1. **Campbell-Baker-Hausdorff** [10 pts]: Let \( X, Y \in \mathfrak{g} \) be elements of a Lie algebra. In class, we claimed that
\[
e^X e^Y = e^{X+Y+...}
\]
where the ellipsis in the above equation denotes terms that only involve commutators of \( X \) and \( Y \) and hence are also elements of \( \mathfrak{g} \). Verify this claim to third order in \( X \) and \( Y \), i.e. up to commutators of the form \([X, [X,Y]]\) and \([Y, [Y,X]]\).

2. **Fun with SU(2)** [15 pts]:
   (a) Let \((z_0, z_1) \in \mathbb{C}^2\). Show that the operators
   \[
   J_x = \frac{1}{2} \left( z_0 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_0} \right) \\
   J_y = \frac{1}{2i} \left( z_0 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_0} \right) \\
   J_z = \frac{1}{2} \left( z_0 \frac{\partial}{\partial z_0} - z_1 \frac{\partial}{\partial z_1} \right)
   \]
satisfy the right commutation relations to generate the Lie algebra for \( \mathfrak{so}_3 \).
   (b) Show that monomials of the form \( z_0^\alpha z_1^\beta \) can be used to form finite dimensional irreps of this Lie algebra. What are the bounds on \( \alpha \) and \( \beta \) for a given irrep? What is the relation between \( \alpha \) and \( \beta \) and eigenvalues of \( J_z \) and \( J^2 = J_x^2 + J_y^2 + J_z^2 \)?

3. **Coherent States of SU(2)** [20 pts]: Consider the so-called “coherent state”
\[
|\vec{z}, j\rangle = \sum_{m=-j}^{j} \frac{\sqrt{2j!}}{\sqrt{(j+m)! (j-m)!}} z_1^{j+m} z_2^{j-m} |j; m\rangle
\]
where \(|j; m\rangle\) is the usual basis of orthonormal angular momentum states from quantum mechanics and \( \vec{z} = (z_1, z_2) \in \mathbb{C}^2 \) is a vector that satisfies the constraint \(|z_1|^2 + |z_2|^2 = 1\).
   (a) Verify that \(|\vec{z}, j\rangle\) is properly normalized and compute the expectation values \( \langle J_x \rangle \), \( \langle J_y \rangle \), and \( \langle J_z \rangle \) in the state \(|\vec{z}, j\rangle\).
   (b) Compute the dispersion \( \Delta J_z = \sqrt{\langle J_z^2 \rangle - \langle J_z \rangle^2} \).
   (c) Use angles on \( S^2 \) to parametrize the ratio \( z_2/z_1 = \tan(\theta/2) e^{i\phi} \). What is \( \langle \vec{J} \rangle \) in terms of \( \theta \) and \( \phi \). Use rotational symmetry to argue what \( \Delta J_x \) and \( \Delta J_y \) must be. Calculate \( (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \). Physically, how do you interpret these results?