Physics 403, Spring 2011
Problem Set 1

due Thursday, February 10

1. **Parts of the Spectral Theorem (20 pts):** Let $V = \mathbb{C}^n$ be a finite dimensional vector space over the complex numbers $\mathbb{C}$ of dimension n with the usual inner product. Let N be a normal operator, i.e. $N^\dagger N = NN^\dagger$, that maps V to itself.

   a) Show that if $|x\rangle$ is an eigenvector of N with eigenvalue λ, then $|x\rangle$ is also an eigenvector of $N^\dagger$ but with eigenvalue λ^*.

   b) For H a Hermitian operator, demonstrate that the eigenvalues of H are real. For U a unitary operator, demonstrate that the eigenvalues of U have modulus 1, i.e. $|\lambda| = 1$.

   c) If $|x\rangle$ and $|y\rangle$ are both eigenvectors of N with eigenvalues $\lambda_x \neq \lambda_y$, demonstrate that $\langle x|y\rangle = 0$.

   d) Why must any linear operator T on V have at least one eigenvalue and one eigenvector?

2. **Orthogonal Polynomials (15 pts):** Given the linearly independent vectors $x(t) = t^n$, for $n = 0, 1, 2, \ldots$, use the Gram-Schmidt process to find the orthonormal polynomials $e_0(t)$, $e_1(t)$, and $e_2(t)$

   a) when the inner product is defined as $\langle x|y\rangle = \int_{-1}^{1} x^*(t)y(t)dt$.

   b) when the inner product is defined with a nontrivial weight function:

   $$\langle x|y\rangle = \int_{-\infty}^{\infty} e^{-t^2} x^*(t)y(t)dt \ .$$

   c) What names are traditionally associated with these polynomials?

3. **The Derivative Operator (15 pts):** Consider $\mathcal{P}_2^c[t]$, i.e. polynomials in the variable t of degree less than or equal to two with complex coefficients.

   a) In the basis $\{1, t, t^2\}$, write the derivative operator $D = d/dt$ as a 3×3 matrix.

   b) Using the inner product from 2(a), construct a matrix representation of the adjoint operator $D^\dagger$. Why is $D^\dagger$ not the conjugate transpose of D?

   c) Using the $e_i(t)$ from 2(a), construct D and $D^\dagger$.

4. **Heisenberg Uncertainty Principle (20 pts):** Denote the expectation value of an operator A in a state $|\psi\rangle$ by $\bar{A} = \langle A\rangle = \langle\psi|A|\psi\rangle$. The uncertainty or deviation in state $|\psi\rangle$ of the operator A is given by

$$\Delta A = \sqrt{\langle\psi|(A - \bar{A})^2|\psi\rangle}$$

a) Show that for any two hermitian operators A and B, we have

$$|\langle\psi|AB|\psi\rangle|^2 \leq \langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle \ .$$

[Hint: Use the Schwarz inequality.]

b) Show that

$$|\langle\psi|[A,B]|\psi\rangle|^2 \leq 4\langle\psi|A^2|\psi\rangle\langle\psi|B^2|\psi\rangle \ .$$

c) Define operators $A' \equiv A - \alpha\,\mathrm{Id}$ and $B' \equiv B - \beta\,\mathrm{Id}$, where α and β are real numbers. Show that A' and B' are hermitian and that $[A', B'] = [A, B]$.

d) Prove the uncertainty relation

$$(\Delta A)(\Delta B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \ .$$