Physics 403, Spring 2011
Problem Set 6

due Thursday, March 31

1. **Green's Function for the Diffusion Equation** [15 pts]:

   (a) The diffusion equation appears in many places in physics, perhaps because it follows from two very modest assumptions. Assume the existence of a conserved current

   $$\partial_t \rho(t,x) - \partial_x j(t,x) = 0$$

   and that the current is proportional to the gradient of the density

   $$j(t,x) = D\partial_x \rho(t,x) \ ,$$

   where D is the diffusion constant. Derive the diffusion equation for $\rho(t,x)$.

   (b) Compute the Fourier transform

   $$\tilde{G}(\omega,k) = \int \frac{d\omega\, dk}{(2\pi)^2} e^{i\omega t - ikx} G(t,x) \ ,$$

   of the Green's function for the diffusion equation.

   (c) Use the inverse Fourier transform to compute $G(t,x)$ from $\tilde{G}(\omega,k)$.

2. **Pantograph Drag** [20 pts]: This beautiful problem I borrowed from Stone and Goldbart. A high-speed train picks up its electrical power via a pantograph from an overhead line. The locomotive travels at a speed U and the pantograph exerts a constant vertical force F on the power line. We make the usual small amplitude approximation and assume (not unrealistically) that the line is supported in such a way that its vertical displacement obeys an inhomogeneous Klein-Gordon equation

   $$\rho\ddot{y} - Ty'' + \rho\Omega^2 y = F\delta(x - Ut) \ ,$$

   with $c^2 = T/\rho$ the velocity squared of propagation of short-wavelength transverse waves on the overhead cable.

   (a) Assume that $U < c$ and solve for the steady state displacement of the cable about the pickup point.

   (b) Now assume that $U > c$. Again find an expression for the displacement of the cable.

   I gather from reading the internet that c is usually the upper bound on the speed of these trains. [Hint: It helps to assume that $y(t,x) = y(x - Ut)$. What is the physical significance of this assumption?]

3. **Sphere Volumes** [10 pts]: Volumes of d-dimensional spheres showed up in prefactors of a number of Green's functions that we computed in class. In this problem, we will compute the volume of a d-dimensional sphere of unit radius. You doubtless know that $\mathrm{Vol}(S^1) = 2\pi$ and $\mathrm{Vol}(S^2) = 4\pi$. You may have seen the following trick for computing the integral $I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$:

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx\, dy = \int_0^{\infty} \int_0^{2\pi} e^{-r^2} r d\theta\, dr = 2\pi \int_0^{\infty} r e^{-r^2} dr = \pi \ .$$

Knowing I, use I^{d+1} to compute $\mathrm{Vol}(S^d)$.

4. **Right and Left Inverses** [10 pts]: In class we considered the operator L on $\mathcal{L}^2_w(a,b)$ where

$$w(x) L[f(x)] = (p(x) f'(x))' + w(x) p_0(x) f(x) \ ,$$

and its putative inverse G where

$$G[g(x)] = \frac{f_2(x)}{Wp} \int_a^x w(y) f_1(y) g(y) dy + \frac{f_1(x)}{Wp} \int_x^b w(y) f_2(y) g(y) dy \ .$$

To be a little more specific, we took the domain of L to be

$$D_L = \{ f, Lf \in \mathcal{L}^2_w(a,b) : \alpha_1 f(a) + \beta_1 f'(a) = 0 \,\&\, \alpha_2 f(b) + \beta_2 f'(b) = 0 \} \ ,$$

such that L was self-adjoint and we assumed that L had no zero eigenvalues. We chose $f_1(x)$ and $f_2(x)$ to satisfy the homogeneous differential equations $Lf_1 = 0$ and $Lf_2 = 0$ where f_1 satisfied the boundary condition described in D_L at $x = a$ and f_2 satisfied the boundary conditions at $x = b$. The Wronskian was then $W(x) = f_1(x) f_2'(x) - f_2(x) f_1'(x)$.

In class we demonstrated that $L[G[g(x)]] = g(x)$. In this problem, we ask you to demonstrate that $G[L[f(x)]] = f(x)$ for $f \in D_L$.