1. Parts of the Spectral Theorem (20 pts): Let $V = \mathbb{C}^n$ be a finite dimensional vector space over the complex numbers $\mathbb{C}$ of dimension $n$ with the usual inner product. Let $N$ be a normal operator, i.e. $N^\dagger N = NN^\dagger$, that maps $V$ to itself.

   a) Show that if $|x\rangle$ is an eigenvector of $N$ with eigenvalue $\lambda$, then $|x\rangle$ is also an eigenvector of $N^\dagger$ but with eigenvalue $\lambda^*$. 
   
   b) For $H$ a Hermitian operator, demonstrate that the eigenvalues of $H$ are real. For $U$ a unitary operator, demonstrate that the eigenvalues of $U$ have modulus 1, i.e. $|\lambda| = 1$.
   
   c) If $|x\rangle$ and $|y\rangle$ are both eigenvectors of $N$ with eigenvalues $\lambda_x \neq \lambda_y$, demonstrate that $\langle x|y \rangle = 0$.
   
   d) Why must any linear operator $T$ on $V$ have at least one eigenvalue and one eigenvector?

2. Orthogonal Polynomials (15 pts): Given the linearly independent vectors $x(t) = t^n$, for $n = 0, 1, 2, \ldots$, use the Gram-Schmidt process to find the orthonormal polynomials $e_0(t), e_1(t),$ and $e_2(t)$

   a) when the inner product is defined as $\langle x|y \rangle = \int_{-1}^{1} x^*(t)y(t)dt$.
   
   b) when the inner product is defined with a nontrivial weight function:

   $$\langle x|y \rangle = \int_{-\infty}^{\infty} e^{-t^2} x^*(t)y(t)dt .$$
   
   c) What names are traditionally associated with these polynomials?

3. The Derivative Operator (15 pts): Consider $P_c^2[t]$, i.e. polynomials in the variable $t$ of degree less than or equal to two with complex coefficients.

   a) In the basis $\{1, t, t^2\}$, write the derivative operator $D = d/dt$ as a $3 \times 3$ matrix.
   
   b) Using the inner product from 2(a), construct a matrix representation of the adjoint operator $D^\dagger$. Why is $D^\dagger$ not the conjugate transpose of $D$?
   
   c) Using the $e_i(t)$ from 2(a), construct $D$ and $D^\dagger$.

4. Heisenberg Uncertainty Principle (20 pts): Denote the expectation value of an operator $A$ in a state $|\psi\rangle$ by $\bar{A} = \langle A \rangle = \langle \psi | A | \psi \rangle$. The uncertainty or deviation in state $|\psi\rangle$ of the operator $A$ is given by

   $$\Delta A = \sqrt{\langle \psi | (A - \bar{A})^2 | \psi \rangle}$$
a) Show that for any two hermitian operators $A$ and $B$, we have

$$|\langle \psi | AB | \psi \rangle |^2 \leq \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle .$$

[ Hint: Use the Schwarz inequality. ]

b) Show that

$$|\langle \psi | [A, B] | \psi \rangle |^2 \leq 4 \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle .$$

c) Define operators $A' \equiv A - \alpha \text{Id}$ and $B' \equiv B - \beta \text{Id}$, where $\alpha$ and $\beta$ are real numbers. Show that $A'$ and $B'$ are hermitian and that $[A', B'] = [A, B]$.

d) Prove the uncertainty relation

$$(\Delta A)(\Delta B) \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle | .$$