back into the lower energy state via spontaneous emission. The probability per unit time for spontaneous emission is independent of $W(\omega)$. Let's call this probability
   $$P_{\text{spont}} = A_{21} \ .$$

   A_{21}, B_{21}, and B_{12} are known as Einstein coefficients.

   a. Write a differential equation for the time dependence of the occupation numbers N_1 and N_2.
   b. What is the lifetime of the excited energy level E_2 at very low temperature?
   c. Determine the distribution $W(\omega)$ in thermal equilibrium as a function of the Einstein coefficients.

   Assume that the ratio N_2/N_1 in thermal equilibrium is given by the Boltzmann factor
   $$\frac{N_1}{N_2} = \exp(\hbar\omega/k_B T) \ .$$

d. By comparing the result of part (c) with the Planck distribution, show that

$$P_{\text{abs}} = P_{\text{stim}} = \langle n \rangle P_{\text{spont}}$$

where $\langle n \rangle = 1/(\exp(\hbar\omega/k_B T) - 1)$ is the average number of photons with frequency ω. Give an interpretation of this formula. When is spontaneous emission dominant? When is stimulated emission dominant?

2. **3S decay of hydrogen (20 pts):** A hydrogen atom in its 3S state decays via a series of electric dipole transitions to its ground state.

   a. What intermediate states may be visited in this multi-step (thus multi-photon) decay?

   b. Of the available decay sequences, what are the probabilities that the atom does decay by this route, for each possibility? You should be able to answer this based on selection rules and other symmetry considerations without having to evaluate precisely any matrix elements.

3. **2P decay of hydrogen (30 pts):** In this problem, you will use Fermi's Golden Rule to calculate the rate of decay of the $m = 0$ 2P ($n = 2$, $l = 1$) excited state of a hydrogenic atom/ion. (The decay rate is the same for all 2P states, but just look at $m = 0$ to be more concrete.) The rate R of decay is given by the Golden Rule:

$$R = \frac{2\pi}{\hbar} |V_{fi}|^2 \mathcal{D}(E) \ .$$

The nucleus has charge Ze and let's ignore the spins of the nucleus and the electron. The electron decays from a 2P state to the 1S ground state. To ease the task of grading this problem and to give you some insight into the relative sizes of the various quantities, we ask you to report all answers in terms of the fine structure constant $\alpha = e^2/4\pi\epsilon_0\hbar c$ instead of in terms of e. Assuming, at first, that the decay rate is slow, so the energy of the emitted photon has small uncertainty,

   a. What is the energy of the emitted photon?

The initial state in this decay is the atom/ion in the $m = 0$ 2P state and no photons, while the final state is the atom/ion in its ground state and one photon. Assume the nucleus is located at the origin and has infinite mass, so the motion of the nucleus may be neglected (a good approximation). Also put the atom in a finite box of volume V (shape $L \times L \times L$ is convenient) with periodic boundary conditions, so the momentum of the photon is a good quantum number that can take on a discrete set of values.

   b. Calculate the density of final states per unit energy, $\mathcal{D}(E)$, for large V at the energy of the emitted photon. It is proportional to V. Remember that for each momentum the photon has two possible polarizations.

The electric field at position r due to the photon j with wavevector $\vec{k}_j$ and polarization $\hat{e}_j$ is

$$\vec{E}_j = \hat{e}_j \sqrt{\frac{\hbar c k_j}{2\epsilon_0 V}} (a_j e^{-i\vec{k}_j\cdot\vec{r}} + a_j^\dagger e^{i\vec{k}_j\cdot\vec{r}}) \ .$$

The full electric field $\vec{E}$ due to the photons is the sum of this over all photon modes. The interaction that causes the transition is $\hat{V}_{\text{int}} = e\vec{E}\cdot\vec{r}$, where $\vec{r}$ is the position of the electron. [We are using the dipole approximation that $kr \ll 1$, which means $e^{i\vec{k}_j\cdot\vec{r}} = 1$ at the electron's position.]

c. What is the matrix element V_{ji} of the interaction $\hat{V}_{\text{int}}$ for the transition that results in emitting the photon j? Note it is proportional to $\sqrt{1/V}$, and you have done the tedious part of the this calculation already in a previous problem set. (Express your answer in terms of the photon wave-vector k_j.)

d. The $|V_{fi}|^2$ that enters Fermi's Golden Rule is actually the average of the matrix element squared over all photons at this energy. What is this average? And then what is the decay rate? Give the decay rate as a formula for general Z and give it in units of 1/sec for the case of hydrogen ($Z = 1$).

e. What is the ratio of the decay rate to the frequency of the emitted photon? This ratio should be small compared to one for the Golden Rule to be a good approximation. What happens to this ratio at large Z?

f. What is the probability distribution of the direction and the polarization of the emitted photon?