1. **Subrepresentations** [10 pts]: In class, we showed that for an inner product space $V$ and a unitary representation $\rho : G \rightarrow GL(V)$, if a subspace $W \subset V$ was stable under $\rho$, the orthogonal complement $W^\perp$ was also stable under $\rho$. Thus $\rho$ decomposed into two subrepresentations given by the restrictions of $\rho$ to $W$ and $W^\perp$ respectively. Here, we would like to remove the assumption that $V$ be an inner product space and that $\rho$ be unitary.

(a) Let $p : V \rightarrow W$ be a linear operator such that $p(x) = x$ if $x \in W$. This operator is called a projection operator. We can define a $p$ dependent orthogonal complement via $W^\perp_p = \ker(p)$. Construct

$$\tilde{p} = \frac{1}{|G|} \sum_{g \in G} \rho(g) \cdot p \cdot \rho(g)^{-1}.$$ 

Show that $\tilde{p} : V \rightarrow W$ is also a projection operator.

(b) Show that $\rho(g) \cdot \tilde{p} = \tilde{p} \cdot \rho(g) \forall g \in G$.

(c) Let $W^\perp = \ker(\tilde{p})$. Show that $W^\perp$ is stable with respect to the action of $G$.

2. **Tensor Products of Character** [10 pts]: Let $\rho : G \rightarrow GL(V)$ be a representation of the group $G$ on the vector space $V$, and let $\chi$ be the corresponding character. Let $g \in G$ be an arbitrary group element.

(a) Show that the characters $\chi_\alpha$ of $\text{Alt}^2(V)$ and $\chi_\sigma$ of $\text{Sym}^2(V)$ can be written in terms of $\chi$ in the following way

$$\chi_\alpha(g) = \frac{1}{2} [\chi(g)^2 - \chi(g^2)],$$

$$\chi_\sigma(g) = \frac{1}{2} [\chi(g)^2 + \chi(g^2)].$$

(b) Let $\rho' : G \rightarrow GL(V')$ be another representation and $\chi'$ the corresponding character. Find the characters of $\text{Alt}^2(V \oplus V')$ and $\text{Sym}^2(V \oplus V')$ in terms of $\chi$ and $\chi'$.

3. **Tensor Product and $S_3$** [10 pts]: Let $V$ be the irreducible representation of the permutation group of three elements, $S_3$, of dimension two. Use the character table of $S_3$ to decompose $V^\otimes n$ into irreducible representations.

4. **Character Tables** [20 pts]: Work out the character table for $S_5$.

5. **Ammonia** [20 pts]: The molecule ammonia NH$_3$ is symmetric with respect to the action of a finite group $G$ that acts by rotations and reflections.

(a) What is this group $G$? Write down the character table for the group.
(b) Consider the 12 dimensional (reducible) representation \( \rho \) obtained by letting \( G \) act on the coordinates of the hydrogen and nitrogen atoms. What is the character \( \phi \) of this representation?

(c) Decompose \( \phi \) into characters of the irreps in your character table.

(d) Remove the characters corresponding to translations and rotations to see which irreps correspond to vibrational modes.