Physics 403, Spring 2011
Problem Set 5
due Thursday, March 24

1. **A Quantum Paradox** [15 pts]: Consider a particle in an infinite potential well on the interval $x \in [0,a]$. Physical considerations impose the boundary conditions $\psi(0) = \psi(a) = 0$. Assume the Hamiltonian operator $H = -\frac{d^2}{dx^2}$ (setting $\hbar = 1$ and $m = 1$) is self-adjoint.

   (a) Solve the eigenvalue equation $H \psi = E \psi$ associated to the boundary conditions $\psi(0) = \psi(a) = 0$. Deduce the spectrum and an orthonormal basis of eigenfunctions.

   (b) Consider the wave function given by $\psi(x) = \mu x(a-x)$ where $\mu$ is a normalization factor so that $\langle \psi | \psi \rangle = 1$. Show that $\langle H \psi | H \psi \rangle > 0$. Isn’t it the case however that $H^2 \psi = 0$ and hence $\langle \psi | H^2 \psi \rangle = 0$? What’s going on?

   (c) Decompose $H^2$ using the eigenfunction basis computed in (a). Use this “matrix” representation of $H^2$ to compute $\langle \psi | H^2 \psi \rangle$.

2. **The Inverse of the Kinetic Energy Operator** [15 pts]: Consider again the operator $H = -\frac{d^2}{dx^2}$ on the domain $D_H = \{ f, f'' \in L^2(0,1) : f(0) = f(1) = 0 \}$. We would like to construct $(H - \omega^2)^{-1}$.

   (a) Find an orthonormal basis of eigenvectors for $H$. Use this basis to construct $(H - \omega^2)^{-1}$.

   (b) Construct solutions $f_1$ and $f_2$ of the equation $(H - \omega^2)f(x) = 0$ such that $f_1(0) = 0$ and $f_2(1) = 0$. Use these homogeneous solutions to find a general solution $f(x)$ to the inhomogeneous equation $(H - \omega^2)f(x) = g(x)$ that satisfies the boundary conditions $f(0) = f(1) = 0$.

   (c) Show that the inverse operators constructed in parts (a) and (b) are the same.

3. **Critical Mass** [15 pts]: An infinite slab of fissile material has thickness $L$. The neutron density $n(x,t)$ in the material obeys the equation

   \[
   \frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} + \lambda n + \mu ,
   \]

   where $n(x,t)$ is zero at the surface of the slab at $x = 0$ and $L$. Here $D$ is the neutron diffusion constant, the term $\lambda n$ describes the creation of new neutrons by induced fission, and the constant $\mu$ is the rate of production per unit volume of neutrons by spontaneous fission.

   (a) Expand $n(x,t)$ as a series

   \[
   n(x,t) = \sum_m a_m(t) \varphi_m(x) ,
   \]

   where $\varphi_m(x)$ are a complete set of functions you think suitable for solving the problem.
(b) Find an explicit expression for the coefficients $a_m(t)$ in terms of their initial values $a_m(0)$.

(c) Determine the critical thickness $L_{\text{crit}}$ above which the slab will explode.