1. **Suppression of higher partial waves (30 pts):** Consider the scattering of a particle of mass \(m\) by a central potential \(V(r)\) which is negligible for \(r > d\).

   a. Show that
   \[
   \tan \delta_l(k) = \frac{k j'_l(kd) - \gamma_l(k) j_l(kd)}{kn'_l(kd) - \gamma_l(k) n_l(kd)},
   \]
   where \(j'_l(x) = dj_l(x)/dx\), \(n'_l(x) = dn_l(x)/dx\) and
   \[
   \gamma_l(k) = \left[ \frac{dR_l^I(k, r)/dr}{R_l^I(k, r)} \right]_{r=d}
   \]
is the value of the logarithmic derivative of the regular, interior solution \(R_l^I(k, r)\), evaluated at \(r = d\) of
   \[
   \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2} V(r) + k^2 \right] R_l(k, r) = 0.
   \]

   b. Using the properties of the functions \(j_l\) and \(n_l\), prove that in the small \(k\) limit
   \[
   \tan \delta_l(k) = \frac{(kd)^{2l+1}}{D_l} \frac{l - \hat{\gamma}_l d}{l + 1 + \hat{\gamma}_l d} + \ldots,
   \]
   where \(D_l = (2l+1)!!\) for \(l > 0\), \(D_0 = 1\), and \(\hat{\gamma} = \lim_{k \to 0} \gamma_l(k)\).

   c. Assuming that \(\hat{\gamma}_l d \neq -(l+1)\), prove that the partial wave amplitudes \(f_l(k)\) exhibit the low-energy behavior
   \[
   f_l(k) \sim k^{2l}, \quad \text{where} \quad f(k, \theta) = \sum_{l=0}^{\infty} f_l(k) P_l(\cos \theta),
   \]
   so that except for the s-wave \((l = 0)\) contribution which in general tends to a non-zero constant, all partial cross sections \(\sigma_l\) \((l \geq 1)\) vanish as \(k^4\). The scattering is therefore isotropic at very low energies. Prove also that the scattering amplitude \(f\) is given as \(k \to 0\) by
   \[
   \lim_{k \to 0} f(k, \theta) = \lim_{k \to 0} \frac{\tan \delta_0(k)}{k}.
   \]
   Typically, this quantity which has dimension of length is defined to be the negative of the scattering length, \(-a\).

2. **Low Energy Atomic Scattering (20 pts):** In problem set 4, we modeled a gas of cold bosonic atoms in a harmonic trap using the following mean-field "Hamiltonian":
   \[
   H = N \left( \frac{\mathbf{p}^2}{2m} + \frac{1}{2} m \omega^2 r^2 + \frac{(N-1)g}{2} |\psi|^2 \right).
   \]
Here $N$ is the number of atoms in the trap. We would like to derive $H$ starting from the actual Hamiltonian $H_{\text{full}}$ describing $N$ identical bosons of mass $m$ in an external spherical harmonic potential of frequency $\omega$ and interacting via a central force potential of the form $V(|\vec{r}_i - \vec{r}_j|)$.

a. What is the actual Hamiltonian $H_{\text{full}}$ describing this $N$-body system of atoms? Use the variational principle and the assumption that the $N$-particle wavefunction can be written as a product of single particle wavefunctions:

$$
\Psi(\{\vec{r}_n\}) = \prod_{n=1}^{N} \psi(\vec{r}_n),
$$

to derive the mean-field result $H$. What is $g$ in terms of $V(r)$? (You may assume that the range of the potential is very small compared to the scales over which the probability density of the particles $|\psi(\vec{r})|^2$ varies.)

b. Use the low energy Born approximation to relate $g$ to the scattering length $a$.

3. Neutrons scattering off a 1D crystal (25 pts): Consider point-like s-wave scatterers with scattering length $a$. The “target” consists of ten such stationary scatterers (the nuclei), equally spaced along the $z$-axis, with spacing $b \gg a$. The neutron is incident along the $z$-axis.

a. What is the differential cross-section as a function of scattering angle?

b. What is the maximum value of the differential cross-section, and at what angle(s) does this maximum value occur?

c. At what angles does the scattering vanish?

d. For $b = 3$ Angstroms, what is the limit on the incident energy of the neutron (in eV) in order to be able to see a scattering maximum at nonzero angle?