

# AdS/CFT

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## **Abstract**

A write up of about ten lectures on the AdS/CFT correspondence given as part of a second semester course on string theory.

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## 1 Preliminary Remarks

The original AdS/CFT correspondence is the equivalence of type IIB string theory in a  $\text{AdS}_5 \times S^5$  background and maximally supersymmetric  $SU(N)$  Yang-Mills theory. (AdS stands for Anti-de Sitter space and CFT for conformal field theory.) The conjectured equivalence was argued for, in large part on the basis of symmetry, in the extraordinarily well cited 1997 paper [1] by Juan Maldacena. Two critical follow up papers, one by Steve Gubser, Igor Klebanov, and Alexander Polyakov [2]

and another by Edward Witten [3] opened up the field by showing how the correspondence could be usefully employed in the calculation of correlation functions. It should be emphasized that Maldacena's 1997 paper did not emerge from the vacuum. Like Athena and the head of Zeus, there is a backstory; the one here involves calculations by Igor Klebanov and collaborators of strings off D-branes and gravitons off of black holes. See for example [4].

These lecture notes are heavily influenced by earlier reviews on the subject. Two important ones are by Maldacena, Aharony, Gubser, Ooguri, and Oz (MAGOO) [5] and by D'Hoker and Freedman [6]. I begin by fleshing out an argument in the MAGOO review, explaining why AdS/CFT might be true. Section 2 is in a weak and vague sense a distillation of the follow-up papers [2] and [3] and explains the central calculational mechanism behind the correspondence. In Section 3, we perform a basic check of the original correspondence between maximally supersymmetric Yang-Mills and type IIB string theory in a  $\text{AdS}_5 \times S^5$  background – matching the Kaluza-Klein spectrum on the string side to a set of protected operators on the gauge theory side. The rest of the review contains various important examples. We explain how to calculate two-point functions of scalar fields. We explore the trace anomaly. We examine the phase diagram as a function of temperature of maximally supersymmetric Yang-Mills on a sphere. We calculate the viscosity of maximally supersymmetric Yang-Mills.

## 2 Why AdS/CFT might be true

The typical motivation for the AdS/CFT correspondence (or more generally gauge/gravity duality) begins with a consideration of type IIA or IIB superstring theory. (We could also rephrase the following discussion using the less well understood M-theory, but for simplicity we stick to ten dimensions in what follows.) Consider a stack of  $N$   $Dp$ -branes where the distance between the branes is much less than the string length scale  $\ell_s$ . There are two types of excitations: open strings that live on the branes and closed strings that live in the bulk. Because of the condition on the distance between the branes, the open strings can be nearly massless.

### 2.1 Decoupling from the open string point of view

We would now like to consider, in very general terms, a low energy effective field theory description of the situation. In particular, we require that the energies  $E \ll 1/\ell_s$  are small compared to the string scale. Only the massless strings can be excited. The closed string sector reduces to ten dimensional supergravity, while the open string sector yields a supersymmetric (SUSY)  $(p+1)$  dimensional gauge theory living on the stack of branes. We may also in principle try to allow for interactions between these sectors:

$$S = S_{\text{closed}} + S_{\text{open}} + S_{\text{int}} . \tag{1}$$

We now argue that these interactions must be irrelevant in the sense of the renormalization group, and that at low energies  $S_{\text{closed}}$  and  $S_{\text{open}}$  decouple.

The Einstein-Hilbert term in the supergravity action is the usual

$$\frac{1}{2\kappa^2} \int \sqrt{-g} R d^{10}x, \quad (2)$$

from which we see that dimensionally  $[\kappa] \sim \ell_s^4$ . The gauge kinetic term in the gauge theory is the standard

$$\frac{1}{4e^2} \int \sqrt{-\tilde{g}} \text{tr} F^2 d^{p+1}x, \quad (3)$$

from which we find that  $[e] \sim \ell_s^{(p-3)/2}$ . (Note that we have recovered the usual fact that in 3+1 dimensions, gauge theories are classically scale invariant. One loop corrections of course can make  $e$  run with scale.) To see how  $\kappa$  and  $e$  enter into  $S_{\text{int}}$ , we first perform the rescaling  $g_{\mu\nu} \rightarrow g_{\mu\nu} + \kappa h_{\mu\nu}$  and  $A_\mu \rightarrow eA_\mu$ . A term in  $S_{\text{int}}$  will take the schematic form

$$\gamma \int d^{p+1}x (\kappa h)^\ell (eA)^m (\partial)^n \quad (4)$$

where  $\partial$  is a derivative. Thus the interaction must scale as

$$[\gamma] \sim \ell_s^{-p-1+m+n}.$$

The physical coupling is not  $\gamma$  but  $\tilde{\gamma} = \gamma \kappa^\ell e^m$  which scales instead as

$$\begin{aligned} [\tilde{\gamma}] &\sim \ell_s^{-p-1+m+n+4\ell+m(p-3)/2} \\ &\sim \ell_s^{(\frac{m}{2}-1)(p-1)+(n-2)+4\ell}. \end{aligned} \quad (5)$$

We would like to argue that the exponent can never be negative. For the term to be an interaction, certainly either  $(m \geq 1 \text{ and } \ell \geq 2)$  or  $(m \geq 2 \text{ and } \ell \geq 1)$ . The worst case, i.e. most relevant, scenario is  $n = 0$ , ignoring issues of gauge and diffeomorphism invariance which anyway would only force us to choose a larger value of  $n$ . Thus we should check the two worst case scenarios: a)  $m = 1, \ell = 2$ , and  $n = 0$ ; and b)  $m = 2, \ell = 1$ , and  $n = 0$ . Choosing larger values of  $m, \ell$  and  $n$  will just make  $\tilde{\gamma}$  more irrelevant. In case (a),  $[\tilde{\gamma}] \sim \ell_s^{6-(p-1)/2}$  which is irrelevant provided  $p < 13$  which is always true for ten dimensional string theories. In case (b),  $[\tilde{\gamma}] \sim \ell_s^2$  which is always irrelevant.<sup>1</sup>

We should of course be more careful and look at the supersymmetric partners of the graviton and the gauge field. One might worry that interactions between some of these partner fields might be relevant and affect the low energy dynamics. We claim they do not and that in the low energy limit  $S_{\text{int}}$  is effectively zero. Thus  $S_{\text{open}}$  and  $S_{\text{closed}}$  decouple.

## 2.2 Decoupling from the closed string point of view

We now change our perspective to a purely closed string point of view. After all open strings traveling between D-branes can equally well be thought of as closed strings exchanged between the D-branes. Martin Roček in his lectures described a set of  $p$ -brane solutions in type IIA and IIB

<sup>1</sup>We are tacitly excluding D0-branes in this discussion, where we do not have a dynamical gauge field anyway.

supergravity. These solutions preserve 16 of the 32 supercharges and describe the back reaction of the geometry and RR-fields to the presence of a stack of  $N$  D $p$ -branes. These  $p$ -brane solutions satisfy the equations of motion that follow from the following gravity action

$$S = \frac{1}{(2\pi)^7 \ell_s^8} \int d^{10}x \sqrt{-\gamma} \left[ e^{-2\Phi} (R + 4(\nabla\Phi)^2) - \frac{1}{2} |F_{p+2}|^2 \right]. \quad (6)$$

where  $F_{p+2} = dA_{p+1}$  and  $\gamma_{\mu\nu}$  is the string frame metric.<sup>2</sup> (See Appendix A for the bosonic pieces of the type IIA and IIB supergravity actions.) As this action is in string frame, the dilaton kinetic term has the wrong sign. We introduce a fluctuation field  $\phi$  such that  $e^\Phi = g_s e^\phi$  and rescale the metric  $g_{\mu\nu} = e^{-\phi/2} \gamma_{\mu\nu}$ . Under this rescaling, the Ricci scalar transforms as

$$R[\gamma_{\mu\nu}] = e^{-\phi/2} \left[ R[g_{\mu\nu}] - \frac{9}{2} \nabla^2 \phi - \frac{9}{2} (\nabla\phi)^2 \right]. \quad (7)$$

We deduce the following Einstein frame action

$$S = \frac{1}{(2\pi)^7 g_s^2 \ell_s^8} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{2} g_s^2 e^{(3-p)\phi/2} |F_{p+2}|^2 \right], \quad (8)$$

which now has a canonically normalized kinetic term for the dilaton field  $\phi$ . The equations of motion that follow from this action are

$$d \star e^{(3-p)\phi/2} F_{p+2} = 0, \quad (9)$$

$$\nabla^2 \phi = \frac{3-p}{2} g_s^2 e^{(3-p)\phi/2} |F_{p+2}|^2, \quad (10)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} (\partial_\mu \phi) (\partial_\nu \phi) + \frac{1}{2(p+1)!} e^{(3-p)\phi/2} g_s^2 F_{\mu\rho_1 \dots \rho_{p+1}} F_{\nu}{}^{\rho_1 \dots \rho_{p+1}} - \frac{1}{4} g_{\mu\nu} \left( (\nabla\phi)^2 + \frac{e^{(3-p)\phi/2}}{(p+2)!} g_s^2 F_{\rho_1 \dots \rho_{p+2}} F^{\rho_1 \dots \rho_{p+2}} \right). \quad (11)$$

The  $p$ -brane causes the metric to back react into the form

$$ds^2 = H^{(p-7)/8} \left( -dt^2 + \sum_{i=1}^p dx_i^2 \right) + H^{(p+1)/8} \sum_{a=1}^{9-p} dy_a^2. \quad (12)$$

The dilaton and RR potential are

$$e^\phi = H^{(3-p)/4}; \quad g_s F_{p+2} = dt \wedge dx_1 \wedge \dots \wedge dx_p \wedge dH^{-1}. \quad (13)$$

where  $H$  is a harmonic function ( $\nabla^2 H = 0$ ). When the  $N$  D $p$ -branes are coincident, the harmonic function takes the form

$$H(r) = 1 + \left( \frac{L}{r} \right)^{7-p}, \quad r^2 = \vec{y}^2. \quad (14)$$

Although the factor of 1 in  $H(r)$  is not needed to have a solution, we include it so that the metric becomes flat  $\mathbb{R}^{1,9}$  in the limit  $r \rightarrow \infty$  (far from the D $p$ -branes).

The quantity  $L$  should reflect the number of D $p$ -branes somehow. It follows from the definitions that

$$e^{(3-p)\phi/2} \star F_{p+2} = \frac{1}{g_s} (7-p) L^{7-p} \text{dvol}(S^{8-p}), \quad (15)$$

<sup>2</sup>In our conventions  $p!|F_p|^2 = F_{\mu_1 \dots \mu_p} F^{\mu_1 \dots \mu_p}$ .

where, thinking of the  $\mathbb{R}^{9-p}$  coordinatized by the  $y_a$  as  $\mathbb{R}^+ \times S^{8-p}$ ,  $\text{dvol}(S^{8-p})$  is the volume form on the  $S^{8-p}$ . Note that the funny factor of  $e^{(3-p)\phi/2}$  would disappear in string frame. The first few editions of Polchinski contain some mistakes regarding the D $p$ -brane quantization condition although the discussion there is well worth reading. From (5) of ref. [7], we have that the RR field strength should be quantized such that

$$\int_{S^{8-p}} e^{(3-p)\phi/2} \star F_{p+2} = (2\pi\ell_s)^{7-p} N . \quad (16)$$

This quantization constraint tells us that  $L$  and  $N$  are related:

$$L^{7-p} = \frac{(2\pi\ell_s)^{7-p} g_s N}{(7-p) \text{Vol}(S^{8-p})} = (4\pi)^{(5-p)/2} \Gamma\left(\frac{7-p}{2}\right) \ell_s^{7-p} g_s N . \quad (17)$$

As we will use it greatly in what follows, we note that for the D3-brane,  $(L/\ell_s)^4 = 4\pi g_s N$ .

The equations of motion are straightforward to check. From the Hodge dual expression (15), it is clear that  $F_{p+2}$  satisfies the first equation of motion (9). Using the fact that  $\nabla^2 H = 0$ , it is also straightforward to check the second equation of motion (10). We check only the trace of Einstein's equations (11):

$$2R = (\nabla\phi)^2 + (3-p)e^{(3-p)\phi/2} g_s^2 |F_{p+2}|^2 . \quad (18)$$

From the form of the metric, we can compute that

$$R = -\frac{p+1}{8} \frac{\nabla^2 H}{H} + \frac{(p+1)(p-3)}{32} \frac{(\nabla H)^2}{H^2} , \quad (19)$$

where the first term on the right hand side must vanish because  $H$  is harmonic. Using the dilaton equation of motion (10) to replace  $|F_{p+2}|^2$ , it is straightforward to check that the equation (18) is satisfied. In appendix A, we give the bosonic pieces of the supergravity actions in ten and eleven dimensions. We leave it as an exercise to the reader to verify that the D $p$ -brane solution above also satisfies the type IIA and IIB equations of motion.

Given this D $p$ -brane solution, we can make an argument that the small and large  $r$  regions decouple in a low energy limit. From the point of view of an observer at  $r = \infty$ , there are two types of low energy excitations. The first are those at large  $r$  which locally have low energy. The second are those at small  $r$  which locally can have any energy but because of the red-shift factor are perceived by the observer at  $r = \infty$  to have low energy. The red-shift factor is given by the time component of the metric  $\sqrt{-g^{tt}(r)} = H(r)^{(7-p)/16}$ . Energies observed at different radii are related via

$$E(r_1) \sqrt{-g_{tt}(r_1)} = E(r_2) \sqrt{-g_{tt}(r_2)} . \quad (20)$$

In particular, for the observer at  $r = \infty$ , we find that  $E(\infty) = E(r) H(r)^{(p-7)/16}$ . As  $r \rightarrow 0$  (and provided  $p < 7$ ),  $E(\infty)$  becomes much, much less than  $E(r)$ .

Now small  $r$  excitations have a difficult time to get to large  $r$ . The warp factor  $H(r)$  acts like a gravitational well, as we can see by studying time-like geodesics traveled by a massive particle.

Suppose we try to throw a massive particle directly up from some small value of  $r$ . The part of the line element that concerns us is (assuming  $p < 7$ )

$$ds^2 = -H^{(p-7)/8} dt^2 + H^{(p+1)/8} dr^2 . \quad (21)$$

To find the geodesic, we consider the equation of motion that follows from the following action

$$S = -m \int dt \sqrt{H^{(p-7)/8} - H^{(p+1)/8} \dot{r}^2} . \quad (22)$$

As the action does not depend explicitly on time, there is a conserved energy

$$\frac{E}{m} = \frac{H^{(p-7)/16}}{\sqrt{1 - H\dot{r}^2}} . \quad (23)$$

If we start with a massive particle at a radius  $r \ll L$  and some initial upward radial velocity  $\dot{r}$ , then the particle will eventually come to rest and start falling back down at a maximal radius given by  $H(r_{\max}) = (m/E)^{16/(7-p)}$ . For energies  $E \ll m$ , this maximal radius will satisfy the inequality  $r_{\max} \ll L$ .

Not only do excitations at small radius have trouble escaping to large radius, but large radius excitations typically will not fall into the gravitational well and end up at small radius. One can calculate the cross-section for these excitations. As the cross-section should scale perturbatively as the gravitational coupling squared, we find that  $\sigma \sim \ell_s^8$ . We should measure the cross section with respect to directions transverse to the  $Dp$ -brane and the radial direction. Thus dimensionally we expect that  $[\sigma] \sim \ell_s^{10-p-2}$ . By dimensional analysis then we find that  $\sigma \sim E^p \ell_s^8$ . As the energy  $E$  goes to zero, the cross-section will go to zero as well. (See ref. [4] for a more careful calculation.)

## 2.3 Making a correspondence

From both the open and closed string description of the  $Dp$ -branes, there is a sector of the theory which corresponds to ten dimensional supergravity in flat space. From the open string point of view, we called this sector  $S_{\text{closed}}$ . From the closed string point of view, this sector was the large  $r$  region of our  $Dp$ -brane gravity solution. Given that the interactions between  $S_{\text{closed}}$  and  $S_{\text{open}}$  were irrelevant from the open string point of view and also given that the large  $r$  and small  $r$  regions of our  $Dp$ -brane gravity solution do not interact, we are naturally led to conjecture that  $S_{\text{open}}$  and the small  $r$  region of the  $Dp$ -brane gravity solution describe the same physics.

The case of D3-branes is special. Consider the behavior of the Ricci scalar curvature (19) at small  $r$ . Plugging in  $H(r)$ , we find that in every dimension except  $p = 3$  (where it vanishes exactly)  $R \sim r^{-(p-3)^2/8}$  diverges in the  $r \rightarrow 0$  limit, signaling the need for higher curvature corrections. A second problem is associated with the dilaton. In the case where  $p < 3$ , the dilaton diverges in the limit  $r \rightarrow 0$ , indicating that the effective string coupling is diverging and signaling the need for higher stringy corrections. These facts single out  $p = 3$ . In all other cases, the gravity solution will need to be corrected in the small  $r$  region of the geometry, either because of  $g_s$  corrections, curvature corrections, or both. Indeed, we will see soon that the  $r \rightarrow 0$  limit for  $p = 3$  is smooth and innocuous.

Given the need for curvature and stringy corrections when  $p \neq 3$ , it is most natural to focus henceforth on the case  $p = 3$ . The small  $r \ll L$  region of the D3-brane solution has the line element

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + d\vec{x}^2) + L^2 \frac{dr^2}{r^2} + L^2 d\Omega_5^2, \quad (24)$$

where  $L^4 = 4\pi g_s N \ell_s^4$ . The line element  $d\Omega_5^2$  gives the metric on a unit  $S^5$ . The remaining pieces of the metric describe the so-called Poincaré patch of five dimensional anti-de Sitter space,  $\text{AdS}_5$ , a smooth homogenous space with no singularities. (See appendix B for various other coordinate systems on AdS and their corresponding metrics.) The constant  $L$  is radius of curvature of both the  $S^5$  and the  $\text{AdS}_5$ . To stay out of the stringy regime, we need that this curvature scale should be large compared to the string length,  $L \gg \ell_s$  which in turn implies that  $g_s N \gg 1$ . To keep the string coupling small, we also need that  $g_s \ll 1$ . In most of the calculations we will use a slightly different radial coordinate  $z = L^2/r$ , for which the line element takes the more compact form

$$ds^2 = L^2 \left[ \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2} + d\Omega_5^2 \right]. \quad (25)$$

Including the one in  $H(r)$ , the original D3-brane solution preserved 16 of the 32 supersymmetries of type IIB supergravity. See appendix C for details. In the near horizon limit, we recover the lost 16, for a total of 32. (On the field theory side of the correspondence, this doubling comes from the fact that the gauge theory is conformal.) The  $\text{AdS}_5$  has an  $\text{SO}(2,4)$  isometry group, which is the conformal group for a field theory in  $\mathbb{R}^{1,3}$ . The sphere  $S^5$  has an  $\text{SO}(6) = \text{SU}(4)/\mathbb{Z}_2$  isometry group. This group is the global R symmetry group of a field theory with  $\mathcal{N} = 4$  supersymmetry in  $\mathbb{R}^{1,3}$ . For an expert,  $S_{\text{open}}$  could only be one thing. The  $\mathcal{N} = 4$  SUSY gauge theory in  $\mathbb{R}^{1,3}$  is unique up to choice of gauge group, and it's conformal. The “obvious” choice of gauge group is  $U(N)$  for a stack of  $N$  D3-branes without orientifolds.<sup>3</sup>

There is also an obvious choice for the identity of the gauge coupling and theta angle in the field theory. Both  $\mathcal{N} = 4$   $\text{SU}(N)$  Yang-Mills and type IIB supergravity are invariant under an  $\text{SL}(2, \mathbb{Z})$ . See appendix A for the details of the type IIB transformation rules. On the Yang-Mills theory, we have

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}, \quad (26)$$

while on the type IIB SUGRA side we have

$$\tau = C_0 + ie^{-\Phi}. \quad (27)$$

Both transform according to the rule

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \quad (28)$$

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<sup>3</sup>There is a subtle issue here of  $U(N)$  vs.  $\text{SU}(N)$ . They are related by a  $U(1)$  subgroup. It is often said that the diagonal  $U(1)$  corresponds to the center of mass degree of freedom of the stack and decouples from the other degrees of freedom.



where  $ad - bc = 1$ . This coincidence suggests the identifications<sup>4</sup>

$$g_{\text{YM}}^2 = 4\pi g_s, \quad \theta = 2\pi C_0. \quad (29)$$

A more careful calculation involving the DBI action of a couple of probe D3-brane bears out these identifications.

$$S_{D_3} = -\frac{1}{(2\pi)^3 \ell_s^4} \int d^4\xi \text{Tr}_f \left\{ e^{-\Phi} [-\det(G_{ab} + B_{ab} + 2\pi\ell_s^2 F_{ab})]^{1/2} \right. \\ \left. + \frac{1}{(2\pi)^3 \ell_s^4} \int \text{Tr}_f [\exp(2\pi\ell_s^2 F_2 + B_2) \wedge \sum_q C_q] + \dots \right. \quad (30)$$

$$= \dots - \frac{1}{(2\pi)^3 \ell_s^4} \int d^4\xi \text{Tr}_f e^{-\Phi} \frac{(2\pi\ell_s^2)^2}{4} F_{ab} F^{ab} \\ + \frac{1}{(2\pi)^3 \ell_s^4} \int \text{Tr}_f \frac{(2\pi\ell_s^2)^2}{2} C_0 F_2 \wedge F_2 + \dots \quad (31)$$

$$= \dots + \int d^4\xi \frac{1}{8\pi e^\Phi} \frac{1}{2} F_{ab}^\alpha F^{\alpha ab} + \frac{1}{4\pi} \frac{1}{2} \int d^4\xi C_0 F_{ab}^\alpha F_{cd}^\alpha \epsilon^{abcd} \frac{1}{4} + \dots \quad (32)$$

The factors of  $1/2$  in the last line come from the trace in the fundamental representation  $\text{Tr}_f$ .

To recap, we have found a gravitational D3-brane solution which should describe  $\mathcal{N} = 4$  SUSY Yang-Mills in a particular parameter regime. To avoid gravitational and stringy corrections, we required that  $g_s N \gg 1$  and  $g_s \ll 1$ . On the gauge theory side, these constraints mean that the 't Hooft parameter  $\lambda = g_{\text{YM}}^2 N = 4\pi g_s N$ , which controls the perturbative expansion of the gauge theory, must be taken large compared to one. Also to keep  $g_s \ll 1$ , we must take a large  $N$  limit,  $N \gg 1$ , on the gauge theory side.

The discussion of massive geodesics and black hole cross sections in the  $Dp$ -brane spacetime foreshadows the identification of the radial coordinate as an energy scale in the field theory. If we for the moment assume this identification is correct, we can interpret the behavior of the Ricci scalar  $R$  and the dilaton  $\Phi$  in string frame in the cases where  $p \neq 3$ . Unlike the Einstein frame discussed above, in string frame

$$R = \frac{(3-p)(1+p)}{4} \frac{(\nabla H)^2}{H^2}$$

only diverges in the limit  $r \rightarrow 0$  when  $p > 3$ . Thus one could interpret the divergence in  $R$  for  $p < 3$  in Einstein frame as really coming from the string coupling. Take  $p < 3$ , where the gauge coupling becoming strong at low energies mirrors the fact that the dilaton blows up at small  $r$ . Having thrown away the one in  $H$  to take the near horizon limit, the Ricci scalar blows up at large  $r$ , indicating the need for higher curvature corrections, suggesting that the 't Hooft coupling is getting weak at high energies. Correspondingly when  $p > 3$ , we have the opposite situation. Having thrown away the one in  $H$ , the dilaton blows up at large  $r$ , corresponding to the fact that the effective gauge coupling gets large at high energy. The Ricci scalar now blows up at small  $r$ , signaling the need for higher curvature corrections and suggesting that the 't Hooft coupling has become small at low energies.

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<sup>4</sup>There seem to be some disagreements in the literature regarding the constant of proportionality relating  $g_{\text{YM}}^2$  and  $g_s$ . For example the relation in the review [6] does not include the factor of  $4\pi$ .

### 3 How AdS/CFT works

The equivalence of  $\mathcal{N} = 4$  SYM and type IIB string theory in  $\text{AdS}_5 \times S^5$  should imply an equality between their respective path integrals. On the gauge theory side, we ought to be able to include gauge invariant sources in the path integral. On the gravity side,  $\text{AdS}_5$  is a space with boundary (at  $z = 0$  in the parametrization (25)). Thus to be well defined, we need to include boundary conditions. Consider probing the stack of D3-branes with wave packets sent in from the asymptotically flat part of the D3-brane geometry. From the near horizon point of view, these wave packets look alternately like sourcing gauge invariant operators on the D-branes or like setting boundary conditions for the  $\text{AdS}_5$  space-time. This line of reasoning leads to the central postulate of the AdS/CFT correspondence, a result [2, 3] whose importance can not be over emphasized:

$$e^{W_{\text{CFT}}[\phi_0]} \equiv \left\langle \exp \int d^4x \phi_0(x) \mathcal{O}(x) \right\rangle_{\text{CFT}} = Z_{\text{string}}[\phi(x, z)|_{z \rightarrow 0} = \phi_0(x)] . \quad (33)$$

In this expression  $\phi(x, z)$  is a field on the string side of the story. Its boundary value  $\phi_0(x)$  can alternately be interpreted as a source for a gauge invariant operator  $\mathcal{O}(x)$  in the conformal field theory. The CFT quantity  $W_{\text{CFT}}$  is then a generating functional for connected correlation functions of  $\mathcal{O}(x)$  in the CFT.

While the correspondence (33) is expected to hold true in general, we will be interested in it primarily in the limit  $g_s$  and  $\ell_s/L \rightarrow 0$ . In field theory terms, this double limit is  $g_{\text{YM}}^2 N$  and  $N \rightarrow \infty$ . Given that we are working in  $\text{AdS}_5$  with a scale set by the radius of curvature  $L$ , we can first replace  $Z_{\text{string}}$  by the corresponding supergravity partition function  $Z_{\text{SUGRA}}$ . Then the effective gravitational coupling constant in the supergravity action (8) we can identify as

$$\frac{(2\pi)^7 g_s^2 \ell_s^8}{L^8} = \frac{8\pi^2}{N^2} \text{Vol}(S^5) . \quad (34)$$

Because  $N$  is large, a saddle point approximation of  $Z_{\text{SUGRA}} \sim e^{-S_{\text{os}}}$  becomes accurate. (We work in Euclidean signature here.) In other words, the on-shell gravitational action  $S_{\text{os}}[\phi_0]$  (i.e. the action evaluated using the equations of motion) is a good approximation to the generating functional,  $W_{\text{CFT}}[\phi_0] \approx -S_{\text{os}}[\phi_0]$ . We can therefore use classical gravity to compute connected correlation functions in the CFT in the limit  $N \rightarrow \infty$ .

We would like to explore the consequences of the postulate (33) for a free scalar field. Consider then the action for a real scalar in the Poincaré patch of (Euclidean)  $AdS_{d+1}$ :

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{-g} \left[ (\partial\phi)^2 + m^2 \phi^2 \right] . \quad (35)$$

We focus here just on the  $\text{AdS}_{d+1}$  geometry and use the line element

$$ds^2 = L^2 \left[ \frac{\delta_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2} \right] . \quad (36)$$

To produce a generating function for the CFT correlation functions, we need to evaluate this action on-shell with a prescribed boundary condition for  $\phi$  at  $z = 0$ . To that end, let us start with the

equation of motion for  $\phi$ :

$$(z^{d+1}\partial_z z^{-d+1}\partial_z + z^2\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2L^2)\phi = 0 . \quad (37)$$

Typically boundary conditions for second order differential equations are either Dirichlet or Neumann. Here, however,  $z = 0$  is a singular point, and the boundary behavior is described instead by two characteristic exponents which satisfy the following indicial equation:

$$\Delta(\Delta - d) = m^2L^2 , \quad (38)$$

as can be seen by plugging  $\phi \sim z^\Delta$  into the equation of motion (37) and expanding the result near  $z = 0$ . Generically, one finds the following behavior for  $\phi$  near  $z = 0$ :

$$\phi = az^{d-\Delta}(1 + O(z^2)) + bz^\Delta(1 + O(z^2)) . \quad (39)$$

(Interesting issues arise when  $\Delta$  is an integer and the series overlap. Extra logarithmic terms appear which we shall ignore.) If we assume  $\Delta > d/2$ , then  $a$  describes the leading small  $z$  behavior and we can tentatively identify  $a = \phi_0$  with the source term in the CFT. The singular behavior at  $z = 0$  means we should really work with a  $z = \epsilon$  cutoff and modify the basic statement (33) to include an  $\epsilon$  dependence,  $\phi|_{z=\epsilon} = \phi_0\epsilon^{d-\Delta}$ , taking the  $\epsilon \rightarrow 0$  limit only at the end.

Given that the boundary  $z = 0$  is a singular point and we cannot use typical Dirichlet or Neumann boundary conditions, it is not obvious that the action (35) has a well defined variational principle. In varying the action, we are left with the following boundary term

$$\begin{aligned} \delta S &= - \int_{z=\epsilon} d^d x \left(\frac{L}{z}\right)^{d-1} \delta\phi(x, z)\partial_z\phi(x, z) \\ &= -L^{d-1} \int_{z=\epsilon} d^d x \frac{1}{z^d} (\delta a z^{d-\Delta} + \delta b z^\Delta + \dots)(a(d-\Delta)z^{d-\Delta} + b\Delta z^\Delta + \dots) \\ &= -L^{d-1} \int_{z=\epsilon} d^d x [(d-\Delta)a\delta a z^{d-2\Delta} + (\Delta\delta a b + (d-\Delta)\delta b a) \\ &\quad + \Delta b\delta b z^{2\Delta-d} + \dots] . \end{aligned} \quad (40)$$

There are really three potentially overlapping power series in the last line. The boundary variation (40) includes only the leading term in each power series; the ellipses denote the subleading terms. In the context of the variational principle, we fix the boundary behavior  $a = \phi_0$ . Thus, we insist that  $\delta a = 0$ . There remains a term proportional to  $\delta b a$  which we need to cancel through the addition of a boundary term. (The  $b\delta b$  term will vanish given our assumption that  $2\Delta > d$ .) We add

$$S_{\text{bry}} = \frac{c}{L} \int_{z=\epsilon} d^d x \sqrt{-\gamma} \phi^2(x, z) \quad (41)$$

where  $\gamma_{\mu\nu}$  is the induced metric on the  $z = \epsilon$  slice of the geometry, and  $c$  is a constant to be determined. The choice of counter-terms is guided by the requirements that  $S_{\text{bry}}$  be local, Lorentz invariant, and depend only intrinsically on the geometry of the boundary. One could imagine also terms of the form  $\phi\Box\phi$  and  $\phi\Box^2\phi$  where  $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$  or even, in the case of a curved boundary,

$R\phi^2$  where  $R$  is the Ricci scalar curvature of the boundary. By dimensional analysis, these higher derivative terms must come with additional powers of  $z$  and cannot cancel the leading  $a\delta b$  term.

Given the boundary term (41), the variation is then

$$\delta S_{\text{bry}} = 2cL^{d-1} \int_{z=\epsilon} d^d x [a\delta a z^{d-2\Delta} + (a\delta b + b\delta a) + b\delta b z^{2\Delta-d} + \dots] , \quad (42)$$

To cancel the  $a\delta b$  term in (40), we should set the constant  $c = (d - \Delta)/2$ .

Having ensured that the on-shell value of the action is indeed an extremum, and thus that the saddle-point approximation is sensible, we can ask what the response of the system is to small changes  $\delta a$  in the source term. The calculation is essentially already done. The leading  $a\delta a$  term cancels and one finds

$$\delta S_{\text{tot}} = \delta S + \delta S_{\text{bry}} = L^{d-1} \int_{z=\epsilon} d^d x (d - 2\Delta)b\delta a . \quad (43)$$

The expectation value of the operator dual to  $\phi$  then follows from the basic postulate (33):

$$\langle \mathcal{O} \rangle = -\frac{\delta S_{\text{tot}}}{\delta \phi_0} = -\frac{\delta S_{\text{tot}}}{\delta a} = L^{d-1}(2\Delta - d)b . \quad (44)$$

We have come to a second omission in the discussion. The ellipses in the variations (40) and (42) contain subleading terms in the  $a\delta a$  series which may be dominant compared to the  $b\delta a$  term considered in (43). In general, we require further counter-terms to cancel these subleading  $a\delta a$  pieces and to prevent  $\langle \mathcal{O} \rangle$  from being UV divergent. As an example, one may consider the subleading term in the  $a\delta a$  series, proportional to  $z^{d-2\Delta+2}\delta a\Box a$ . Assuming  $2\Delta > d + 2$ , this term is dominant compared to  $b\delta a$ , but it can be canceled by adding a  $\phi\Box\phi$  boundary term to the action. That these counterterms can be identified in general and that  $\langle \mathcal{O} \rangle$  can be renormalized is discussed in more detail in for example ref. [8]. The procedures described above for scalar fields can be generalized for higher spin fields. These techniques usually go by the name of ‘‘holographic renormalization’’.

Note that the characteristic exponent  $\Delta$  is also the scaling dimension of the operator  $\mathcal{O}$ . The transformation rule  $x \rightarrow \Lambda x$  and  $z \rightarrow \Lambda z$  is a symmetry of the line element (36) and of the geometry of  $\text{AdS}_{d+1}$ . The restriction of the scaling symmetry to the boundary  $z = 0$  corresponds to scale transformations of the CFT. Under this scale transformation, the field  $\phi$  transforms as  $\phi'(z, x) = \phi(\Lambda z, \Lambda x)$ . Thus we find that

$$\langle \mathcal{O}' \rangle = \Lambda^\Delta \langle \mathcal{O} \rangle . \quad (45)$$

Primary scalar operators in CFT satisfy a unitarity bound [9],  $\Delta > (d - 2)/2$ , saturated by the free field case. The assumption  $\Delta > d/2$  thus leaves out a set of operators with scaling dimension in the range  $(d - 2)/2 < \Delta < d/2$ . To close this gap, let us now assume that  $\Delta < d/2$  and repeat the exercise we went through above. We still freeze the value of  $a$  and thus set  $\delta a = 0$ . Now, in addition to canceling the  $a\delta b$  term in the variation (40), we also need to cancel the  $b\delta b$  term, which no longer vanishes in the limit  $z \rightarrow 0$ . Breaking from our rule that counterterms should depend only on the intrinsic geometry of the boundary, we add a Gibbons-Hawking like term that depends on a

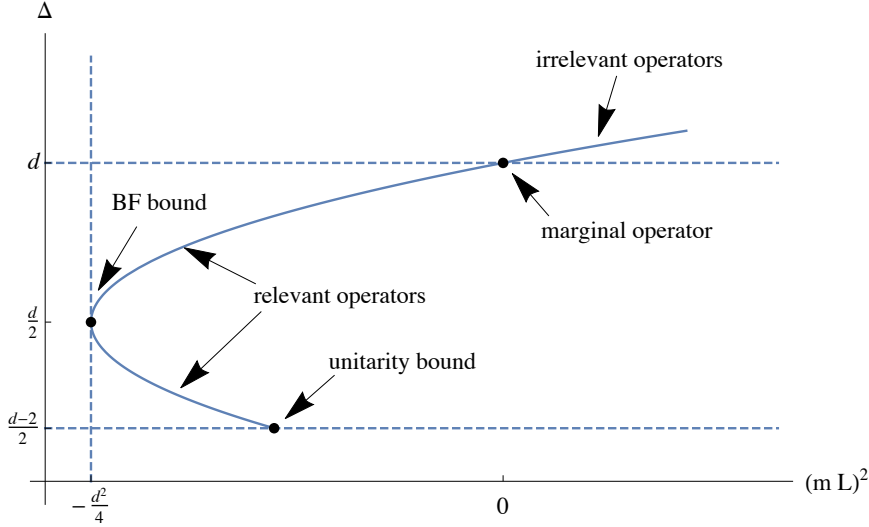


Figure 1: A plot of the scaling dimension  $\Delta$  of  $\mathcal{O}$  versus the mass  $m$  of the  $\text{AdS}_{d+1}$  scalar  $\phi$ .

normal derivative

$$S_{\text{bry}} = \int_{z=\epsilon} d^d x \sqrt{-\gamma} \left( \frac{c}{L} \phi^2 + c' n^\mu \phi \partial_\mu \phi \right) . \quad (46)$$

where  $c$  and  $c'$  are constants and  $n^\mu = (0, z/L)$  is a unit normal to the boundary. We leave it as an exercise to show that  $c' = 1$  and  $c = -\Delta/2$  for a good variational principle. Just as we did earlier, we can then consider the response of the system to a small  $\delta a$ . We find that  $\langle \mathcal{O} \rangle = L^{d-1} (2\Delta - d) b$ , just as before. In the window  $(d-2)/2 < \Delta < d/2$ , there are no subleading divergences in the  $b \delta b$  series, and no further counter-terms are needed.

The set of scalar fields considered in this lecture is summarized pictorially in figure 1. The point  $\Delta = d/2$  where the curve turns around is known as the Breitenlohner-Freedman (BF) bound. It is the smallest mass-squared for a scalar field in  $\text{AdS}_{d+1}$  that allows for a sensible stress-energy tensor [10, 11].

While for simplicity, we have focused on the simplest case of the Poincaré patch, the techniques here generalize to situations where the space is only asymptotically, in the limit  $z \rightarrow 0$ , of AdS type. From a CFT point of view, this restriction on the asymptotics means keeping the UV behavior of the field theory the same. One could imagine, for example, providing a nonzero source  $\phi_0 \neq 0$  for a relevant operator  $\Delta < d$ , in which case the large  $z$  (i.e. low energy) geometry will generally be modified. The small  $z$  asymptotics remain the same, and now we may calculate correlation functions in the presence of the source. On the other hand, if we add a source for an irrelevant operator  $\Delta > d$ , the small  $z$  (i.e. high energy) geometry will be modified and the preceding results can no longer be applied.

### 3.1 Scalar Two-Point Functions in Pure AdS<sub>d+1</sub>

Above, in expanding the field  $\phi(x, z)$  near the boundary

$$\phi(x, z) = z^{d-\Delta}a(1 + \dots) + z^\Delta b(1 + \dots)$$

and positing  $a = \phi_0$ , we found that the one-point function  $\langle O \rangle \sim b$  was determined by the coefficient of the second series. Here, we will use a second boundary condition to find a relation between  $b$  and the source  $a$ . Given that relation, we can then compute a two-point correlation function  $\langle OO \rangle$  by varying  $\langle O \rangle$  with respect to  $a = \phi_0$ .

In pure AdS<sub>d+1</sub>, we can find an explicit solution of the equations of motion (37) for the scalar field. We first make a plane wave ansatz,  $\phi \sim e^{k \cdot x} \phi(z)$ . The equation of motion simplifies to an ordinary differential equation

$$z^{d+1}(z^{1-d}\phi')' - (z^2k^2 + m^2L^2)\phi = 0, \quad (47)$$

where  $'$  denotes  $\partial_z$ . Next, we make the substitution  $\phi(z) = z^{d/2}H(z)$ ,

$$z^2H'' + zH' - \left(k^2z^2 + m^2L^2 + \frac{d^2}{4}\right)H = 0, \quad (48)$$

and recognize a second order differential equation of Bessel type. In the Euclidean or space-like case where  $k^2 > 0$ , we find a solution in terms of Hankel functions:

$$H = c_1H_\nu^{(1)}(ikz) + c_2H_\nu^{(2)}(ikz), \quad (49)$$

where we have defined  $\nu \equiv \sqrt{m^2L^2 + d^2/4}$ . To fix the second boundary condition, consider the large  $z$  behavior where  $H_\nu^{(1)}(ikz) \sim e^{-kz}$  and  $H_\nu^{(2)}(ikz) \sim e^{kz}$ , allowing us to set  $c_2 = 0$  and throw out the second, exponentially growing solution.

To extract the two-point function, consider the small  $z$  expansion of the solution, assuming  $\Delta > d/2$  and that  $\nu$  is not an integer,

$$\phi = c_1 \left[ z^{d-\Delta} \left( -\frac{i}{\pi} \left( \frac{2}{ik} \right)^\nu \Gamma(\nu) + \dots \right) + z^\Delta \left( \left( \frac{ik}{2} \right)^\nu \frac{1 + i \cot(\pi\nu)}{\Gamma(1+\nu)} + \dots \right) \right]. \quad (50)$$

From the leading and subleading coefficients of the series expansion, we can read off the values of  $\phi_0$  and  $\langle O \rangle$ :

$$\phi_0 = c_1 \left( -\frac{i}{\pi} \right) \left( \frac{2}{ik} \right)^\nu \Gamma(\nu), \quad (51)$$

$$\langle O \rangle = (2\Delta - d)L^{d-1}c_1 \left( \frac{ik}{2} \right)^\nu \frac{1 + i \cot(\pi\nu)}{\Gamma(1+\nu)}. \quad (52)$$

The (Fourier transform of the) two-point function can then be extracted by varying the one-point function:

$$G^{\mathcal{O}\mathcal{O}}(k) = \frac{\delta\langle O \rangle}{\delta\phi_0} = \frac{\langle \mathcal{O} \rangle}{\phi_0} = (-2\nu) \left( \frac{ik}{2} \right)^{2\nu} (i\pi) \frac{1 + i \cot(\pi\nu)}{\Gamma(\nu)\Gamma(1+\nu)} L^{d-1} \quad (53)$$

We need now to Fourier transform back to position space. Focusing on the  $k^{2\nu} = k^{2\Delta-d}$  behavior, note that by translational symmetry and dimensional analysis, the only possible result is that

$$\langle \mathcal{O}(x_2)\mathcal{O}(x_1) \rangle = \int \frac{d^d k}{(2\pi)^d} G^{\mathcal{O}\mathcal{O}}(k) e^{ik \cdot (x_2 - x_1)} \sim \frac{1}{|x_2 - x_1|^{2\Delta}} . \quad (54)$$

Two-point functions in CFT are indeed constrained to have precisely this form.

### 3.2 Gauge fields in the bulk, global symmetries in the boundary

Having gained some experience with scalar fields, we move on to gauge fields in  $\text{AdS}_{d+1}$ , which in the context of the holographic renormalization are actually somewhat simpler, requiring fewer counter-terms. Consider the following abelian gauge field in the bulk:

$$S = -\frac{1}{4e^2} \int d^{d+1}x \sqrt{-g} F_{AB} F^{AB} . \quad (55)$$

The equations of motion are simply  $\partial_A \sqrt{-g} F^{AB} = 0$ . To keep the discussion simple, we pick a radial gauge  $A_z = 0$ . The equations of motion  $\partial_A \sqrt{-g} F^{A\mu}$  expand, using the line element (36), to give

$$\partial_z z^{3-d} \partial_z A_\mu + z^{3-d} \partial_\lambda \eta^{\lambda\nu} F_{\nu\mu} = 0 . \quad (56)$$

In analogy to the scalar discussion, we consider a small  $z$  expansion of the gauge field,  $A_\mu \sim z^\Delta$ . The corresponding indicial equation

$$\Delta(\Delta + 2 - d) = 0 , \quad (57)$$

has the two roots  $\Delta = 0$  and  $\Delta = d - 2$ , leading to the following small  $z$  series solution

$$A_\mu = a_\mu(1 + \dots) + b_\mu z^{d-2}(1 + \dots) . \quad (58)$$

We should also consider the remaining equation of motion  $\partial_A \sqrt{-g} F^{Az} = 0$  which expands to give

$$\partial_\mu z^{3-d} \partial_z \eta^{\mu\nu} A_\nu = 0 . \quad (59)$$

Inserting the small  $z$  series solution into this equation of motion produces the constraint  $\partial_\mu \eta^{\mu\nu} b_\nu = 0$ . In other words,  $\eta^{\mu\nu} b_\nu$  satisfies a current conservation condition.

In determining the equations of motion, we produced a boundary term which we now consider more carefully:

$$\delta S = \frac{L^{d-3}}{e^2} \int_{z=\epsilon} d^d x z^{3-d} \eta^{\mu\nu} \delta A_\mu \partial_z A_\nu \quad (60)$$

$$= \frac{L^{d-3}}{e^2} \int_{z=\epsilon} d^d x z^{3-d} \eta^{\mu\nu} (\delta a_\mu + \delta b_\mu z^{d-2}) ((d-2)b_\nu z^{d-3} + \dots) \quad (61)$$

$$= \frac{L^{d-3}}{e^2} \int_{z=\epsilon} d^d x (d-2) \eta^{\mu\nu} \delta a_\mu b_\nu . \quad (62)$$

To get a good variational principle, where we set  $\delta a_\mu = 0$ , we need no further counter-terms. To extract the one-point function however, we may find that even though the leading  $a \delta a$  term cancels

because of the  $\partial_z$  derivative, there could be subleading divergences that are nonetheless dominant compared to the  $\delta a b$  term. In fact the situation here is further complicated by the fact that  $d - 2$  is integer and the two series may overlap, generating logarithms. There is a  $z \rightarrow -z$  symmetry of the equations of motion which implies that the series expansion is in even powers of  $z$ . Thus, the series only overlap when  $d$  is an even integer. While in  $d = 3$ , we may take the variation (62) at face value, in  $d = 4$  a logarithmic singularity appears which requires more careful treatment. In  $d > 4$ , there can be further complications. Ignoring these gritty details, we take the variation (62) at face value and compute the one-point function:

$$\langle J^\mu \rangle = \frac{\delta S}{\delta a_\mu} = \frac{(d-2)L^{d-3}}{e^2} \eta^{\mu\nu} b_\nu . \quad (63)$$

We are now in a position to identify the operator  $J^\mu$ . From the point of view of the CFT, it is sourced by an external gauge field  $a_\mu$  and satisfies a current conservation condition  $\partial_\mu J^\mu = 0$ . Thus it must be a conserved current. Note that  $a_\mu$  is not dynamical both from the gravity and CFT point of view.

### 3.3 The stress tensor

The stress-tensor operator in the CFT is one of the more difficult fields to study through AdS/CFT but also one of the most useful and interesting. It naturally couples to the boundary value of the metric. To analyze this case, let us first set some notation. The bulk metric shall be  $G_{AB}$ . We will pick a gauge where the line-element is

$$ds^2 = \frac{L^2}{z^2} dz^2 + \gamma_{\mu\nu} dx^\mu dx^\nu , \quad (64)$$

where  $\gamma_{\mu\nu}$  is the boundary metric. We further define

$$g_{\mu\nu} \equiv \frac{z^2}{L^2} \gamma_{\mu\nu} . \quad (65)$$

In general,  $g_{\mu\nu}$  will have a nontrivial  $z$  dependence which we can write for small  $z$  as

$$g_{\mu\nu} = \begin{cases} g_{\mu\nu}^{(0)} + z^2 g_{\mu\nu}^{(2)} + \dots + z^d g_{\mu\nu}^{(d)} + z^{d+2} g_{\mu\nu}^{(d+2)} + \dots , & \text{odd } d \\ g_{\mu\nu}^{(0)} + z^2 g_{\mu\nu}^{(2)} + \dots + z^d g_{\mu\nu}^{(d)} + z^d \log z h_{\mu\nu}^{(d)} + \dots , & \text{even } d \end{cases} \quad (66)$$

Note that the CFT metric is not  $g_{\mu\nu}$  but the boundary value  $g_{\mu\nu}^{(0)}$ . The full tensor structure  $g_{\mu\nu}$  contains more information, as we will see.

Given the earlier discussion of scalars and gauge fields, we can anticipate that the action will contain a bulk contribution, a boundary contribution to have a good variational principle, and further counter-terms to render the correlation functions finite:

$$S = S_{\text{EH}} + S_{\text{GH}} + S_{\text{ctr}} . \quad (67)$$

The bulk term is Einstein-Hilbert plus a negative cosmological constant, required so that AdS $_{d+1}$  is a solution of the equations of motion:

$$S_{\text{EH}} = \frac{1}{2\kappa^2} \int_M d^{d+1}x \sqrt{-G} \left( R + \frac{d(d-1)}{L^2} \right) . \quad (68)$$



However, anti-de Sitter space has a boundary and second derivatives  $R \sim \partial^2 G_{AB}$  in the action will generate boundary terms of the form  $\partial_A(\delta g_{BC})$  which need to be canceled. The standard procedure is to add a Gibbons-Hawking term

$$S_{\text{GH}} = \frac{1}{\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} K , \quad (69)$$

where  $K = G^{AB} \nabla_A n_B$  is the trace of the extrinsic curvature and  $n_B$  is an outward pointing unit normal vector. Such a boundary term will cancel normal derivatives of the metric variation  $n^A \partial_A(\delta g_{BC})$ .

The variation of the Einstein-Hilbert term gives

$$\delta S_{\text{EH}} = \frac{1}{2\kappa^2} \int_M d^{d+1} x \left[ \sqrt{-G} (\delta R_{AB}) G^{AB} + \sqrt{-G} R_{AB} \delta G^{AB} + \left( R + \frac{d(d-1)}{L^2} \right) \delta(\sqrt{-G}) \right] . \quad (70)$$

The variation of the second two terms produces Einstein's equation, which vanish on-shell. The variation of the Ricci tensor is a covariant derivative

$$\delta R_{AB} = -(\delta \Gamma_{AC}^C)_{;B} + (\delta \Gamma_{AB}^C)_{;C} , \quad (71)$$

a result sometimes known as the Palatini identity. Inside the action, this variation becomes a total derivative

$$\sqrt{-G} G^{AB} \delta R_{AB} = -(\sqrt{-G} G^{AB} \delta \Gamma_{AC}^C)_{;B} + (\sqrt{-G} G^{AB} \delta \Gamma_{AB}^C)_{;C} \quad (72)$$

Skipping some steps which we will flesh out in the next section, this total derivative reduces to the boundary term

$$\delta S_{\text{EH}} = -\frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} (n^A \gamma^{CD} \delta G_{CD;A} - K n^A n^B \delta G_{AB} + K^{AB} \delta G_{AB}) . \quad (73)$$

Meanwhile, varying the Gibbons-Hawking term leads to

$$\delta S_{\text{GH}} = \frac{1}{\kappa^2} \int_{\partial M} d^d x [\sqrt{-\gamma} \delta K - K \delta(\sqrt{-\gamma})] . \quad (74)$$

Again skipping some steps, the variation of the extrinsic trace produces

$$\delta K = \frac{1}{2} \gamma^{CD} \delta G_{CD;A} n^A - \frac{K}{2} n^A n^B \delta G_{AB} .$$

Assembling the pieces, the boundary variation is then

$$\delta S_{\text{EH}} + \delta S_{\text{GH}} = -\frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} (K^{\mu\nu} - K \gamma^{\mu\nu}) \delta \gamma_{\mu\nu} \quad (75)$$

where  $K_{AB} = \nabla_{(A} n_{B)}$ . Thus the ‘‘bare’’ stress tensor will be<sup>5</sup>

$$(T_{\text{bare}})^{\mu\nu} \frac{\sqrt{-g^{(0)}}}{2} = \frac{\delta S}{\delta g_{\mu\nu}^{(0)}} = -\frac{L^{d+2}}{2\kappa^2} \sqrt{-g} \frac{1}{z^{d+2}} (K^{\mu\nu} - K \gamma^{\mu\nu}) . \quad (76)$$

This stress tensor appears in the early AdS/CFT paper [12]. The factor of  $z^{-d-2}$  in this expression suggests that the bare stress tensor may be divergent. Indeed, combined with an inverse metric

<sup>5</sup>In Lorentzian signature, conventionally the variation of the action is proportional to the stress tensor. In Euclidean, there should be a relative minus sign. We are implicitly working in Lorentzian signature here.

factor  $\gamma^{\mu\nu}$ , there will in general be divergent terms starting at order  $z^{-d}$ . These terms need to be regulated. The form of the counter terms in  $d \leq 6$  is

$$S_{\text{ctr}} = \frac{1}{\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} \left[ \frac{d-1}{L} + \frac{L}{2(d-2)} \mathcal{R} + \frac{L^3}{(d-4)(d-2)^2} \left( \mathcal{R}^{\mu\nu} \mathcal{R}_{\mu\nu} - \frac{d}{4(d-1)} \mathcal{R}^2 \right) + \dots \right]. \quad (77)$$

The Ricci tensor  $\mathcal{R}_{\mu\nu}$  is computed with the boundary metric  $\gamma_{\mu\nu}$ . We include as many of these counter-terms as are necessary to cancel the divergences. A term of the form  $\sqrt{-\gamma} \mathcal{R}^n$  can cancel a divergence of order  $z^{-d+2n}$ . As a result, we need to include counter terms up to but not including  $O(\mathcal{R}^{d/2})$  to cancel potential divergences. (In even  $d$ , there is an ambiguity in the definition of the stress tensor that comes from including terms of precisely  $O(\mathcal{R}^{d/2})$ . This ambiguity parallels a similar ambiguity on the the field theory side. In  $d = 4$ , for example, there is an analogous ambiguity in the coefficient of the  $\square R$  term in the trace anomaly.) In  $\text{AdS}_3$ , only the first term is needed. For  $\text{AdS}_4$  and  $\text{AdS}_5$ , the first and second are needed. The second term proportional to  $\mathcal{R}$  can be thought of as an analog of the  $\phi \square \phi$  counter term we needed for the scalar field. For  $\text{AdS}_6$  and  $\text{AdS}_7$ , all three are needed, and higher order terms we have not written down would need to be constructed to regulate the divergences in  $d > 6$ .

## Deriving the Boundary Stress Tensor

Similar discussions to the following can be found in textbooks on general relativity, for example appendix E.1 of Wald's book. However, in most of the general relativity literature, the variation of the metric on the boundary is set to zero,  $\delta G_{AB}|_{z=0} = 0$ . Like in the the case of the scalar we studied before, we would like to discover the response of the system to small variations in the boundary value of  $\delta G_{AB}$ . Thus we need to redo the classic textbook calculations, keeping a nonzero value for the metric fluctuations on the boundary.

We begin by studying the term proportional to  $\delta R_{AB}$  in the variation of the Einstein-Hilbert action (70). Using that  $\delta R_{AB}$  becomes a total derivative (72) inside the integral, the variation (70) becomes

$$\delta S_{\text{EH}} = -\frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} [G^{AB} \delta \Gamma_{AC}^C n_B - G^{AB} \delta \Gamma_{AB}^C n_C] \quad (78)$$

$$= -\frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} G^{AB} G^{CD} (\delta G_{CD;A} n_B - \delta G_{AD;B} n_C) \quad (79)$$

$$= -\frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} n^A G^{CD} (\delta G_{CD;A} - \delta G_{CA;D}) . \quad (80)$$

We can write the boundary metric as an operator  $\gamma^{AB} = G^{AB} - n^A n^B$  that projects onto the subspace orthogonal to  $n^A$ . In the variation, we can replace  $G^{CD}$  with  $\gamma^{CD}$  as the terms proportional to  $n^A n^C n^D$  will drop out of the difference:

$$\delta S_{\text{EH}} = -\frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} n^A \gamma^{CD} (\delta G_{CD;A} - \delta G_{CA;D}) . \quad (81)$$

But now  $\gamma^{CD} \delta G_{CA;D}$  becomes almost a total tangential derivative which we can integrate by parts. In more detail, we have the identity

$$\gamma^{ED} (\gamma_D^C n^A \delta G_{AC})_{;E} = -K n^A n^C \delta G_{AC} + K^{AC} \delta G_{AC} + \gamma^{CD} n^A \delta G_{AC;D} , \quad (82)$$

where now the quantity on the left really is a total boundary derivative because the covariant derivative acts on a quantity with projected indices. In this identity we have replaced the covariant derivative of the unit normal with the extrinsic curvature,  $n^{A;C} = K^{AC}$ . This identity combined with the intermediate result (81) leads to

$$\delta S_{\text{EH}} = -\frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} (n^A \gamma^{CD} \delta G_{CD;A} - K n^A n^C \delta G_{AC} + K^{AC} \delta G_{CA}) . \quad (83)$$

Next we consider the variation of the Gibbons-Hawking term:

$$\delta S_{\text{GH}} = \frac{1}{\kappa^2} \int_{\partial M} d^d x (\sqrt{-\gamma} \delta K + K \delta(\sqrt{-\gamma})) . \quad (84)$$

Rewriting  $\delta K$  in terms of the connection leads to

$$\delta K = (\delta \nabla_A) n^A + \nabla_A \delta n^A .$$

The first term in this variation can be simplified straightforwardly:

$$\begin{aligned} (\delta \nabla_A) n^A &= (\delta \nabla_A) n^A \\ &= (\delta \Gamma_{AC}^A) n^C \\ &= \frac{1}{2} G^{AD} (\delta G_{AD;C} + \delta G_{CD;A} - \delta G_{AC;D}) n^C \\ &= \frac{1}{2} G^{AD} \delta G_{AD;C} n^C . \end{aligned}$$

The constraint  $n^A n_A = 1$  implies that the variation of the unit normal must take the form

$$\delta n_A = \left( \frac{1}{2} n_A n^B n^C + c \gamma_A^B n^C \right) \delta G_{BC} , \quad (85)$$

where  $c$  is an as yet undetermined constant. To fix  $c = 0$ , we know that the tangent vectors  $\partial X^A / \partial x^\mu$  do not depend on the metric and must be orthogonal to  $\delta n_A$ . But to vary  $K$ , we need  $\delta n^A = \delta(g^{AB} n_B)$  which must then be

$$\delta n^A = - \left( \frac{1}{2} n^A n^B n^C + \gamma^{AB} n^C \right) \delta G_{BC} . \quad (86)$$

The variation of the trace of the extrinsic curvature is thus

$$\delta K = \frac{1}{2} \gamma^{AD} \delta G_{AD;C} n^C - \frac{K}{2} n^B n^C \delta G_{BC} - \nabla_A (\gamma^{AB} n^C \delta G_{BC}) . \quad (87)$$

The variation of the Gibbons-Hawking term then becomes

$$\delta S_{\text{GH}} = \frac{1}{2\kappa^2} \int_{\partial M} d^d x \sqrt{-\gamma} (n^A \gamma^{BC} \delta G_{BC;A} - K n^A n^B \delta G_{AB} + K \gamma^{AB} \delta G_{AB}) , \quad (88)$$

where we have discarded a total boundary derivative. As is well known, the normal derivatives in  $\delta S_{\text{EH}}$  and  $\delta S_{\text{GH}}$  cancel. As is less well known, the terms proportional to  $K n^A n^B \delta G_{AB}$  cancel as well, leaving the boundary stress tensor (75).

## 4 A Basic Check of AdS/CFT

In this lecture, we perform a consistency check of the correspondence between  $\mathcal{N} = 4$   $SU(N)$  SYM and type IIB string theory in an  $AdS_5 \times S^5$  background. Recall the basic field content of  $\mathcal{N} = 4$  SYM. There is a gauge field  $A_\mu$ , four Weyl fermions  $\lambda_i$ , and six real scalar fields  $\phi^I$ , all transforming in the adjoint of  $SU(N)$ .

Previously, we learned of a series of scalar fields in the KK reduction of  $AdS_5 \times S^5$ . These were linear combinations of  $h^a{}_a$  and  $C_{abcd}$  with all legs in the  $S^5$ . They transformed in a symmetric traceless representation of  $SO(6)$ . Recall that  $SO(6) = SU(4)/\mathbb{Z}_2$ . In  $SU(4)$  language, they have the Young tableaux  $(0, p, 0)$  for  $p > 1$ :

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \dots \quad (89)$$

We also saw that they had masses  $(mL)^2 = p(p-4)$ .

We claim that the boundary values of these supergravity fields act as sources for traceless symmetric products of the  $\phi^I$  of  $\mathcal{N} = 4$   $SU(N)$  SYM,  $\mathcal{O}^p = \text{tr}(\phi^{I_1} \cdots \phi^{I_p}) c_{I_1 \dots I_p}$  where the tensor  $c_{I_1 \dots I_p}$  is symmetric and traceless. In the limit  $g_{YM}^2 N \rightarrow 0$ , the  $\phi^I$  are free fields with scaling dimension  $\Delta(\phi^I) = 1$ . It then follows that  $\Delta(\mathcal{O}^p) = p$ , consistent with the mass-scaling dimension relation (38). But this argument is too fast. At this point, it is not at all clear why  $\Delta(\mathcal{O}^p)$  should be independent of  $g_{YM}^2 N$ . There is a long story here, the short version of which follows directly. The full version will take us some time.

The rough answer is that  $\mathcal{N} = 4$  SYM is a superconformal field theory. The superconformal algebra (SCA) has two types of SUSY generators, call them  $S$  and  $Q$ , 16 of each. That  $\mathcal{O}^p$  is a superconformal primary means that it is annihilated by the  $S$ . That  $\mathcal{O}^p$  belongs to a 1/2 BPS multiplet implies that it is annihilated by half of the  $Q$ 's. The SCA has the anticommutation relation, schematically,

$$\{Q, S\} = M + R + D, \quad (90)$$

where  $M$  is a Lorentz generator,  $R$  is an R-symmetry generator, and  $D$  is a dilation operator, whose eigenvalues are proportional to  $\Delta$ . This anticommutation relation thus implies that the R-charge of the scalar operator is proportional to the scaling dimension  $\Delta$ . In other words,  $\Delta$  is protected by the structure of the SCA. We have then a nontrivial check of the AdS/CFT correspondence.

### 4.1 The Long Version

The discussion in this section draws heavily from ref. [13].

The conformal group is the Poincaré group extended by dilations and special conformal trans-

formations. The Lie algebra commutation relations are as follows:

$$[P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) , \quad [K_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) , \quad (91)$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = i(\eta_{\nu\lambda}M_{\mu\rho} + \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\mu\lambda}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\lambda}) , \quad (92)$$

$$[D, P_\mu] = iP_\mu , \quad [D, K_\mu] = -iK_\mu , \quad [K_\mu, P_\nu] = -2iM_{\mu\nu} + 2i\eta_{\mu\nu}D , \quad (93)$$

where  $P_\mu$  generate translations,  $M_{\mu\nu}$  Lorentz transformations,  $K_\mu$  special conformal transformations, and  $D$  dilations. If the Poincaré group acts on  $\mathbb{R}^{1,d-1}$ , then the conformal group is isomorphic to  $SO(2,d)$ , as can be seen by considering the linear combinations  $P_\mu \pm K_\mu$ . In the context of 2d CFT from last semester, we may identify the Virasoro generators  $L_1$  and  $\tilde{L}_1$  with linear combinations of the  $K_\mu$ ;  $L_{-1}$  and  $\tilde{L}_{-1}$  with linear combinations of the  $P_\mu$ ; and  $L_0$  and  $\tilde{L}_0$  with  $D$  and the rotation generator  $M_{01}$ .

A standard differential operator representation of this algebra is

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) , \quad (94)$$

$$P_\mu = -i\partial_\mu , \quad D = -ix^\mu\partial_\mu , \quad (95)$$

$$K_\mu = i(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu) . \quad (96)$$

Note that  $D$  has eigenvalues  $i\Delta$  when acting on monomials in  $x^\mu$  of degree  $-\Delta$ .<sup>6</sup>

Toward the goal of studying superconformal primary operators, we review first the definition of conformal primaries. Note from the relation (93) it follows that for eigenvectors of  $D$ ,  $P_\mu$  increases  $\Delta$  by one while  $K_\mu$  lowers it by one. A conformal primary is a lowest weight state of  $D$ , or equivalently a state annihilated by  $K_\mu$  that is not  $P_\mu$  of something else. (States that are  $P_\mu$  of something else we typically call descendants.) Let  $\mathcal{O}(x)$  be a conformal primary operator and  $|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle$  the corresponding conformal primary state. We have that

$$K_\mu|\mathcal{O}\rangle = 0 , \quad D|\mathcal{O}\rangle = i\Delta|\mathcal{O}\rangle . \quad (97)$$

To define a superconformal algebra, we specialize to  $d = 4$  dimensions and add the new generators

$$Q^i{}_\alpha , \bar{Q}_{i\dot{\alpha}} , S_i{}^\alpha , \bar{S}^{i\dot{\alpha}} , \quad (98)$$

where  $i = 1, \dots, N$  and  $\alpha, \dot{\alpha} = 1, 2$ . We will see in a moment that there are also  $R$ -symmetry generators  $R^i{}_j$ . These additions yield the superconformal algebra  $su(2, 2|N)$ . The anticommutation

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<sup>6</sup>We have presented the conformal algebra in a way suggesting that all the generators are Hermitian, in which case they should have real eigenvalues. It may seem strange then that the operator  $D$  can have a pure imaginary eigenvalue. One comment is that  $D = -ix^\mu\partial_\mu$  is not Hermitian with the usual inner product  $\langle u, v \rangle = \int u(x)^*v(x) dx$ . Another is that monomials in  $x^\mu$  of degree  $\Delta$  are not normalizable eigenvectors under this inner product anyway. A third is that the group  $SO(2,4)$  is non-compact with a Lorentzian metric that is not positive definite.

relations among the ordinary supercharges  $Q$  and superconformal charges  $S$  are as follows:

$$\{Q^i_\alpha, \bar{Q}_{j\dot{\alpha}}\} = 2\delta_j^i P_{\alpha\dot{\alpha}}, \quad \{Q^i_\alpha, Q^j_\beta\} = 0 = \{\bar{Q}_{i\dot{\alpha}}, \bar{Q}_{j\dot{\beta}}\}, \quad (99)$$

$$\{\bar{S}^{i\dot{\alpha}}, S_j^\alpha\} = 2\delta_j^i \bar{K}^{\dot{\alpha}\alpha}, \quad \{\bar{S}^{i\dot{\alpha}}, \bar{S}^{j\dot{\beta}}\} = 0 = \{S_i^\alpha, S_j^\beta\}, \quad (100)$$

$$\{Q^i_\alpha, \bar{S}^{j\dot{\alpha}}\} = 0, \quad \{S_i^\alpha, \bar{Q}_{j\dot{\alpha}}\} = 0, \quad (101)$$

$$\{Q^i_\alpha, S_j^\beta\} = 4 \left[ \delta_j^i (M_\alpha^\beta - \frac{i}{2} \delta_\alpha^\beta D) - \delta_\alpha^\beta R^i_j \right], \quad (102)$$

$$\{\bar{S}^{i\dot{\alpha}}, \bar{Q}_{j\dot{\beta}}\} = 4 \left[ \delta_j^i (\bar{M}^{\dot{\alpha}\beta} + \frac{i}{2} \delta_\beta^{\dot{\alpha}} D) - \delta_\beta^{\dot{\alpha}} R^i_j \right], \quad (103)$$

where we have defined

$$P_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu, \quad \bar{K}^{\dot{\alpha}\alpha} = (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} K_\mu, \quad (104)$$

$$M_\alpha^\beta = -\frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu)_\alpha^\beta M_{\mu\nu}, \quad \bar{M}^{\dot{\alpha}\beta} = -\frac{i}{4} (\bar{\sigma}^\mu \sigma^\nu)^{\dot{\alpha}\beta} M_{\mu\nu}. \quad (105)$$

The most important relations here are eqs. (99), (100), (102) and (103). In the same way that the ordinary supercharges  $Q^i_\alpha$  and  $\bar{Q}_{j\dot{\beta}}$  function morally as the square root of the momentum generator  $P_\mu$ , in the superconformal extension of the super-algebra,  $S_i^\alpha$  and  $\bar{S}^{j\dot{\beta}}$  function as the square root of the superconformal generator  $K_\mu$ . In defining superconformal primary operators, this fact means we can replace  $K_\mu$  by  $S_i^\alpha$  and  $\bar{S}^{j\dot{\beta}}$ . One other thing to note: The anti-commutation relations (102) and (103) are the precise version of the relation (90), to be used in the last and crucial step of demonstrating that the  $\mathcal{O}^p$  are protected operators.

We then need also to specify how  $Q^i_\alpha$ ,  $\bar{Q}_{j\dot{\beta}}$ ,  $S_i^\alpha$ , and  $\bar{S}^{j\dot{\beta}}$  fail to commute with the generators of the conformal group:

$$[M_\alpha^\beta, Q^i_\gamma] = \delta_\gamma^\beta Q^i_\alpha - \frac{1}{2} \delta_\alpha^\beta Q^i_\gamma, \quad [M_\alpha^\beta, S_i^\gamma] = -\delta_\alpha^\gamma S_i^\beta + \frac{1}{2} \delta_\alpha^\beta S_i^\gamma, \quad (106)$$

$$[\bar{M}^{\dot{\alpha}\beta}, \bar{Q}^{i\dot{\gamma}}] = -\delta_\gamma^{\dot{\alpha}} \bar{Q}_{i\dot{\beta}} + \frac{1}{2} \delta_\beta^{\dot{\alpha}} \bar{Q}^{i\dot{\gamma}}, \quad [\bar{M}^{\dot{\alpha}\beta}, \bar{S}^{i\dot{\gamma}}] = \delta_\beta^{\dot{\gamma}} \bar{S}^{i\dot{\alpha}} - \frac{1}{2} \delta_\beta^{\dot{\alpha}} \bar{S}^{i\dot{\gamma}}, \quad (107)$$

$$[D, Q^i_\alpha] = \frac{i}{2} Q^i_\alpha, \quad [D, \bar{Q}_{i\dot{\alpha}}] = \frac{i}{2} \bar{Q}_{i\dot{\alpha}}, \quad [D, S_i^\alpha] = -\frac{i}{2} S_i^\alpha, \quad [D, \bar{S}^{i\dot{\alpha}}] = -\frac{i}{2} \bar{S}^{i\dot{\alpha}}, \quad (108)$$

$$[K_\mu, Q^i_\alpha] = -(\sigma_\mu)_{\alpha\dot{\alpha}} \bar{S}^{i\dot{\alpha}}, \quad [K_\mu, \bar{Q}_{i\dot{\alpha}}] = S_i^\alpha (\sigma_\mu)_{\alpha\dot{\alpha}}, \quad (109)$$

$$[P_\mu, \bar{S}^{i\dot{\alpha}}] = -(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} Q^i_\alpha, \quad [P_\mu, S_i^\alpha] = \bar{Q}^{i\dot{\alpha}} (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}. \quad (110)$$

The most important relations here for us are eqs. (108). From these relations it follows immediately that  $Q^i_\alpha$  and  $\bar{Q}_{j\dot{\beta}}$  act to raise  $\Delta$  by 1/2, while  $S_i^\alpha$  and  $\bar{S}^{j\dot{\beta}}$  lower  $\Delta$  by 1/2. Superconformal primary states, which are also lowest weight states of  $D$ , are annihilated by  $S_i^\alpha$  and  $\bar{S}^{j\dot{\beta}}$  and are not  $Q^i_\alpha$  or  $\bar{Q}_{j\dot{\beta}}$  of something else.

Finally, we need to specify how the  $R$ -symmetry generators act:

$$[R^i_j, R^k_l] = \delta_j^k R^i_l - \delta_l^i R^k_j, \quad (111)$$

$$[R^i_j, Q^k_\alpha] = \delta_j^k Q^i_\alpha - \frac{1}{4} \delta_j^i Q^k_\alpha, \quad [R^i_j, \bar{Q}_{k\dot{\alpha}}] = -\delta_k^i \bar{Q}_{j\dot{\alpha}} + \frac{1}{4} \delta_j^i \bar{Q}_{k\dot{\alpha}}, \quad (112)$$

$$[R^i_j, S_k^\alpha] = -\delta_k^i S_j^\alpha + \frac{1}{4} \delta_j^i S_k^\alpha, \quad [R^i_j, \bar{S}^{k\dot{\alpha}}] = \delta_j^k \bar{S}^{i\dot{\alpha}} - \frac{1}{4} \delta_j^i \bar{S}^{k\dot{\alpha}}. \quad (113)$$

If we take the  $R$ -symmetry generators to be traceless  $R^i_i = 0$ , replacing the  $R$ -symmetry group  $U(4)$  with  $SU(4)$ , then the superalgebra becomes  $psu(2, 2|4)$ .

Now let us consider the SUSY transformations of the fields in  $\mathcal{N} = 4$  SYM. Recall that  $\mathcal{N} = 4$  SYM has six real scalar fields  $\phi^I$  ( $I = 1, \dots, 6$ ); four Weyl fermions  $\lambda_{i\alpha}$  and  $\bar{\lambda}_{j\dot{\alpha}}$  ( $i = 1, \dots, 4$  and  $\alpha, \dot{\alpha} = 1, 2$ ); and a gauge field  $A_\mu$ , all transforming in the adjoint representation of the gauge group  $G$ , which we here take to be  $SU(N)$ . The  $R$ -symmetry group is  $SU(4)$ . Thus the  $\phi^I$  transform in the defining representation of  $SO(6)$  (or equivalently an antisymmetric irreducible representation of  $SU(4)$ ). The  $\lambda_i$  and  $\bar{\lambda}_j$  transform in the fundamental and antifundamental of  $SU(4)$  (or equivalently spinor representations of  $SO(6)$ ). The SUSY rules are as follows [6]:

$$\delta\phi^I = [Q^i_\alpha, \phi^I] = (C^I)^{ij}\lambda_{j\alpha}, \quad (114)$$

$$\delta\lambda_{j\beta} = \{Q^i_\alpha, \lambda_{j\beta}\} = F_{\mu\nu}^+(\sigma^{\mu\nu})_{\alpha\beta}\delta_j^i + [\phi^I, \phi^J]\epsilon_{\alpha\beta}(C_{IJ})^i_j, \quad (115)$$

$$\delta\bar{\lambda}_{j\dot{\beta}} = \{Q^i_\alpha, \bar{\lambda}_{j\dot{\beta}}\} = (C_I)^{ij}(\bar{\sigma}^\mu)_{\alpha\dot{\beta}}D_\mu\phi^I, \quad (116)$$

$$\delta A_\mu = [Q^i_\alpha, A_\mu] = (\sigma_\mu)_\alpha^{\dot{\beta}}\bar{\lambda}_{\dot{\beta}}^i. \quad (117)$$

The  $C^I$  and  $C_{IJ}$  are constructed from bilinears of the  $SO(6)$   $\gamma$  matrices. The superscript  $+$  indicates the self-dual part of  $F_{\mu\nu}$ . There are analogous relations involving the action of  $\bar{Q}$ .

We have written these SUSY transformation down to make it clear that  $\lambda_{i\alpha}$ ,  $F_{\mu\nu}$ ,  $[\phi^I, \phi^J]$ ,  $D_\mu\phi^I$ , and  $\bar{\lambda}_{j\dot{\alpha}}$  all appear on the right hand side of the transformations. In other words, they are all  $Q^i_\alpha$  or  $\bar{Q}_{j\dot{\beta}}$  of something else and cannot correspond to superconformal primary operators. Consider then a symmetrized product of the  $\phi^I$ . From the structure of the SUSY transformations, such an object cannot be  $Q^i_\alpha$  or  $\bar{Q}_{j\dot{\beta}}$  of something else. Furthermore, note that the  $S_i^\alpha$  and  $\bar{S}^{j\dot{\alpha}}$  generators reduce the scaling dimension of a field by  $1/2$ . As there is nothing in the fundamental multiplet with scale dimension lower than 1,  $S_i^\alpha$  and  $\bar{S}^{j\dot{\alpha}}$  must annihilate the fields  $\phi^I$ . Thus a symmetrized product of the  $\phi^I$  is also annihilated by  $S_i^\alpha$  and  $\bar{S}^{j\dot{\beta}}$ . It is, in other words, a superconformal primary. To make irreducible representations of  $SO(6)$ , one should then remove the traces from these symmetrized products.

Next we consider the anticommutation relation (102) between the  $Q$ 's and the  $S$ 's:

$$\{Q^i_\alpha, S_j^\beta\}|\mathcal{O}^p\rangle = 4\left[\delta_j^i(M_\alpha^\beta - \frac{i}{2}\delta_\alpha^\beta D) - \delta_\alpha^\beta R^i_j\right]|\mathcal{O}^p\rangle. \quad (118)$$

By construction  $|\mathcal{O}^p\rangle$  is a scalar, and the relation above reduces to

$$\{Q^i_\alpha, S_j^\beta\}|\mathcal{O}^p\rangle = 4\delta_\alpha^\beta\left[-\frac{i}{2}\delta_j^i D - R^i_j\right]|\mathcal{O}^p\rangle. \quad (119)$$

To understand  $R^i_j$ , we need a brief interlude on the representation theory of  $SU(4)$ . The Lie group has three generators  $H_i$  in the Cartan sub-algebra and also three pairs of raising and lower operators  $E_i^\pm$ ,  $i = 1, 2, 3$ . Representations of  $SU(4)$  are characterized by highest weight states  $|\lambda_1, \lambda_2, \lambda_3\rangle$  where

$$H_i|\lambda_1, \lambda_2, \lambda_3\rangle = \lambda_i|\lambda_1, \lambda_2, \lambda_3\rangle. \quad (120)$$

Moreover, we have

$$E_i^+ |\lambda_1, \lambda_2, \lambda_3\rangle = 0, \quad (121)$$

$$E_i^- |\lambda_1, \lambda_2, \lambda_3\rangle = 0 \text{ if } \lambda_i = 0. \quad (122)$$

The dimension of a representation is given by

$$d(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{12}(\lambda_1 + \lambda_2 + \lambda_3 + 3)(\lambda_1 + \lambda_2 + 2)(\lambda_2 + \lambda_3 + 2)(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_3 + 1). \quad (123)$$

For the symmetric traceless products  $|\mathcal{O}^p\rangle$ , the highest weight state is  $|0, p, 0\rangle$ .

A nice way of writing the matrix  $R^i_j$  is as follows:

$$[R^i_j] = \begin{pmatrix} \frac{1}{4}(3H_1 + 2H_2 + H_3) & E_1^+ & [E_1^+, E_2^+] & [E_1^+, [E_2^+, E_3^+]] \\ E_1^- & \frac{1}{4}(-H_1 + 2H_2 + H_3) & E_2^+ & [E_2^+, E_3^+] \\ -[E_1^-, E_2^-] & E_2^- & -\frac{1}{4}(H_1 + 2H_2 - H_3) & E_3^+ \\ [E_1^-, [E_2^-, E_3^-]] & -[E_2^-, E_3^-] & E_3^- & -\frac{1}{4}(H_1 + 2H_2 + 3H_3) \end{pmatrix} \quad (124)$$

Acting on the state  $|\mathcal{O}^p\rangle$ , this matrix has the following behavior:

$$[R^i_j]|\mathcal{O}^p\rangle = \begin{pmatrix} \frac{p}{2} & 0 & 0 & 0 \\ 0 & \frac{p}{2} & 0 & 0 \\ * & * & -\frac{p}{2} & 0 \\ * & * & 0 & -\frac{p}{2} \end{pmatrix} |\mathcal{O}^p\rangle \quad (125)$$

where  $*$  indicates a nonzero element of the matrix whose precise form does not concern us.

Now at weak coupling, we know that  $\Delta = p$  for the states  $|\mathcal{O}^p\rangle$ . From the action (125) of  $R^i_j$  and the anticommutation relation (119), it follows then that  $Q^1_\alpha$  and  $Q^2_\alpha$  annihilate  $|\mathcal{O}^p\rangle$  at weak coupling.<sup>7</sup> Running through a similar argument starting with the anticommutation relation (103), we leave it as an exercise to show that  $\bar{Q}_{3\dot{\alpha}}$  and  $\bar{Q}_{4\dot{\alpha}}$  also annihilate  $|\mathcal{O}^p\rangle$ . Not only then is  $|\mathcal{O}^p\rangle$  annihilated by the  $S$  and  $\bar{S}$  superconformal charges. It is also annihilated by half of the ordinary  $Q$  and  $\bar{Q}$  supercharges. Such a state is called half BPS.

There is a larger story here. By acting on the lowest weight state  $|\mathcal{O}^p\rangle$  with the  $Q$  and  $\bar{Q}$  supercharges, we generate a superconformal multiplet. The fact that half of the  $Q$  and  $\bar{Q}$  charges annihilate  $|\mathcal{O}^p\rangle$  means that the multiplet will be smaller than usual; it will be a shortened multiplet. Every state in the multiplet is also annihilated by half of the  $Q$  and  $\bar{Q}$  charges. (The highest weight state of a given irrep of  $SU(4)$  will always be annihilated  $Q^1$ ,  $Q^2$ ,  $\bar{Q}^3$ , and  $\bar{Q}^4$ . Lower weight states, however, may be annihilated by a different selection of the  $Q$  and  $\bar{Q}$ .) Thus every state in the multiplet will have a conformal dimension prescribed by the anticommutation relation (118) to be a linear combination of its spin and  $R$ -charge.

Imagine now that the condition  $\Delta = p$  could be changed by slowly increasing  $g_{\text{YM}}$ . By the anticommutation relation (118), some of the  $Q$  and  $\bar{Q}$  charges must no longer annihilate the  $|\mathcal{O}^p\rangle$ ,

<sup>7</sup>Actually we have only shown that  $S$  annihilates the state created by  $Q^1$  or  $Q^2$  acting on  $|\mathcal{O}^p\rangle$ . Such a state is called a null state and will not correspond to a physical excitation.



which in turn means that the multiplet must lengthen. States are not created and destroyed by tuning  $g_{\text{YM}}$  and so the only way for lengthening to happen is for two short multiplets to combine together to form a longer one. Sometimes this lengthening does indeed occur. However, a careful study of the representations of the  $\mathcal{N} = 4$  superconformal algebra shows that there is no way for the multiplet generated by  $|\mathcal{O}^p\rangle$  to combine with anything else [13]. Thus the conformal dimensions of everything in the  $|\mathcal{O}^p\rangle$  superconformal multiplet are protected, i.e. independent of  $g_{\text{YM}}$ .

Consider the example  $|\mathcal{O}^2\rangle = c_{IJ}\phi^I\phi^J|0\rangle$  which is a set of 20 scalar operators with dimension  $\Delta = 2$ . If we act on these states using a single  $Q^i$  or  $\bar{Q}^j$ , the SUSY transformation rule (114) tells us we get spinors of the schematic form  $\lambda^i\phi^J$  and  $\bar{\lambda}^i\phi^J$  both with  $\Delta = 5/2$ . Acting again with a single  $Q^i$  or  $\bar{Q}^j$ , we get a larger variety of possibilities with  $\Delta = 3$ . We distinguish the possibilities by their representation under the Lorentz group. There are complex scalar operators of the schematic form  $\lambda_i\lambda_j + [\phi^I, \phi^J]\phi^K$ , vectors of the form  $\phi^I D_\mu\phi^J + \lambda_j\sigma^\mu\lambda_i$ , and an antisymmetric two form  $F_{\mu\nu}\phi^I$ . The vector we recognize as the conserved  $R$ -symmetry current. With scaling dimension  $\Delta = 7/2$ , there is a spinor  $\lambda^j[\phi^I, \phi^J]$  and a spin 3/2 object  $\lambda^j D_\mu\phi^I$ . This spin 3/2 object is the supercurrent. Finally, with  $\Delta = 4$ , we find a complex scalar along with the stress tensor. The complex scalar turns out to be the Lagrangian density and the theta term  $\epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}$ . Given that the stress tensor and conserved current can be found in the  $\mathcal{O}^2$  multiplet, it should not be too surprising that the conformal dimensions here are protected.

We can be more precise about the representations involved. Label a state by its weights under  $\text{SU}(4)$  and the Lorentz group  $\text{SU}(2)\times\text{SU}(2)$ ,  $|\lambda_1, \lambda_2, \lambda_3\rangle_{(s_1, s_2)}$ , for example  $|\mathcal{O}^p\rangle = |0, 2, 0\rangle_{(0,0)}$ . The supercharges have the weights

$$Q^1 \sim [1, 0, 0]_{(\pm\frac{1}{2}, 0)}, \quad Q^2 \sim [-1, 1, 0]_{(\pm\frac{1}{2}, 0)}, \quad Q^3 \sim [0, -1, 1]_{(\pm\frac{1}{2}, 0)}, \quad Q^4 \sim [0, 0, -1]_{(\pm\frac{1}{2}, 0)}, \quad (126)$$

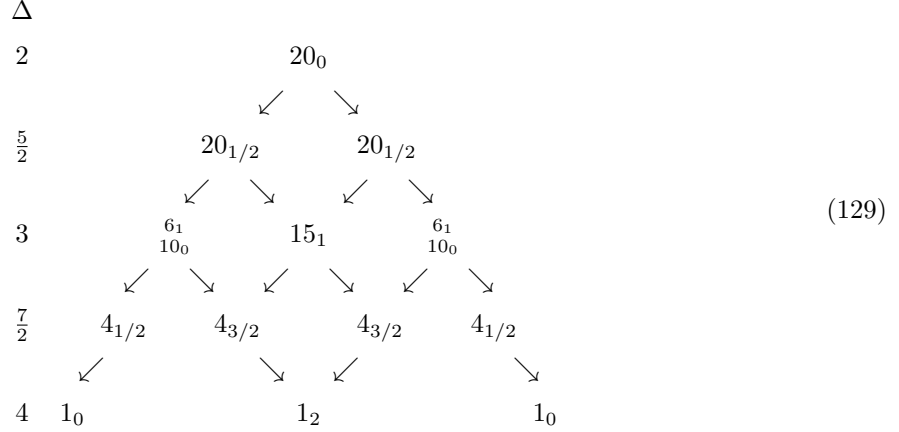
$$\bar{Q}_1 \sim [-1, 0, 0]_{(0, \pm\frac{1}{2})}, \quad \bar{Q}_2 \sim [1, -1, 0]_{(0, \pm\frac{1}{2})}, \quad \bar{Q}_3 \sim [0, 1, -1]_{(0, \pm\frac{1}{2})}, \quad \bar{Q}_4 \sim [0, 0, 1]_{(0, \pm\frac{1}{2})}. \quad (127)$$

The structure of the  $\mathcal{O}^2$  multiplet can be summarized pictorially. Each entry in the table gives the highest weight state of the  $\text{SU}(4)\times\text{SU}(2)\times\text{SU}(2)$  representation. Arrows pointing to the left indicate the action of  $Q^3$  or  $Q^4$  while arrows pointing to right indicate  $\bar{Q}_1$  or  $\bar{Q}_2$ :

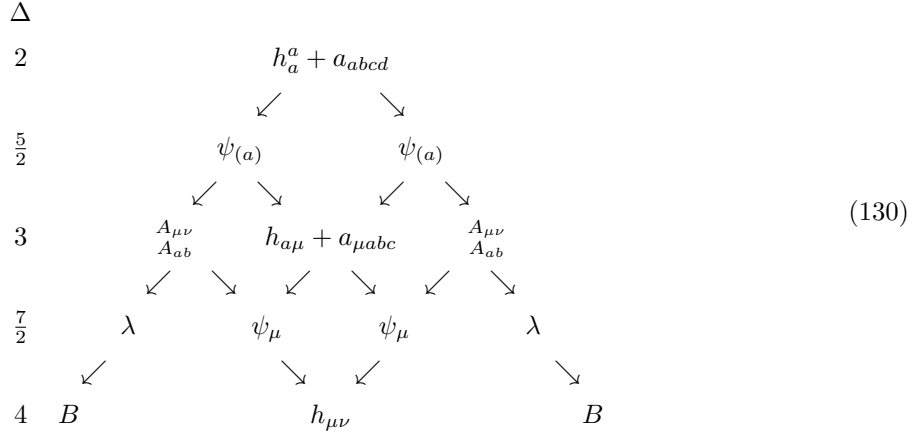
$$\begin{array}{ccccccc}
\Delta & & & & & & \\
2 & & & & & & |0, 2, 0\rangle_{(0,0)} \\
& & & \swarrow & & \searrow & \\
\frac{5}{2} & & & |0, 1, 1\rangle_{(\frac{1}{2}, 0)} & & |1, 1, 0\rangle_{(0, \frac{1}{2})} & \\
& & & \swarrow & & \searrow & \\
3 & & & \begin{array}{l} |0, 1, 0\rangle_{(1,0)} \\ |0, 0, 2\rangle_{(0,0)} \end{array} & & |1, 0, 1\rangle_{(\frac{1}{2}, \frac{1}{2})} & & \begin{array}{l} |0, 1, 0\rangle_{(0,1)} \\ |2, 0, 0\rangle_{(0,0)} \end{array} & (128) \\
& & & \swarrow & & \searrow & \swarrow & & \searrow \\
\frac{7}{2} & & & |0, 0, 1\rangle_{(\frac{1}{2}, 0)} & & |1, 0, 0\rangle_{(1, \frac{1}{2})} & & |0, 0, 1\rangle_{(\frac{1}{2}, 1)} & & |1, 0, 0\rangle_{(0, \frac{1}{2})} \\
& & & \swarrow & & \searrow & \swarrow & & \searrow & \\
4 & & & |0, 0, 0\rangle_{(0,0)} & & |0, 0, 0\rangle_{(1,1)} & & & & |0, 0, 0\rangle_{(0,0)}
\end{array}$$

From this table and the dimension formula (123), we learn the dimensions of the various  $\text{SU}(4)$

representations involved. We can write the table in a perhaps more transparent form, replacing  $|\lambda_1, \lambda_2, \lambda_3\rangle_{(s_1, s_2)}$  with  $\dim(\lambda_1, \lambda_2, \lambda_3)_{s_1+s_2}$



In Professor van Nieuwenhuizen's lectures, we considered the Kaluza-Klein reduction of type IIB supergravity on  $\text{AdS}_5 \times S^5$ . Let  $\mu, \nu$  index  $\text{AdS}_5$  and  $a, b$  index the  $S^5$ . Comparing with table III of [14], we can make the following replacements in our table above



In this table  $h_{AB}$  are metric fluctuations,  $a_{ABCD}$  are fluctuations of the RR four-form potential, and  $A_{AB}$  are fluctuations of a complex combination of the NSNS and RR two-form potentials. The axio-dilaton is  $B$ , the gravitino is  $\psi_A$ , and the dilatino is  $\lambda$ .

In the context of the AdS/CFT correspondence, the complex scalar couples to the boundary value of the axio-dilaton field in the KK reduction. The stress tensor couples to the boundary value of the metric. The spin 3/2 field couples to the gravitino. The antisymmetric two-form couples to fluctuations of the complexified two-form potential  $A_2 + iB_2$ . The vector field and the scalars in the 20 couple to linear combinations of fluctuations of the RR four-form and the metric. And so on.

Thus concludes a basic check of the correspondence between  $\mathcal{N} = 4$   $SU(N)$  SYM and type IIB string theory on  $\text{AdS}_5 \times S^5$ .

## 4.2 An Aside on $\mathcal{N} = 1$ theories

There exist many generalizations of the original AdS/CFT correspondence at this point. A number of them involve four dimensional gauge theories with only  $\mathcal{N} = 1$  supersymmetry. There is a much simpler version of the check that we performed above in this case. Superconformal theories with  $\mathcal{N} = 1$  supersymmetry have a U(1) R-symmetry. The eight supercharges are then conventionally denoted  $Q_\alpha, \bar{Q}_{\dot{\beta}}, S^\alpha$  and  $\bar{S}^{\dot{\beta}}$ . The supercharges  $Q_\alpha$  and  $\bar{Q}_{\dot{\alpha}}$  can be written with the aid of Grassman variables  $\theta$  and  $\bar{\theta}$ ,

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i \sigma_{\alpha\dot{\alpha}}{}^m \bar{\theta}^{\dot{\alpha}} \partial_m, \quad (131)$$

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i \theta^\alpha \sigma_{\alpha\dot{\alpha}}{}^m \partial_m. \quad (132)$$

Conventionally the R-symmetry is normalized such that  $R(Q) = -1$ , which then implies  $R(\theta) = 1$ . As  $Q$  is morally the square-root of  $P_\mu$ , it must also be true that  $\Delta(Q) = 1/2$ .

We can write the  $\mathcal{N} = 4$  theory in  $\mathcal{N} = 1$  notation. There are three chiral superfields conventionally labeled  $X, Y$ , and  $Z$ , the lowest components of which are  $x = \phi^1 + i\phi^2, y = \phi^3 + i\phi^4$  and  $z = \phi^5 + i\phi^6$ . The charges  $Q$  and  $\bar{Q}$  function something like holomorphic and anti-holomorphic derivatives in this case. Thus  $X, Y$ , and  $Z$  would be annihilated by  $\bar{Q}$  and also by  $S$  and  $\bar{S}$ . Thus they are the lowest weight states of a shortened superconformal multiplet. Indeed, a symmetrized product  $x^q y^r z^{p-q-r}$  is already traceless and is an example of the operators of the type  $\mathcal{O}^p$  we discussed before.

There is an F-term in the action for a  $\mathcal{N} = 1$  theory of the form

$$\int d^4x d^2\theta W,$$

where  $W$  is the superpotential. It is clear that for a superconformal theory with unbroken U(1) R-symmetry and scale invariance,  $W$  must have R-charge two and conformal dimension three. In  $\mathcal{N} = 4$  SYM, the superpotential is cubic  $W \sim \text{tr} X[Y, Z]$ . Thus by symmetry one finds that  $3R_X = 2\Delta_X$  and similarly for  $Y$  and  $Z$  and the operator  $x^q y^r z^{p-q-r}$ . But for free fields,  $\Delta_x = 1$  and indeed because  $\mathcal{O}^p$  is protected,  $\Delta(\mathcal{O}^p) = p$ . The subsector of operators  $x^q y^r z^{p-q-r}$  are often called chiral primaries; they always satisfy the relation  $2\Delta = 3R$  in four space-time dimensions. Indeed, with an inner product on the state space, one can prove more generally the so-called BPS inequality  $\Delta \geq 3|R|/2$  for scalar operators.

## 5 Trace Anomalies

Recall from last semester that in two dimensional CFTs the trace of the stress tensor need not vanish on curved manifolds:

$$\langle T^\mu{}_\mu \rangle = \frac{c}{24\pi} R. \quad (133)$$

In fact, the condition that this anomaly vanish led us to consider bosonic string theory only in 26 dimensions and superstring theory only in ten. This trace anomaly is in fact a general feature of

CFTs in even numbers of space-time dimensions. Given diffeomorphism invariance and hence a covariantly conserved stress tensor, it is equivalent to the statement that the dilation current  $x^\mu T_\mu^\nu$  is not conserved.

In four dimension, the structure of the trace anomaly is more complicated. There are several candidate curvature invariants –  $R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda}$ ,  $R^{\mu\nu}R_{\mu\nu}$ ,  $R^2$  and  $\square R$  – which all have the correct scaling dimension to sit on the right hand side of the trace anomaly. It is convenient to replace  $R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda}$  and  $R^{\mu\nu}R_{\mu\nu}$  with the square of the Weyl curvature and the Euler density

$$I \equiv W^{\mu\nu\lambda\rho}W_{\mu\nu\lambda\rho} = R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2, \quad (134)$$

$$E_4 \equiv \frac{1}{4}\delta_{\alpha\beta\gamma\delta}^{\mu\nu\rho\lambda}R^{\alpha\beta\gamma\delta}R_{\mu\nu\rho\lambda} = R^{\mu\nu\rho\lambda}R_{\mu\nu\rho\lambda} - 4R^{\mu\nu}R_{\mu\nu} + R^2. \quad (135)$$

We then write the trace anomaly in the (almost) conventional form

$$\langle T_\mu^\mu \rangle = \frac{1}{(4\pi)^2}(cI - aE_4 + d\square R + eR^2). \quad (136)$$

I say almost because we can set  $e = 0$ . The reason is something called Wess-Zumino consistency. If we imagine that this trace anomaly comes from varying an effective action with respect to scale transformations  $g_{\mu\nu} \rightarrow e^{2\sigma}g_{\mu\nu}$ , then like partial derivatives, variations should commute. By assumption, the trace anomaly equation can be derived from the variation  $\delta_1 W = \int d^4x \langle T_\mu^\mu \rangle \delta\sigma_1$ . But then since  $[\delta_1, \delta_2] = 0$  and since the variation  $\delta R \sim \square\sigma$ , it follows that an  $R^2$  term is not Wess-Zumino consistent.

The values of  $d$ ,  $c$ , and  $a$  are theory dependent. A one loop calculation is required to determine them. While  $d$  turns out to be sensitive to the regularization scheme chosen,  $a$  and  $c$  are physical and can be related to coefficients in the two and three point correlation functions of the stress tensor. For superconformal theories,  $a$  and  $c$  are also related to correlation functions of the R-symmetry current – not perhaps too surprising since we saw above that  $J^\mu$  and  $T^{\mu\nu}$  sit inside the same supermultiplet. To avoid a one-loop calculation, we look up the results for free fields for example in Chapter 6.3 of Birrell and Davies [15]. The free fields of interest are a conformally coupled scalar  $\phi$ , satisfying the equation  $(\square + R/6)\phi = 0$  in  $d = 4$ , a massless Weyl fermion  $\lambda$ , and a gauge field  $A_\mu$ .

	$180(a - c)$	$360c$	degeneracy
$\phi$	-1	1	$6N^2$
$\lambda$	$-\frac{7}{4}$	$\frac{11}{2}$	$4N^2$
$A_\mu$	13	62	$N^2$
total	0	$90N^2$	

(137)

We have also included in this table the degeneracy of these fields in  $\mathcal{N} = 4$  SU( $N$ ) SYM in the large  $N$  limit. We may then conclude that at weak coupling in the large  $N$  limit

$$a = c = \frac{N^2}{4}. \quad (138)$$

Thus for  $\mathcal{N} = 4$  SU( $N$ ) SYM in the large  $N$  limit, the trace anomaly takes the somewhat simpler

form

$$\langle T_\mu^\mu \rangle = \frac{N^2}{32\pi^2} \left( R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right) . \quad (139)$$

These anomaly coefficients cannot depend on marginal operators in a CFT, and so they remain the same for all values of  $g_{\text{YM}}^2 N$ . Thus if we compute the trace anomaly using AdS/CFT, we should get the same answer. Let us check.<sup>8</sup> From the result for the bare holographic stress tensor (76) and the counter-terms (77), we can read off the renormalized stress tensor:

$$-\sqrt{-g_0} T_\nu^\mu = \lim_{z \rightarrow 0} \frac{\sqrt{-\gamma}}{\kappa^2} \left[ K_\nu^\mu - K \gamma_\nu^\mu + \frac{3}{L} \gamma_\nu^\mu + \frac{1}{4} L \mathcal{R} \gamma_\nu^\mu - \frac{1}{2} \mathcal{R}_\nu^\mu \right] . \quad (140)$$

The calligraphic font is used to remind us to use the boundary metric  $\gamma_{\mu\nu}$  (defined in (64)) in constructing these curvature invariants. In what follows, it is more convenient to work directly with the field theory metric  $g_{\mu\nu}$ , defined in (65). Because  $\mathcal{R}_{\mu\nu}$  is invariant under global rescaling of the metric, we have that  $\mathcal{R}_{\mu\nu} = \tilde{R}_{\mu\nu}$ , where  $\tilde{R}_{\mu\nu}$  is constructed from  $g_{\mu\nu}$ . We also have that  $\mathcal{R} = \frac{z^2}{L^2} \tilde{R}$ .

Note that we could have added further finite counter-terms of the form  $\mathcal{O}(\mathcal{R}^2)$  to the boundary gravity action. This type of ambiguity parallels the scheme dependent possibility of having a  $\square \tilde{R}$  in the trace of the stress tensor. A  $\square \tilde{R}$  in the trace can come from varying a  $\tilde{R}^2$  counter-term in the effective action.

Given the holographic stress-tensor (140), it is then straightforward to take a trace:

$$-\sqrt{-g_0} T_\mu^\mu = \lim_{z \rightarrow 0} \frac{\sqrt{-\gamma}}{\kappa^2} \left[ -3K + \frac{12}{L} + \frac{1}{2} L \mathcal{R} \right] . \quad (141)$$

We unpack this trace result using the boundary expansion of the metric (66) in the particular case  $d = 4$ . Analogous to the scalar case, boundary conditions allow us to set  $g_{\mu\nu}^{(0)}$  and  $g_{\mu\nu}^{(4)}$ . The other coefficients in the expansion are fixed in terms of these two coefficients by Einstein's equations. We will need the following relations in what follows.

**Claim One:**

$$\mathcal{R}L = \frac{z^2}{L} \left[ -6 \text{tr} g_{(0)}^{-1} g_{(2)} + 2z^2 \text{tr}(g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)}) + z^2 (\text{tr} g_{(0)}^{-1} g_{(2)})^2 + \dots \right] , \quad (142)$$

**Claim Two:**

$$\tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} - \frac{1}{3} \tilde{R}^2 = 4 \left( \text{tr} g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} - (\text{tr} g_{(0)}^{-1} g_{(2)})^2 \right) + \mathcal{O}(z^2) , \quad (143)$$

The curvature  $\tilde{R}_{\mu\nu}$  is computed with the metric  $g_{\mu\nu}$ . Note that  $\tilde{R}_{\mu\nu}|_{z=0} = R[g_{\mu\nu}^{(0)}]$ .

**Claim Three:**

$$\text{tr} g_{(0)}^{-1} g_{(4)} = \frac{1}{4} \text{tr}(g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)}) , \quad (144)$$

**Claim Four:**

$$\text{tr} g_{(0)}^{-1} h_{(4)} = 0 . \quad (145)$$

---

<sup>8</sup>This check was first performed in another early AdS/CFT paper [16].

We will derive these four relations later.

We analyze each of the three terms in the trace (141) separately. Note first that we have

$$\sqrt{-\gamma} = \frac{L^4}{z^4} \sqrt{-g} . \quad (146)$$

Given that an outward pointing normal is  $n = (0, -z/L)$ , the trace of the extrinsic curvature is

$$\begin{aligned} K &= \frac{z^5}{L^5} \frac{1}{\sqrt{-g}} \partial_z \frac{L^5}{z^5} \sqrt{-g} \left( -\frac{z}{L} \right) \\ &= -\frac{1}{L} z^5 \frac{1}{\sqrt{-g}} \partial_z \frac{\sqrt{-g}}{z^4} . \end{aligned} \quad (147)$$

The square root of the determinant  $\sqrt{-g}$  expands then as

$$\begin{aligned} \sqrt{-g} &= \sqrt{-g_{(0)}} \left[ 1 + \frac{z^2}{2} \text{tr } g_{(0)}^{-1} g_{(2)} + \frac{z^4}{2} \text{tr } g_{(0)}^{-1} g_{(4)} - \frac{z^4}{4} \text{tr } g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} + \right. \\ &\quad \left. + \frac{z^4}{8} (\text{tr } g_{(0)}^{-1} g_{(2)})^2 + \frac{z^4}{2} \log z \text{tr } g_{(0)}^{-1} h_{(4)} + \dots \right] . \end{aligned} \quad (148)$$

We're making use of the familiar fact that  $\det g = \exp \text{tr} \log g$  and the Taylor series expansion  $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots$

The first term in the trace is

$$\begin{aligned} -\frac{3\sqrt{-\gamma}K}{\kappa^2} &= \frac{3L^3}{\kappa^2} z \partial_z \frac{1}{z^4} \sqrt{-g} , \\ &= \frac{3L^3}{\kappa^2} \sqrt{-g_{(0)}} z \partial_z \left[ \frac{1}{z^4} + \frac{1}{2z^2} \text{tr } g_{(0)}^{-1} g_{(2)} + \frac{1}{2} \log z \text{tr } g_{(0)}^{-1} h_{(4)} + \dots \right] \\ &= \frac{3L^3}{\kappa^2} \left[ -\frac{4}{z^4} - \frac{1}{z^2} \text{tr } g_{(0)}^{-1} g_{(2)} + O(z^2) \right] \sqrt{-g_{(0)}} . \end{aligned} \quad (149)$$

To go from the second to the third line, we used Claim Four. The second term in the trace is

$$\begin{aligned} \frac{\sqrt{-\gamma}}{\kappa^2} \frac{12}{L} &= \frac{12L^3}{\kappa^2} \sqrt{-g_{(0)}} \left[ \frac{1}{z^4} + \frac{1}{2z^2} \text{tr } g_{(0)}^{-1} g_{(2)} + \frac{1}{2} \text{tr } g_{(0)}^{-1} g_{(4)} - \frac{1}{4} \text{tr } g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} + \right. \\ &\quad \left. + \frac{1}{8} (\text{tr } g_{(0)}^{-1} g_{(2)})^2 + \frac{1}{2} \log z \text{tr } g_{(0)}^{-1} h_{(4)} + \dots \right] . \end{aligned} \quad (150)$$

Notice that the log term will again vanish by Claim Four. The third term is

$$\begin{aligned} \frac{\sqrt{-\gamma}}{\kappa^2} \frac{\mathcal{R}L}{2} &= \frac{1}{2\kappa^2} \frac{L^5}{z^4} \sqrt{-g_{(0)}} \left[ 1 + \frac{z^2}{2} \text{tr } g_{(0)}^{-1} g_{(2)} + \dots \right] \frac{z^2}{L^2} \\ &\quad \times \left[ -6 \text{tr } g_{(0)}^{-1} g_{(2)} + 2z^2 \text{tr} (g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)}) + z^2 (\text{tr } g_{(0)}^{-1} g_{(2)})^2 + \dots \right] , \end{aligned} \quad (151)$$

where we have used Claim One. Assembling the three pieces, we get

$$\begin{aligned} -\langle T_\mu^\mu \rangle &= \frac{L^3}{\kappa^2} \left[ \frac{-12+12}{z^4} + \frac{-3+6-3}{z^2} \text{tr } g_{(0)}^{-1} g_{(2)} + 6 \text{tr } g_{(0)}^{-1} g_{(4)} + (-3+1) \text{tr } g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} \right. \\ &\quad \left. + \left( \frac{3}{2} + \frac{1}{2} - \frac{3}{2} \right) (\text{tr } g_{(0)}^{-1} g_{(2)})^2 + O(z^2) \right] \\ &= \frac{L^3}{\kappa^2} \left[ 6 \text{tr } g_{(0)}^{-1} g_{(4)} - 2 \text{tr } g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} + \frac{1}{2} (\text{tr } g_{(0)}^{-1} g_{(2)})^2 \right] \\ &= \frac{L^3}{2\kappa^2} \left[ -\text{tr } g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} + (\text{tr } g_{(0)}^{-1} g_{(2)})^2 \right] \\ &= \frac{L^3}{2\kappa^2} \frac{1}{4} \left[ -\tilde{R}^{\mu\nu} \tilde{R}_{\mu\nu} + \frac{1}{3} \tilde{R}^2 \right] \Big|_{z=0} . \end{aligned} \quad (152)$$

In the first line, we see how crucial the counter-terms (77) are for getting a finite result. In some sense, we can view the first line as a derivation of the coefficients in the counter-term action (77). To go from the second to the third equality, we use Claim Three. To go from the third to the fourth equality, we use Claim Two.

The gravity computation (152) is then consistent with the field theory one (139) provided we make the identification

$$\frac{L^3}{8\kappa^2} = \frac{N^2}{32\pi^2} . \quad (153)$$

Does this identification make sense? We can read off the ten dimensional gravitational coupling constant from the SUGRA action (8):

$$\frac{1}{2\kappa_{10}^2} = \frac{1}{(2\pi)^7 \ell_s^8 g_s^2} . \quad (154)$$

By usual KK reasoning, the five dimensional coupling should then be

$$\begin{aligned} \frac{1}{2\kappa^2} &= \frac{\text{Vol}(S^5)L^5}{2\kappa_{10}^2} \\ &= \frac{\text{Vol}(S^5)L^5}{(2\pi)^7 \ell_s^8 g_s^2} \\ &= \frac{\text{Vol}(S^5)}{(2\pi)^7} (4\pi g_s N)^2 \frac{1}{L^3} \frac{1}{g_s^2} . \end{aligned}$$

where in the third line, we used that  $(L/\ell_s)^4 = 4\pi g_s N$ . Using also that  $\text{Vol}(S^5) = \pi^3$ , we find indeed the relation (153).

That there be some relation between  $a$  and  $c$  for theories with Einstein gravity duals is a foregone conclusion. There is only one parameter these coefficients can depend on,  $L^3/\kappa^2$ . That relation we have seen is  $a = c$ .

## Justifying the Claims

The proof of the claims requires writing down the Einstein Equations ,

$$R_{AB} = -\frac{d}{L^2} G_{AB} , \quad (155)$$

in a gauge fixed background where  $G_{\mu z} = 0$ . The Christoffel symbols are

$$\Gamma_{zz}^z = -\frac{1}{z} , \quad \Gamma_{\mu z}^z = \Gamma_{zz}^\mu = 0 , \quad (156)$$

$$\Gamma_{\mu\nu}^z = -\frac{1}{2} g_{\mu\nu,z} + \frac{1}{z} g_{\mu\nu} , \quad (157)$$

$$\Gamma_{z\nu}^\mu = \frac{1}{2} g^{\mu\lambda} g_{\nu\lambda,z} - \frac{1}{z} \delta_\nu^\mu , \quad (158)$$

$$\Gamma_{\nu\lambda}^\mu = \gamma_{\nu\lambda}^\mu \quad (159)$$

where  $\gamma_{\nu\lambda}^\mu$  are the Christoffel symbols for the metric  $g_{\mu\nu}$  along a constant  $z$ -slice. The Ricci curvature takes the usual form

$$R_{AB} = R^C{}_{ACB} = -\Gamma_{AC,B}^C + \Gamma_{AB,C}^C - \Gamma_{AC}^E \Gamma_{EB}^C + \Gamma_{AB}^E \Gamma_{EC}^C . \quad (160)$$

The  $zz$ -component of the Ricci curvature is then

$$R_{zz} = -\frac{1}{2} \operatorname{tr} g^{-1} \left( g'' - \frac{1}{z} g' \right) + \frac{1}{4} \operatorname{tr} g^{-1} g' g^{-1} g' - \frac{d}{z^2} . \quad (161)$$

Specializing to  $d = 4$ , the  $zz$ -component of Einstein's equations  $R_{zz} = -\frac{d}{L^2} G_{zz}$  then leads directly to claims three and four. For the remaining claims, we also need the other components of the Ricci curvature:

$$R_{\mu z} = \frac{1}{2} g^{\mu\lambda} (\nabla_\nu g'_{\lambda\mu} - \nabla_\lambda g'_{\mu\nu}) , \quad (162)$$

$$\begin{aligned} R_{\mu\nu} &= \tilde{R}_{\mu\nu} - \frac{1}{2} g''_{\mu\nu} + \frac{d-1}{2z} g'_{\mu\nu} - \left( \frac{1}{4} g' - \frac{1}{2z} g \right)_{\mu\nu} \operatorname{tr} g^{-1} g' \\ &\quad + \frac{1}{2} (g' g^{-1} g')_{\mu\nu} - \frac{d}{z^2} g_{\mu\nu} . \end{aligned} \quad (163)$$

Note that  $\tilde{R}_{\mu\nu}$  is computed with  $g_{\mu\nu}$ , or since  $R_{\mu\nu}$  is invariant under global scale transformations, equivalently with  $\gamma_{\mu\nu}$ . Thus we could also call it  $\mathcal{R}_{\mu\nu}$ .

Claim two follows from the first order terms in a small  $z$  expansion of  $R_{\mu\nu} = -\frac{d}{L^2} G_{\mu\nu}$ :

$$\tilde{R}_{\mu\nu} + 2g_{\mu\nu}^{(2)} + g_{\mu\nu}^{(0)} \operatorname{tr} g_{(0)}^{-1} g_{(2)} + \dots = 0 . \quad (164)$$

From this expression, we can read off both  $\tilde{R}_{\mu\nu}$  and its trace:

$$\tilde{R}_{\mu\nu} = -2g_{\mu\nu}^{(2)} - g_{\mu\nu}^{(0)} \operatorname{tr} g_{(0)}^{-1} g_{(2)} , \quad (165)$$

$$\tilde{R} = -6 \operatorname{tr} g_{(0)}^{-1} g_{(2)} . \quad (166)$$

From these two expressions, claim two follows after a short computation. Note that the result for  $\tilde{R}$  is the leading order term in Claim One. To compute the subleading terms, we need to look at the subleading terms in the  $\mu\nu$  trace of Einstein's equations:

$$0 = \tilde{R} - \frac{1}{2} \operatorname{tr} g^{-1} g'' + \frac{2d-1}{2z} \operatorname{tr} g^{-1} g' - \frac{1}{4} (\operatorname{tr} g^{-1} g')^2 + \frac{1}{2} \operatorname{tr} g^{-1} g' g^{-1} g' , \quad (167)$$

from which we find that

$$\begin{aligned} \tilde{R}_{(2)} &- \frac{1}{2} \cdot 12 \operatorname{tr} g_{(0)}^{-1} g_{(4)} + \frac{1}{2} \cdot 2 \operatorname{tr} g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} \\ &\quad + \frac{7}{2} \cdot 4 \operatorname{tr} g_{(0)}^{-1} g_{(4)} - \frac{7}{2} \cdot 2 \operatorname{tr} g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} \\ &\quad - \frac{1}{4} \cdot 2^2 (\operatorname{tr} g_{(0)}^{-1} g_{(2)})^2 + \frac{1}{2} \cdot 2^2 \operatorname{tr} g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} = 0 . \end{aligned} \quad (168)$$

We've not collected terms so that the reader can more easily follow the algebra. Collecting terms now yields

$$0 = \tilde{R}_{(2)} + 8 \operatorname{tr} g_{(0)}^{-1} g_{(4)} - 4 \operatorname{tr} g_{(0)}^{-1} g_{(2)} g_{(0)}^{-1} g_{(2)} - (\operatorname{tr} g_{(0)}^{-1} g_{(2)})^2 . \quad (169)$$

By claim three, this result then immediately yields the subleading terms in Claim One.



## The Two Dimensional Case

To fix the normalization of Newton's constant in terms of the central charge  $c$ , let us also review the two dimensional computation. The two dimensional stress tensor follows as a special case from (76) and (77):

$$-\sqrt{-g_0}T_\nu^\mu = \lim_{z \rightarrow 0} \frac{\sqrt{-\gamma}}{\kappa^2} \left[ K_\nu^\mu - K\gamma_\nu^\mu + \frac{1}{L}\gamma_\nu^\mu \right]. \quad (170)$$

Taking the trace then straightforwardly yields

$$-\sqrt{-g_0}T_\mu^\mu = \lim_{z \rightarrow 0} \frac{\sqrt{-\gamma}}{\kappa^2} \left[ -K + \frac{2}{L} \right]. \quad (171)$$

We follow the same steps as we did in the four dimensional case, computing small  $z$  expansions of  $K$  and  $\sqrt{-\gamma}$ :

$$\begin{aligned} K &= \frac{z^3}{L^3} \frac{1}{\sqrt{-g}} \partial_z \frac{L^3}{z^3} \sqrt{-g} \left( -\frac{z}{L} \right) \\ &= -\frac{1}{L} z^3 \frac{1}{\sqrt{-g}} \partial_z \frac{\sqrt{-g}}{z^2}, \end{aligned} \quad (172)$$

$$\sqrt{-g} = \sqrt{-g_{(0)}} \left[ 1 + \frac{z^2}{2} \text{tr} g_{(0)}^{-1} g_{(2)} + \frac{z^2}{2} \log z \text{tr} g_{(0)}^{-1} h_{(2)} + \dots \right]. \quad (173)$$

The first term in (171) is then

$$\begin{aligned} -\frac{\sqrt{-\gamma}K}{\kappa^2} &= \frac{L}{\kappa^2} z \partial_z \frac{1}{z^2} \sqrt{-g}, \\ &= \frac{L}{\kappa^2} \sqrt{-g_{(0)}} \left[ -\frac{2}{z^2} + \frac{1}{2} \text{tr} g_{(0)}^{-1} h_{(2)} + O(z^2) \right], \end{aligned} \quad (174)$$

while the second is

$$\frac{\sqrt{-\gamma}}{\kappa^2} \frac{2}{L} = \frac{2L}{\kappa^2} \sqrt{-g_{(0)}} \left[ \frac{1}{z^2} + \frac{1}{2} \text{tr} g_{(0)}^{-1} g_{(2)} + \frac{1}{2} \log z \text{tr} g_{(0)}^{-1} h_{(2)} + \dots \right]. \quad (175)$$

Putting the two pieces together yields

$$-T_\mu^\mu = \frac{L}{\kappa^2} \text{tr} g_{(0)}^{-1} g_{(2)}, \quad (176)$$

where we have used that  $\text{tr} g_{(0)}^{-1} h_{(2)} = 0$ , something that follows from a small  $z$  expansion of the  $\mu\nu$  components of Einstein's equations (see (163)). We now need to relate this expression to the Ricci scalar  $\tilde{R}$  computed from  $g_{\mu\nu}$ . The trace of the  $\mu\nu$  components of Einstein's equations (167) now gives

$$\tilde{R} = -2 \text{tr} g_{(0)}^{-1} g_{(2)} + \dots \quad (177)$$

Inserting this expression into (176) yields

$$T_\mu^\mu = \frac{L}{2\kappa^2} \tilde{R}, \quad (178)$$

which agrees with the field theory result (133) provided we make the identification

$$\frac{L}{2\kappa^2} = \frac{c}{24\pi}. \quad (179)$$

## 6 Hawking-Page Phase Transition

[These notes are loosely based on Witten's paper [17] which appeared only a month after his seminal paper on AdS/CFT [3].]

In 1983, working purely from gravity considerations, Hawking and Page [18] argued that AdS in global coordinates, when the temperature is raised above a critical value  $T_c$ , undergoes a first order phase transition to a space-time containing a black hole. Given our knowledge of the AdS/CFT correspondence, there must be some interesting dual interpretation of this phase transition for  $\mathcal{N} = 4$  SYM on  $S^3 \times S^1$ .

Let the radius of  $S^3$  be  $b$  and the circumference of the  $S^1$ , which is also the inverse temperature, be  $\beta$ . Because of the  $R\phi^2$  coupling, the conformal scalars in  $\mathcal{N} = 4$  SYM will all get masses proportional to  $1/b$ . There are no zero modes for the fermions or gauge field on the  $S^3$ ; like for the scalar, the minimum energy modes are proportional to  $1/b$ . Thus at zero temperature,  $\mathcal{N} = 4$  SYM becomes trivial. There are no excitations. But for temperatures large compared to the inverse radius of the  $S^3$ ,  $\beta < b$ , one anticipates something interesting might happen as the low energy modes become thermally populated. Indeed, in the zero coupling limit, one finds a Hagedorn type transition in a large  $N$  limit [Bo Sundborg, hep-th/9908001]. In view of the AdS/CFT correspondence, the Hawking and Page result implies there is also a phase transition in the large  $N$ , large  $g_{YM}^2 N$  limit. Presumably, the phase transition exists for all  $g_{YM}^2 N$  although its precise nature is not known.

N.B.: Because we are at finite volume, one would naively expect no non-analyticities in the free energy as a function of  $T$ . We get a phase transition precisely because we have taken a large  $N$  limit.

Let us review the Hawking-Page calculation using AdS/CFT machinery. We begin with the metric ansatz

$$\frac{ds^2}{L^2} = \frac{1}{z^2} \left( -f(z)dt^2 + b^2 d\Omega_3^2 + \frac{dz^2}{f(z)} \right). \quad (180)$$

Einstein's equations are  $R_{AB} = -4G_{AB}/L^2$ . The  $\theta_1\theta_1$  component in a parametrization of the  $S^3$  where

$$d\Omega_3^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\phi^2,$$

implies

$$f' - \frac{4}{z}f + \frac{2z}{b^2} + \frac{4}{z} = 0.$$

The most general solution for  $f$  is then

$$f(z) = 1 + \frac{z^2}{b^2} + cz^4. \quad (181)$$

(One may check that the remaining Einstein equations are also satisfied.) AdS in global coordinates is recovered by setting  $c = 0$ . The conformal boundary is located at  $z = 0$  and we keep only the region  $z > 0$ . (We can recover the Poincaré patch by further sending  $b \rightarrow \infty$  and rescaling the  $S^3$  coordinates to “zoom in” on a small region of the sphere.) We may also introduce a black hole

horizon with a nonzero value of  $c$ . The horizon is located at  $g_{tt}(z = z_h) = 0$ , which then implies

$$c = -\frac{1 + z_h^2/b^2}{z_h^4} . \quad (182)$$

We now use the usual trick of extracting the Hawking temperature by insisting that the Euclidean metric be regular at  $z = z_h$ . The relevant pieces of the Euclidean line element are

$$ds_E^2 = \frac{L^2}{z_h^2} \left( f'(z_h)(z - z_h)d\tau^2 + \frac{dz^2}{f'(z_h)(z - z_h)} + \dots \right) , \quad (183)$$

where we have only kept the leading terms in an expansion of the line element near  $z = z_h$ . We introduce a new radial variable  $r$  such that

$$\frac{L^2 dz^2}{z_h^2 f'(z_h)(z - z_h)} = dr^2 \Rightarrow r = \frac{2L}{z_h \sqrt{|f'(z_h)|}} \sqrt{z_h - z} . \quad (184)$$

(Note we have to be a little careful with signs since  $f'(z_h) < 0$  and  $z < z_h$ .) The Euclidean metric can now be written in terms of  $r$ :

$$\begin{aligned} ds_E^2 &= \frac{L^2}{z_h^2} |f'(z_h)| r^2 \frac{z_h^2 |f'(z_h)|}{4L^2} d\tau^2 + dr^2 + \dots \\ &= \frac{f'(z_h)^2}{4} r^2 d\tau^2 + dr^2 + \dots \end{aligned}$$

We now introduce an angular variable  $\theta$  such that

$$\frac{|f'(z_h)| d\tau}{2} = d\theta , \quad (185)$$

and where  $\theta$  has period  $2\pi$ . Thus, identifying the Hawking temperature  $T_H$  with one over the periodicity of  $\tau$ , we find

$$\begin{aligned} T_H &= \frac{|f'(z_h)|}{4\pi} \\ &= \frac{\left| \frac{2z_h}{b^2} - \frac{1+z_h^2/b^2}{z_h^4} 4z_h^3 \right|}{4\pi} \\ &= \frac{\frac{4}{z_h} + \frac{2z_h}{b^2}}{4\pi} . \end{aligned} \quad (186)$$

This result for the temperature shows that for a given value of  $T_H$ , there are generically either no or two corresponding values of  $z_h$ . See figure 2. The larger value of  $z_h$  corresponds to a small black hole because the horizon is further from the boundary and has a smaller area. The smaller value of  $z_h$  corresponds to a large black hole. The small black holes are similar to black holes in flat space while we will see in a moment that the large black holes actually have positive specific heat.

To compute which phase is stable, AdS in global coordinates or the black hole, we need to compute a free energy. AdS/CFT tells us that the free energy can be computed from the log of the CFT partition function:

$$F = -T \log Z = -T \log e^{S_E} = TS_E \quad (187)$$

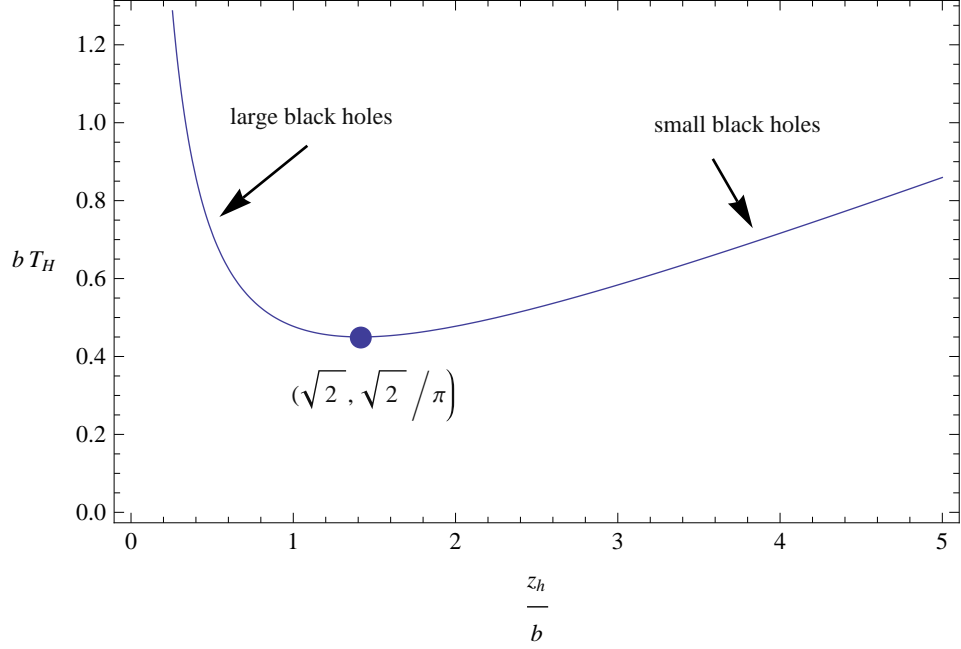


Figure 2: Temperature as a function of horizon location.

where we use our usual formula for the regulated gravity action

$$S_{\text{grav}} = S_{\text{EH}} + S_{\text{GH}} + S_{\text{ctr}} , \quad (188)$$

and we set  $T = T_H$ . (I believe the only real difference here for us between  $S_E$  and  $S_{\text{grav}}$  is that for  $S_E$ , we integrate over a circle rather than over all time.)

First the Einstein-Hilbert term:

$$\begin{aligned} S_{\text{EH}} &= -\frac{1}{2\kappa^2} \int d^5x \sqrt{-G} \left( R + \frac{12}{L^2} \right) \\ &= -\frac{L^3}{2\kappa^2} \text{Vol}(S^3) \frac{1}{T} \int_{\epsilon}^{z_h} dz \left( \frac{-8}{z^5} \right) b^3 \\ &= -\frac{L^3}{\kappa^2} \text{Vol}(S^3) \frac{1}{T} \left( \frac{-1}{z_h^4} + \frac{1}{\epsilon^4} \right) b^3 . \end{aligned}$$

In the second line, we used that  $R = -20/L^2$ .

Next the Gibbons-Hawking term:

$$S_{\text{GH}} = -\frac{1}{\kappa^2} \int d^4x \sqrt{-\gamma} K$$

Note that

$$\begin{aligned} K &= \frac{1}{\sqrt{-G}} \partial_z \sqrt{-G} \left( \frac{-z}{L} f^{1/2} \right) \\ &= -z^5 \partial_z \left( \frac{f^{1/2}}{z^4} \frac{1}{L} \right) . \end{aligned}$$

We then find

$$\begin{aligned} S_{\text{GH}} &= \frac{L^3}{\kappa^2} b^3 \text{Vol}(S^3) \frac{1}{T} \left( z f^{1/2} \partial_z \left( \frac{f^{1/2}}{z^4} \right) \right) \Big|_{z=\epsilon} \\ &= -\frac{L^3}{\kappa^2} b^3 \text{Vol}(S^3) \frac{1}{T} \left( \frac{4}{\epsilon^4} + \frac{3}{b^2 \epsilon^2} - \frac{2(1 + z_h^2/b^2)}{z_h^4} + \dots \right) \end{aligned}$$

Third, the counter term:

$$\begin{aligned} S_{\text{ctr}} &= \frac{1}{\kappa^2} \int d^4x \sqrt{-\gamma} \left( \frac{3}{L} + \frac{L}{4} \mathcal{R} \right) \\ &= \frac{L^3}{\kappa^2} b^3 \text{Vol}(S^3) \frac{1}{T} \frac{1}{z^4} f^{1/2} \left( 3 + \frac{3}{2} \frac{z^2}{b^2} \right) \Big|_{z=\epsilon} \\ &= \frac{L^3}{\kappa^2} b^3 \text{Vol}(S^3) \frac{1}{T} \left( \frac{3}{\epsilon^4} + \frac{3}{b^2 \epsilon^2} + \frac{3}{8} \left( \frac{1}{b^4} - \frac{4}{z_h^4} - \frac{4}{b^2 z_h^2} \right) + \dots \right) \end{aligned}$$

The divergent  $1/\epsilon^4$  and  $1/\epsilon^2$  terms vanish leaving the result

$$S_{\text{grav}} = \frac{L^3}{\kappa^2} \text{Vol}(S^3) b^3 \frac{1}{T} \left[ \frac{3}{8b^4} - \frac{1}{2z_h^4} + \frac{1}{2b^2 z_h^2} \right]. \quad (189)$$

This result has an amazing amount of physics that we can now extract. From this on-shell action, we deduce the free energy of the black hole and thermal AdS phases:

$$F_{\text{bh}} = \frac{L^3}{\kappa^2} \text{Vol}(S^3) b^3 \left[ \frac{3}{8b^4} - \frac{1}{2z_h^4} + \frac{1}{2b^2 z_h^2} \right], \quad (190)$$

$$F_{\text{AdS}} = \frac{L^3}{\kappa^2} \text{Vol}(S^3) b^3 \frac{3}{8b^4}. \quad (191)$$

- The  $F_{\text{AdS}}$  is essentially a Casimir energy and can be deduced from the conformal anomaly  $a$ . (See for example my paper with Kuo-Wei Huang [19].) In more detail, we assume that the pressure is independent of the volume. Given that  $P = -(\partial F/\partial V)_T$ , it follows that  $F = -PV$ . This same pressure appears in the diagonal spatial entries of the stress tensor at equilibrium. By a Schwarzian type derivative transformation, the pressure can be related to the conformal anomaly  $a$ .
- Consider the free energy difference

$$\Delta F = F_{\text{bh}} - F_{\text{AdS}} = \frac{L^3}{2\kappa^2} b^3 \text{Vol}(S^3) \frac{1}{z_h^4} \left[ \frac{z_h^2}{b^2} - 1 \right] \quad (192)$$

In other words, the black hole is favored when  $z_h < b$  and thermal AdS is favored when  $z_h > b$ .

- Because  $z_h < \sqrt{2}z_h$ , we see that the large black holes become favored at high temperature. The small black holes – the ones that are similar to Schwarzschild black holes in flat space – are thermodynamically unstable. Given that we know Schwarzschild black holes become hotter the smaller they get, i.e. they have negative heat capacity, that the small black holes are not thermodynamically favored should not be that surprising.
- For fun, we can consider the  $b \rightarrow \infty$  limit where the  $S^3$  is replaced by  $\mathbb{R}^3$ . In this case, we should divide by the volume factor and consider the energy density instead:

$$f_{\text{bh}} \equiv \frac{F_{\text{bh}}}{\text{Vol}(S^3) b^3} = -\frac{L^3}{2\kappa^2} (\pi T)^4, \quad (193)$$

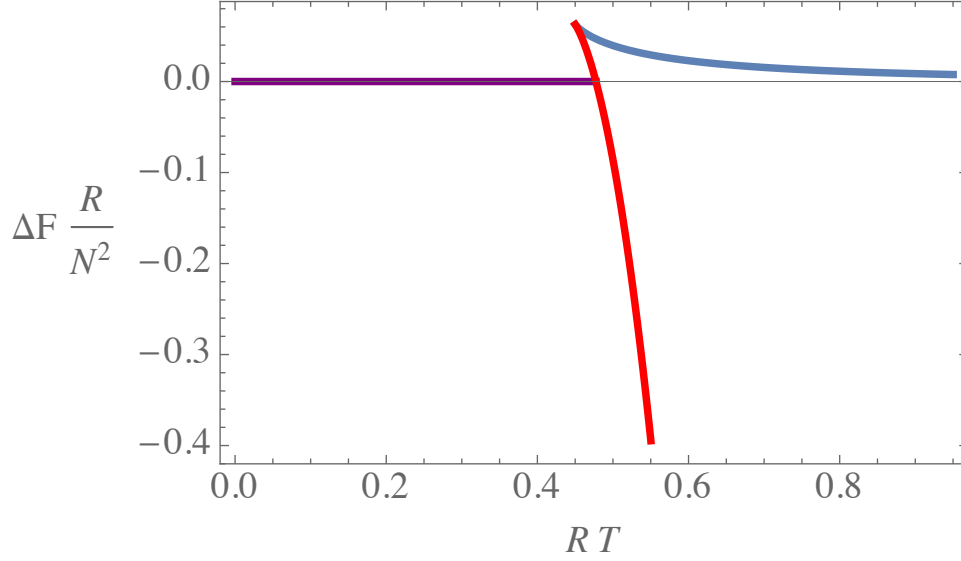


Figure 3: Free energy difference as a function of temperature.

where we have used the fact that  $T = 1/\pi z_h$  in the limit  $b \rightarrow \infty$ . Using the fact that  $\pi^2 L^3/\kappa^2 = N^2/4$  that we used in computing the conformal anomaly  $a$  holographically, we find that

$$f_{\text{bh}} = -\frac{\pi^2}{8} N^2 T^4 .$$

For free  $\mathcal{N} = 4$  SYM, one can compute the free energy density  $f_{\text{free}}$ . (See for example the paper by Gubser, Klebanov, and Tseytlin [20].) One finds that the strong interactions reduce the effective number of degrees of freedom.

$$f_{\text{bh}} = \frac{3}{4} f_{\text{free}} .$$

Much has been made of this factor of  $3/4$  in the literature. It is at last an interesting prediction for  $\mathcal{N} = 4$  SYM in the strong coupling limit. Lattice calculations involving pure YM produce similar looking factors relating  $f$  at temperatures slightly above confinement to perturbative calculations in the ultra-high  $T$  limit.

- To end this lecture, we can verify the thermodynamic relation between free energy and entropy:

$$\begin{aligned}
S_{\text{bh}} &= -\frac{dF_{\text{bh}}}{dT} = -\frac{dF_{\text{bh}}}{dz_h} \left( \frac{dT}{dz_h} \right)^{-1} \\
&= -\frac{L^3}{\kappa^2} \text{Vol}(S^3) b^3 \left( \frac{2}{z_h^5} - \frac{1}{b^2 z_h^3} \right) 4\pi \left( -\frac{4}{z_h^2} + \frac{2}{b^2} \right)^{-1} \\
&= \frac{L^3}{\kappa^2} \text{Vol}(S^3) b^3 \frac{4\pi}{z_h^3} \\
&= \frac{A}{4G_N} , \tag{194}
\end{aligned}$$

where  $A$  is the area of the event horizon and  $G_N$  is Newton's constant. [Conventionally  $\frac{1}{2\kappa^2} = \frac{1}{16\pi G_N}$ .] One could consider this short calculation a sort of holographic derivation of black hole entropy.

## 7 Current-Current Correlation Functions and Conductivity

Let us start by discussing current-current correlation functions in field theory more generally. In the vacuum state of a conformal field theory, current conservation and scale invariance together constrain the two-point function to take the form

$$\langle J^\mu(x)J^\nu(0) \rangle = \frac{c}{x^{2(d-1)}} \left( \delta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2} \right). \quad (195)$$

We work in Euclidean signature, at least to start, to keep the discussion a bit simpler. In vacuum, then, the correlation function is determined by symmetry and the choice of an overall constant  $c$ . If we now consider the Fourier transform

$$G^{\mu\nu}(k) = \int d^d x e^{ikx} \langle J^\mu(x)J^\nu(0) \rangle, \quad (196)$$

the result is in general UV divergent, requiring some type of regularization. The Fourier transformed Green's function can be written as

$$G_{\mu\nu}(k) = c \left( \delta_{\mu\nu} I_{-1}(k) + 2\frac{\partial^2}{\partial k^\mu \partial k^\nu} I_0(k) \right), \quad (197)$$

where we have defined

$$I_n(k) \equiv \int d^d x \frac{e^{ikx}}{x^{2(d+n)}} = k^{d+2n} (2\pi)^{d/2} \int_{k\epsilon}^\infty \frac{J_{\frac{d}{2}-1}(kx)}{(kx)^{\frac{3d}{2}+2n}} d(kx). \quad (198)$$

We regulated the Fourier transform by introducing a small distance cut-off  $\epsilon$ . The radial integral evaluates to

$$\int_{k\epsilon}^\infty \frac{J_{\frac{d}{2}-1}(kx)}{(kx)^{\frac{3d}{2}+2n}} d(kx) = \frac{\Gamma(-n-\frac{d}{2})}{2^{d/2}} \left( \frac{1}{2^{d+2n}\Gamma(d+n)} - \frac{1}{(k\epsilon)^{d+2n}} \right). \quad (199)$$

The final expression for the Fourier transform of the current-current correlation function is then

$$G^{\mu\nu}(k) = c \frac{\pi^{d/2}\Gamma(2-\frac{d}{2})}{2^{d-3}\Gamma(d)} k^{d-2} \left( \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + O(\epsilon^{2-d}). \quad (200)$$

The UV divergent term of order  $\epsilon^{2-d}$  is  $k$ -independent and violates the Ward identity, so we throw it away. In even dimensions, the divergence in  $\Gamma(2-d/2)$  can be swapped for a logarithmic term  $\log(k\epsilon)$ . If we think of the Green's function as a way of propagating a disturbance away from a localized source, an important point here is that even powers of  $k$  can be replaced in position space by derivatives acting on a Dirac delta function. They thus affect the behavior of the system locally at the source and do not affect the long distance behavior. From this point of view, it is clear that in even dimensions, a Green's function proportional to  $k^{d-2}$  must be supplemented by a log to give the requisite non-analyticity.

Knowing what to expect we now switch back to AdS/CFT and the action (55) in the specific case of  $AdS_5$ . Consider a small, transverse, plane wave fluctuation of the gauge field

$$A_1 = a(z)e^{iqx^2} + c.c. \quad (201)$$

The equation of motion for the gauge field constrains  $a(z)$  to obey the second order differential equation

$$(z^{-1}a')' = \frac{q^2}{z}a, \quad (202)$$

where  $'$  denotes  $\partial_z$ . The small  $z$  expansion (58) has the specific form

$$a = a_0 \left( 1 + \frac{q^2}{2}z^2 \log z - \frac{3}{64}q^4z^4 + \frac{q^4}{16}z^4 \log z + \dots \right) + a_2z^2 \left( 1 + \frac{q^2}{8}z^2 + \dots \right) + c.c., \quad (203)$$

where now because the two series overlap, logarithms have appeared.

In varying the action, we find the boundary term

$$\delta S = \frac{1}{e^2} \int_{z=\epsilon} d^4x \frac{L}{z} \delta A_1 \partial_z A_1. \quad (204)$$

Plugging in the series expansion, we find that

$$\delta S = \frac{L}{e^2} \int_{z=\epsilon} d^4x \delta a_0^* \left( 2a_2 + \frac{q^2}{2}a_0 + q^2a_0 \log z \right) + c.c. \quad (205)$$

The one point function of the current is thus proportional to the term in parentheses:

$$\langle J^1 \rangle = \frac{L}{e^2} \left( 2a_2 + \frac{q^2}{2}a_0 + q^2a_0 \log(q\epsilon) \right) e^{iqx^2} + c.c. \quad (206)$$

The log independent term proportional to  $q^2a_0$  is ambiguous and can be shifted away by an appropriate choice of cut-off inside the argument of the log. We thus find the result (63) we had before, that the one-point function of the current is proportional to the coefficient of the  $z^{d-2}$  term in the expansion of  $A_\mu$ . Naively, the  $a_2$  term seems to be ambiguous as well. While the real part of  $a_2$  is indeed rather meaningless, an imaginary part cannot be shifted away and has implications for dissipative behavior in the field theory.

The pure AdS geometry is simple enough that the differential equation (202) has an analytic solution. Picking the solution that exponentially damps out in the interior, we find

$$a = a_0 qz K_1(qz) = a_0 \left( 1 + \frac{q^2z^2}{4}(2\gamma - 1 + 2 \log \frac{qz}{2}) \right) + \dots \quad (207)$$

Referring to the expression for the one-point function of the current (206), we can extract the Fourier transform of the two-point current correlation function by varying the one-point function with respect to the external gauge field:

$$G^{11}(k) = \frac{\delta \langle J^1 \rangle}{\delta a_0} \sim q^2 \log q\epsilon. \quad (208)$$

This  $q^2 \log q\epsilon$  scaling is precisely what we expect from our earlier field theory discussion and the general result (200).



## Conductivity

If we consider a more general state with less symmetry, the current correlation functions become less constrained and more interesting. Consider the case of  $d = 3$  and nonzero temperature, for which there exists a black brane metric with line element

$$ds^2 = \frac{L^2}{z^2} \left( -f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2 \right) , \quad (209)$$

where  $f(z) = 1 - \frac{z^3}{z_h^3}$ . The horizon is at  $z = z_h$ , while the boundary as usual is at  $z = 0$ . The Hawking temperature of this black brane is  $T = 1/\pi z_h$ , as one can for example check by looking at the periodicity of the Euclidean time  $t_E = it$ .

On the field theory side, a nonzero temperature produces a plasma and a preferred rest frame. One can write two tensor structures that are compatible with the remaining symmetries and the Ward identity  $k_\mu G_R^{\mu\nu}(k) = 0$ :

$$G_R^{\mu\nu}(k) = ik^{d-2}(P_{\mu\nu}^{(1)}K^{(1)}(\omega, \mathbf{k}) + P_{\mu\nu}^{(2)}K^{(2)}(\omega, \mathbf{k})) \quad (210)$$

where

$$P_{00}^{(1)} = P_{0i}^{(1)} = P_{i0}^{(1)} = 0 , \quad P_{ij}^{(1)} = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} , \quad P_{\mu\nu}^{(2)} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} . \quad (211)$$

The functions  $K^{(1)}$  and  $K^{(2)}$  can only depend on the wave vector  $k$  through the dimensionless ratios  $\omega/T$  and  $|\mathbf{k}|/T$ . In the limit  $T \rightarrow 0$ , we must recover the vacuum result  $K^{(1)} = 0$  and  $K^{(2)} = c$ . It turns out that in the limit  $\mathbf{k} \rightarrow 0$ ,  $K^{(1)}$  also vanishes while  $K^{(2)}$  will in general be a nontrivial function of  $\omega/T$ .

To use AdS/CFT to compute these correlation functions in the limit  $\mathbf{k} \rightarrow 0$ , consider a small perturbation

$$A_x = a(z)e^{-i\omega t} . \quad (212)$$

Such a gauge field corresponds to an external electric field of strength  $E_x = -\partial_t A_x|_{z=0} = i\omega a(0)e^{-i\omega t}$  and a current  $J^x = \frac{1}{e^2} a'(0)e^{-i\omega t}$ . The equation of motion is remarkably simple in  $d = 3$ ,

$$f(z)\partial_z(f(z)\partial_z a) = -\omega^2 a . \quad (213)$$

We solve this equation by defining a new variable  $\zeta$  such that  $\partial_\zeta = f(z)\partial_z$ , leading to plane wave solutions

$$a = c_+ e^{i\omega\zeta} + c_- e^{-i\omega\zeta} . \quad (214)$$

Note that as  $\zeta \rightarrow \infty$ ,  $z \rightarrow z_h$ , and thus the  $+$  solution, which we will keep, corresponds to a plane wave moving into the horizon. The small  $z$  expansion for  $a(z)$  is

$$a(z) = c_+(1 + i\omega z + \dots) , \quad (215)$$

leading us to identify  $K^{(2)}(\omega, 0) = \frac{1}{e^2}$ . There is a more physical way to think about these response functions. We've computed a current  $J^x$  and an external electric field  $E_x$ . Through Ohm's Law, their ratio should be a conductivity:

$$\sigma = \frac{J^x}{E_x} = \frac{1}{e^2} \frac{a'(0)}{(i\omega)a(0)} = \frac{1}{e^2}. \quad (216)$$

Thus,  $K^{(2)}(\omega, 0)$  is also the optical conductivity. In this holographic example, remarkably it is independent of temperature. This independence can be traced to a classical electromagnetic duality of the action for an abelian gauge field in four dimensions (55) [21]. The paper [21] which calculated the conductivity of this  $d = 3$  holographic plasma, sometimes called the ABJM plasma, has some historical significance. It is one of the first, if not the first, application of AdS/CFT to condensed matter systems.

To see the relevance of electromagnetic duality, let us write the optical conductivity in terms of the boundary values of the electric and magnetic fields:

$$\sigma = \frac{1}{e^2} \frac{B_y(0)}{E_x(0)}. \quad (217)$$

But we should also be able to measure the same conductivity in the dual theory where the electric and magnetic fields are swapped. As the theory is free, we can leave  $e$  untouched; we do not need to send  $e \rightarrow 1/e$ . As the fields obey exactly the same equations in the dual theory, the only way for this independence to occur is if the boundary values are equal.

## 8 Viscosity

In 2001, Policastro, Son, and Starinets [22] computed the viscosity  $\eta$  of SYM in the large  $N$ , large  $g_{\text{YM}}^2 N$  limit using gauge/gravity duality. Their answer, compactly expressed in terms of the entropy density  $s$ , was that

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}. \quad (218)$$

(We set  $\hbar = k_B = 1$  in what follows.) This answer has turned out to be remarkable in a number of respects that are worthy of comment.

- The ratio is remarkably small. Common substances such as air and water have ratios which depend on temperature and pressure but are at least several dozen times higher. The smallness of the ratio should be viewed as an effect of strong coupling. Viscosity is proportional to the rate of momentum diffusion. In strongly coupled systems, many scattering events and a corresponding substantial amount of time is required for momentum to diffuse over distances large compared to the mean free path.
- This ratio was later shown to be a universal for any QFT in a rotationally symmetric state holographically dual to Einstein gravity [23, 24, 25].<sup>9</sup>

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<sup>9</sup>For anisotropic systems, the viscosity becomes a tensor, introducing a nonuniversal aspect to the story. For the

- With current techniques, it is extremely difficult to compute the viscosity for strongly coupled systems by any other means. For instance, to compute the viscosity numerically using lattice gauge theory, one must either define the gauge theory in real time and overcome the sign problem or analytically continue a numerically determined Euclidean Green’s function (see for example [27]).
- The idea of measuring viscosity in terms of entropy density has found experimental application. Two experimental hydrodynamic systems which seem to have viscosities that approach the value (218) from above are the quark-gluon plasma formed in the collision of heavy nuclei [28] and “fermions at unitarity”, i.e. clouds of fermionic atoms tuned to a strong coupling limit using a magnetic field and Feshbach resonance [29].

The smallness of the ratio and its universality led Kovtun, Son, and Starinets [23] to conjecture that the value (218) might actually be a lower bound. From a theoretical point of view, the possibility of arbitrary sign corrections to  $1/4\pi$  from higher curvature terms in the dual gravity action has greatly weakened the case that the lower bound is  $1/4\pi$  [30, 31, 32]. In fact, there exist certain gauge/gravity duality constructions where the  $1/N$  corrections are under control and where the bound is violated by  $1/N$  effects [32]. The fact that there are as yet no experimental counter-examples and the uncertainty principle argument put forth in the original paper [23] keep hope alive that there may indeed be a bound although perhaps with a somewhat lower value.

Our first task is to define precisely what we mean by viscosity for SYM. At  $T > 0$ , SYM is a neutral relativistic plasma. At scales smaller than the system size but larger than the mean free path, which by conformal invariance must be proportional to  $1/T$ , we expect that the plasma admits a hydrodynamic description. (A good reference here is chapter 15 of [33].) At these scales, the system is close to thermal equilibrium and can be well described by the variation of conserved quantities, in this case energy and momentum. Locally, the statement of energy and momentum conservation is that

$$\nabla_\mu T^{\mu\nu} = 0 . \tag{219}$$

Given that  $T^{\mu\nu}$  is slowly varying, compared to the scale set by the temperature  $T$ , we can expand  $T^{\mu\nu}$  in gradients. At zeroth order in a local rest frame, the stress tensor is diagonal:

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} , \tag{220}$$

where  $\epsilon$  and  $p$  are the energy density and pressure respectively. With knowledge of the equation of state, we can think of the system as existing at local thermal equilibrium and express  $\epsilon(T)$  and  $p(T)$  as functions of a slowly varying temperature  $T(x)$ .

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p-wave holographic superfluid, which is dual to Einstein gravity, the different components of  $\eta$  have different values in the ordered phase [26].

Hydrodynamics should also describe energy and momentum flow. To that end we introduce a slowly varying velocity field  $u^\mu$  such that  $u^2 = -1$  and for the fluid at rest  $u^\mu = (1, \vec{0})$ . At zeroth order in gradients, the only possible structure for the stress tensor is

$$T^{\mu\nu} = \alpha u^\mu u^\nu + \beta g^{\mu\nu} . \quad (221)$$

Comparing with the fluid at rest, one finds that  $\alpha = \epsilon + p$  and  $\beta = p$ . Given the equation of state and the constraint on  $u^\mu$ , the conservation conditions are four evolution equations for the four unknowns  $T$  and  $\vec{u}$ .

Next, we consider first order gradient corrections,

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} - \sigma^{\mu\nu} . \quad (222)$$

To constrain the form of  $\sigma^{\mu\nu}$ , we fix a point  $x$  and frame  $\vec{u}(x) = 0$ . We then define  $T$  and  $\vec{u}$  such that  $\sigma^{0\mu} = 0$ . As properties of the system at thermal equilibrium,  $T$  and  $\vec{u}$  are only well defined up to gradient corrections anyway. By rotational symmetry, at the point  $x$ , the gradient corrections must take the form

$$\sigma_{ij} = \eta \left( \partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \partial_k u^k \right) - \zeta \delta_{ij} \partial_k u^k . \quad (223)$$

The coefficient  $\eta$  is conventionally called shear viscosity while  $\zeta$  is the bulk viscosity. In our indexing conventions,  $\mu$  and  $\nu$  include the time direction while  $i$  and  $j$  do not. The tensor structures  $\partial_i u_j$  and  $\delta_{ij} \partial_k u^k$  have been divided up such that the structure multiplying  $\eta$  is traceless. For conformal field theories, among which SYM is an example, the trace of the stress tensor must vanish.<sup>10</sup> Thus, the bulk viscosity is zero in our case.

By general covariance, we can write this tensor structure for a general metric and point  $x$ :

$$\sigma^{\mu\nu} = \eta P^{\mu\lambda} P^{\nu\rho} \left( \nabla_\lambda u_\rho + \nabla_\rho u_\lambda - \frac{2}{3} g_{\lambda\rho} \nabla \cdot u \right) , \quad (224)$$

where we have defined the projector  $P^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$  onto directions orthogonal to  $u^\mu$ . (Note that curvature corrections involve second derivatives of the metric and would appear at higher order in our gradient expansion of  $T^{\mu\nu}$ .)

In holography, we will extract the viscosity by varying the CFT metric. Let us see how that works first purely from the CFT side. Consider a metric fluctuation of the form

$$g_{ij}(t, \vec{x}) = \delta_{ij} + h_{ij}(t) , \quad (225)$$

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<sup>10</sup>There could be a trace anomaly, but the trace anomaly is proportional to the curvature and thus higher order in our gradient expansion.

where  $h_i^i = 0$ ,  $g_{00} = -1$ , and  $g_{0i} = 0$ . Let us also take a static situation where  $u^\mu = (1, \vec{0})$ . In this case, we find that

$$\begin{aligned}\nabla_x u_y &= \partial_x u_y - \Gamma_{xy}^\mu u_\mu \\ &= \Gamma_{xy}^0 \\ &= \frac{1}{2} g^{00} (-\partial_t h_{xy}) \\ &= \frac{1}{2} \partial_t h_{xy} .\end{aligned}$$

Along with the fact that  $\nabla \cdot u = \partial_t \log \sqrt{-g} = 0$  by tracelessness of the metric fluctuation, we obtain

$$\sigma_{xy} = \eta \partial_t h_{xy} . \quad (226)$$

On the gravity side, I will begin with a non-trivial assertion about the form of the CFT stress tensor, defined on a constant  $z = z_0$  hypersurface in the limit  $z_0 \rightarrow 0$  [12]:

$$T_\nu^\mu = - \lim_{z_0 \rightarrow 0} \frac{\sqrt{-\gamma}}{8\pi G_N} \left[ K_\nu^\mu - K \delta_\nu^\mu + \frac{3}{L} \delta_\nu^\mu + \frac{\mathcal{R}L}{4} \delta_\nu^\mu - \frac{1}{2} \mathcal{R}_\nu^\mu \right] , \quad (227)$$

where  $\gamma_{\mu\nu}$  is the induced metric on the  $z = \epsilon$  slice of the geometry, and we have defined the extrinsic curvature  $K_{AB} = \nabla_{(A} n_{B)}$  and its trace  $K = K_\mu^\mu$  in terms of a unit normal vector  $n_A$  to the constant  $z = z_0$  hypersurface. The Ricci curvature  $\mathcal{R}_{\mu\nu}$  and scalar  $\mathcal{R}$  are defined on the same  $z = z_0$  hypersurface. In our indexing conventions,  $A$  and  $B$  include the radial direction  $u$  while  $\mu$  and  $\nu$  do not.

The terms in  $T_{\mu\nu}$  that depend on  $K_{\mu\nu}$  follow from (??) and a variation of the on-shell gravity action with respect to the boundary metric  $\delta S_{\text{grav}}/\delta g_{\mu\nu}$ .<sup>11</sup> As is well known, for classical gravity theories defined on spacetimes with boundary, in order to have a well defined variational principle, the usual Einstein action must be supplemented by the Gibbons-Hawking term. The same calculation that demonstrates the necessity of the Gibbons-Hawking term will produce the  $K_{\mu\nu}$  dependent terms in the CFT stress tensor. The last three terms in (227) are counterterms required to cancel off UV divergences in  $\delta S_{\text{grav}}/\delta g_{\mu\nu}$ . The counterterms are uniquely determined by requiring them to be local, covariant, and of smallest engineering dimension. A careful treatment of these counterterms is often referred to as holographic renormalization. See ref. [8] for a more in depth discussion.

Given the stress tensor (227), our strategy will be to vary the boundary metric in the same way as we did before using hydrodynamics and isolate the term in (227) that is proportional to the viscosity. One big difference from what we did before is that gravity will specify the value of  $\eta$ .

Consider the line element

$$\frac{ds^2}{L^2} = \frac{1}{z^2} \left( -f(z) dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right) + 2\tilde{g}_{xy} \frac{dx dy}{z^2} , \quad (228)$$

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<sup>11</sup>There are some subtleties about gauge fixing that I am busy sweeping under the rug. It is usually most straightforward to work in a gauge where  $g_{Az} = 0$ , in which case varying with respect to  $g_{\mu\nu}$  is unambiguous. In general, we should (and can) be more careful.

where  $f(z) = 1 - (z/z_h)^4$  and  $\tilde{g}_{xy} = e^{-i\omega t}\phi(z)$ . We insist that  $\phi(z) \ll 1$  and look at the first order term in  $\phi(z)$  in Einstein's equations  $R_{AB} = -4g_{AB}/L^2$ . Remarkably, at first order Einstein's equations impose the condition that

$$\square\tilde{g}_{xy} = 0, \quad (229)$$

where the Laplacian  $\square$  is defined using the unperturbed metric. In other words,  $\tilde{g}_{xy}$  behaves like a massless scalar in AdS. There is a theorem about the low energy absorption cross section for massless scalars in black hole spacetimes which can be used to explain why  $1/4\pi$  is a universal value for  $\eta/s$  for QFTs dual to Einstein gravity [23].

Putting universal considerations aside, let us solve  $\square\tilde{g}_{xy} = 0$  in the hydrodynamic limit  $\omega \ll T$ . First we set boundary conditions at the horizon. Assuming that  $\phi \sim (z - z_h)^\alpha$  when  $z \approx z_h$ , we find that  $\alpha = \pm i\omega z_h/4$ . We want causal boundary conditions, which correspond to waves traveling into the black brane. This causality constraint means we must choose the minus sign.

We leave it as an exercise to demonstrate that  $\phi(z) = f(z)^{-i\omega z_h/4}$  satisfies  $\square\tilde{g}_{xy} = 0$  up to  $O(\omega^2)$ . To evaluate the stress tensor (227), I usually use a computer. With a couple of keystrokes, optimally, one then discovers that

$$T_y^x = \lim_{z \rightarrow 0} \frac{L^3}{8\pi G_N} \left( \frac{\phi}{2z_h^4} + \frac{\phi'}{2z^3} \right). \quad (230)$$

Comparing with the hydrodynamic form (222) of the stress tensor, one finds the pressure and viscosity in terms of geometric quantities,

$$p = \frac{L^3}{16\pi G_N} \frac{1}{z_h^4}, \quad \eta = \frac{L^3}{16\pi G_N} \frac{1}{z_h^3}. \quad (231)$$

Recalling that  $1/z_h = \pi T$  and  $G_N/L^3 = \pi/2N^2$ , the pressure and viscosity can be expressed in terms of field theory quantities,

$$p = \frac{\pi^2 N^2 T^4}{8}, \quad \eta = \frac{\pi N^2 T^3}{8}. \quad (232)$$

The entropy density follows either from the thermodynamic relation  $s = dp/dT$  or from the black hole entropy formula. Either way, the answer is that  $s = \pi^2 N^2 T^3/2$  and hence that

$$\frac{\eta}{s} = \frac{1}{4\pi}.$$

## 9 Entanglement Entropy

I would like to end this set of lectures with a brief discussion of entanglement entropy. Entanglement entropy is a measure of the entanglement between two quantum sub-systems. It is a notion first developed within the quantum information community but that has taken on a life of its own in a much broader physics context. It has been proposed as a way of understanding black hole entropy [34, 35]. It can serve as an order parameter for certain exotic phase transitions [36, 37]. In relativistic field theories, it provides a measure of renormalization group flow [38, 39]. The Ryu-Takayanagi

formula [40, 41] relates the black hole and field theory applications. It is a simple method for calculation entanglement entropy for field theories with dual classical gravity descriptions.

To define the entanglement entropy, we assume that the Hilbert space may be factorized  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ . The sub-systems in a many-body context are usually taken to be a spatial region  $A$  and its complement  $\bar{A} = B$ . That such a factorization exists can be a problematic assumption. For example, for a lattice gauge theory, the gauge invariant observables are not local quantities, and a careful treatment needs to be made of the observables that get cut into pieces by  $\partial A$ . Assuming that such a factorization exists, we can perform a partial trace of the density matrix  $\rho$  to obtain a reduced density matrix  $\rho_A \equiv \text{tr}_B \rho$  that retains information about all of the local observables in region  $A$ . The entanglement entropy is then the von Neumann entropy of the reduced density matrix:

$$S_E \equiv -\text{tr} \rho_A \log \rho_A . \quad (233)$$

For field theories, the entanglement entropy is typically badly UV divergent. In the vacuum, it is widely expected if not universally proven that

$$S \sim \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} , \quad (234)$$

where  $\epsilon$  is a short distance UV regulator. Despite the divergent behavior, we can extract two useful pieces of information from this result. The first is the scaling with  $\text{Area}(\partial A)$  which suggests that the correlations in the vacuum are mostly local. The second is the relation to black hole entropy where perhaps  $G_N$  provides some kind of universal UV regulator.

The entanglement entropy is quite often extremely difficult to calculate. Surprisingly, there is a remarkably simple way to measure entanglement of field theories with gravity duals, at least in a large  $N$  limit. The idea, proposed by Ryu and Takayanagi [40, 41] and later proven [42], is to construct a minimal spatial surface  $\Sigma$  in AdS along a constant time slice such that the boundary of the surface is the same as the boundary of the region  $A$ :  $\partial \Sigma = \partial A$ . Inspired by the black hole entropy formula, Ryu and Takayanagi suggested that

$$S_E = \frac{\text{Area}(\Sigma)}{4G_N} . \quad (235)$$

This area is divergent because the boundary of hyperbolic space is infinitely far away. But the entanglement entropy is also UV divergent. The point is really that a one-to-one map can be made between the divergent terms and finite terms in a regulated computation of  $S_E$ .

The entanglement entropy satisfies the property of strong sub-additivity:

$$S_E(A) + S_E(B) \geq S_E(A \cup B) + S_E(A \cap B) . \quad (236)$$

The rather technical field theory proof [43] has a beautifully simple geometric counter-part using the RT formula [44] (see figure 4). Just by cutting and gluing the minimal surfaces  $\Sigma_A$  and  $\Sigma_B$ , we can obtain candidate surfaces  $\tilde{\Sigma}_{A \cup B}$  and  $\tilde{\Sigma}_{A \cap B}$  which share boundaries with  $A \cup B$  and  $A \cap B$  such that

$$\text{Area}(\Sigma_A) + \text{Area}(\Sigma_B) = \text{Area}(\tilde{\Sigma}_{A \cup B}) + \text{Area}(\tilde{\Sigma}_{A \cap B}) \quad (237)$$

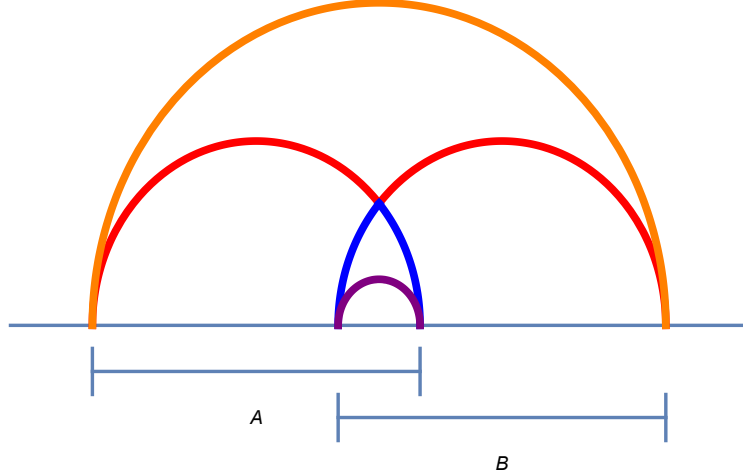


Figure 4: An illustration of the strong sub-additivity argument for disk shaped regions in  $\text{AdS}_{d+1}$ . The red curve  $\tilde{\Sigma}_{A \cup B}$  and blue curve  $\tilde{\Sigma}_{A \cap B}$  correspond to surfaces constructed from cutting and pasting the minimal surfaces  $\Sigma_A$  and  $\Sigma_B$ . The orange curve  $\Sigma_{A \cup B}$  and purple curve  $\Sigma_{A \cap B}$  are the actual minimal surfaces.

However, these surfaces  $\tilde{\Sigma}_{A \cup B}$  and  $\tilde{\Sigma}_{A \cap B}$  will not be minimal because we have not yet minimized their area. Indeed, they will in general have cusps and sharp corners. Minimizing their area leads immediately to strong sub-additivity.

Let us see how the RT formula works in the simplest nontrivial example,  $d = 2$ . Consider the line element for  $\text{AdS}_3$  in the Poincaré patch:

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + dx^2 + dz^2) . \quad (238)$$

We would like to work out a minimal length geodesic along a constant time slice  $t = 0$ . We parametrize the geodesic via  $z(x)$  and compute it by minimizing the length:

$$\text{Area}(\Sigma) = \int \frac{L}{z} \sqrt{1 + (z')^2} dx . \quad (239)$$

There is a “conserved” energy which will allow us to solve a first order differential equation rather than the second order Euler-Lagrange equations:

$$E = \frac{1}{z} \sqrt{1 + (z')^2} - z' \frac{\partial}{\partial z'} \frac{1}{z} \sqrt{1 + (z')^2} , \quad (240)$$

which simplifies to

$$E = \frac{1}{z} (1 + (z')^2)^{-1/2} . \quad (241)$$

Inverting this expression gives

$$z' = \pm \frac{1}{Ez} \sqrt{1 - (Ez)^2} . \quad (242)$$



Integrating, one finds

$$x - c = \mp \frac{1}{E} \sqrt{1 - (Ez)^2} , \quad (243)$$

or equivalently

$$(x - c)^2 + z^2 = \frac{1}{E^2} . \quad (244)$$

In other words, a geodesic is a half circle with radius  $1/E$  that ends on the boundary  $z = 0$ . Let us replace  $1/E$  with  $r$ , the radius of the circle.

Next, let's compute the length of this geodesic (for a circle with  $c = 0$ )

$$\text{Area}(\Sigma) = Lr \int_{-R}^R \frac{dx}{z^2} \quad (245)$$

$$= \frac{L}{R} \int_{-R}^R \frac{dx}{1 - (\frac{x}{R})^2} \quad (246)$$

$$= \frac{L}{2} \log \frac{1 + \frac{x}{R}}{1 - \frac{x}{R}} \Big|_{-R}^R . \quad (247)$$

The integral is logarithmically divergent, and so we introduce a regulator, integrating only to a distance  $\epsilon$  from the  $z = 0$  boundary. The regulated length is then

$$\text{Area}(\Sigma) = 2L \log \frac{2R}{\epsilon} . \quad (248)$$

The Ryu-Takayanagi formula for the entanglement entropy therefore gives

$$S_E = \frac{\text{Area}(\Sigma)}{4G_N} = \frac{8\pi}{2\kappa^2} \cdot L \log \frac{2r}{\epsilon} = \frac{c}{3} \log \frac{2r}{\epsilon} , \quad (249)$$

where the normalization of Newton's constant is fixed by the 2d trace anomaly (179). This result should hold for CFTs in a "large  $N$ " limit. Large  $N$  here means that  $c \gg 1$ . Remarkably, the entanglement entropy for a single interval in a 2 dimensional CFT can be calculated in full generality for any value of  $c$ . (See for example the review [45].) The result is exactly the same.

It is not too much more difficult to compute the entanglement entropy of a  $(d - 1)$ -dimensional ball  $B^{d-1}$  for a field theory in  $d$  dimensions. Working in polar coordinates with the line element

$$\frac{ds^2}{L^2} = \frac{1}{z^2} (-dt^2 + dr^2 + r^2 d\Omega^2 + dz^2) , \quad (250)$$

we obtain the functional for the area

$$\text{Area}(\Sigma) = L^{d-1} \text{Vol}(S^{d-2}) \int \frac{r^{d-2}}{z^{d-1}} \sqrt{1 + (z')^2} dr . \quad (251)$$

The Euler-Lagrange equations following from this functional are satisfied by a hemisphere  $z^2 + r^2 = R^2$ . The area is then

$$\text{Area}(\Sigma) = L^{d-1} R \text{Vol}(S^{d-2}) \int_0^R r^{d-2} (R^2 - r^2)^{d/2} dr \quad (252)$$

which is badly divergent at  $r = R$ . We regulate by integrating only to a distance  $z = \epsilon$  from the boundary:

$$\text{Area}(\Sigma) \sim L^{d-1} \frac{1}{d-2} \frac{R^{d-2} \text{Vol}(S^{d-2})}{\epsilon^{d-2}}, \quad (253)$$

which yields the area law scaling (235) described above.

In the case  $d = 4$ , we would get

$$S_E = \frac{L^3}{4G_N} 4\pi \left( \frac{R^2}{2\epsilon^2} - \frac{1}{2} \log \frac{R}{\epsilon} + \dots \right) \quad (254)$$

The coefficients of the log term is nothing but  $4a$  (see (138) and (153)), which does have an invariant meaning [46].

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## A Bosonic Supergravity Actions and Equations of Motion

### A.1 Type IIA

The bosonic piece of the IIA Einstein frame action is

$$\begin{aligned} S_{\text{IIA}} = & \frac{1}{2\kappa^2} \int d^{10}x (-g)^{1/2} R - \frac{1}{4\kappa^2} \int \left( d\phi \wedge \star d\phi + e^{-\phi} H_3 \wedge \star H_3 \right. \\ & \left. + g_s^2 e^{3\phi/2} F_2 \wedge \star F_2 + g_s^2 e^{\phi/2} \tilde{F}_4 \wedge \star \tilde{F}_4 + g_s^2 B_2 \wedge F_4 \wedge F_4 \right), \end{aligned} \quad (255)$$

where

$$\tilde{F}_4 = F_4 - C_1 \wedge H_3, \quad F_4 = dC_3, \quad F_2 = dC_1. \quad (256)$$

We define the Einstein metric by  $(g_{\mu\nu})_{\text{Einstein}} = e^{-\phi/2} (\gamma_{\mu\nu})_{\text{string}}$  where  $e^\Phi = g_s e^\phi$ . As a result  $g_s$  appears in the action, explicitly and also through  $2\kappa^2 = (2\pi)^7 \ell_s^8 g_s^2$ .

The field equations are [47]

$$\begin{aligned}
d\star d\phi &= -\frac{e^{-\phi}}{2}H_3 \wedge \star H_3 + \frac{3g_s^2 e^{3\phi/2}}{4}F_2 \wedge \star F_2 + \frac{g_s^2 e^{\phi/2}}{4}\tilde{F}_4 \wedge \star \tilde{F}_4 \\
d(e^{3\phi/2}\star F_2) &= e^{\phi/2}H_3 \wedge \star \tilde{F}_4 \\
d(e^{\phi/2}\star \tilde{F}_4) &= -F_4 \wedge H_3 \\
\frac{g_s^2}{2}F_4 \wedge F_4 &= d(e^{-\phi}\star H_3 + g_s^2 e^{\phi/2}C_1 \wedge \star \tilde{F}_4) \\
R_{MN} &= \frac{1}{2}(\partial_M \phi)\partial_N \phi + \frac{e^{-\phi}}{4}(H_M{}^{PQ}H_{NPQ} - \frac{1}{12}G_{MN}H^{PQR}H_{PQR}) \\
&\quad + \frac{g_s^2 e^{3\phi/2}}{2}(F_M{}^P F_{NP} - \frac{1}{16}G_{MN}F^{PQ}F_{PQ}) \\
&\quad + \frac{g_s^2 e^{\phi/2}}{12}(\tilde{F}_M{}^{PQR}\tilde{F}_{NPQR} - \frac{3}{32}G_{MN}\tilde{F}^{PQRS}\tilde{F}_{PQRS}). \tag{257}
\end{aligned}$$

We use indices  $M, N, \dots$  in ten dimensions. The Bianchi identities are

$$d\tilde{F}_4 = -F_2 \wedge H_3 \quad , \quad dF_2 = 0.$$

## A.2 Type IIB

The bosonic piece of the IIB Einstein frame action [48] is

$$\begin{aligned}
S_{\text{IIB}} &= \frac{1}{2\kappa^2} \int d^{10}x (-g)^{1/2} R - \frac{1}{4\kappa^2} \int \left( d\phi \wedge \star d\phi + g_s^2 e^{2\phi} dC \wedge \star dC + \right. \\
&\quad \left. e^{-\phi} H_3 \wedge \star H_3 + g_s^2 e^\phi \tilde{F}_3 \wedge \star \tilde{F}_3 + \frac{g_s^2}{2} \tilde{F}_5 \wedge \star \tilde{F}_5 + g_s^2 C_4 \wedge H_3 \wedge F_3 \right), \tag{258}
\end{aligned}$$

supplemented by the self-duality condition

$$\star \tilde{F}_5 = \tilde{F}_5. \tag{259}$$

Here

$$\begin{aligned}
\tilde{F}_3 &= F_3 - CH_3, \quad F_3 = dC_2, \\
\tilde{F}_5 &= F_5 - C_2 \wedge H_3, \quad F_5 = dC_4. \tag{260}
\end{aligned}$$

The field equations are [49]

$$\begin{aligned}
d\star d\phi &= g_s^2 e^{2\phi} dC \wedge \star dC - \frac{e^{-\phi}}{2}H_3 \wedge \star H_3 + \frac{g_s^2 e^\phi}{2}\tilde{F}_3 \wedge \star \tilde{F}_3, \\
d(e^{2\phi}\star dC) &= -e^\phi H_3 \wedge \star \tilde{F}_3, \\
d\star(e^\phi \tilde{F}_3) &= F_5 \wedge H_3, \\
d\star(e^{-\phi}H_3 - g_s^2 C e^\phi \tilde{F}_3) &= -g_s^2 F_5 \wedge F_3, \\
d\star \tilde{F}_5 &= -F_3 \wedge H_3, \\
R_{MN} &= \frac{1}{2}\partial_M \phi \partial_N \phi + \frac{g_s^2 e^{2\phi}}{2}\partial_M C \partial_N C + \frac{g_s^2}{96}\tilde{F}_{MPQRS}\tilde{F}_N{}^{PQRS} \\
&\quad + \frac{1}{4}(e^{-\phi}H_{MPQ}H_N{}^{PQ} + g_s^2 e^\phi \tilde{F}_{MPQ}\tilde{F}_N{}^{PQ}) \\
&\quad - \frac{1}{48}G_{MN}(e^{-\phi}H_{PQR}H^{PQR} + g_s^2 e^\phi \tilde{F}_{PQR}\tilde{F}^{PQR}). \tag{261}
\end{aligned}$$

The Bianchi identities are

$$\begin{aligned} d\tilde{F}_3 &= -dC \wedge H_3 , \\ d\tilde{F}_5 &= -F_3 \wedge H_3 . \end{aligned} \quad (262)$$

Type IIB supergravity is invariant under the action of  $SL(2, \mathbb{R})$ , as we can make more manifest by writing the action in the following way

$$\begin{aligned} S_{\text{IIB}} &= \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[ R - \frac{(\partial_\mu \bar{\tau})(\partial^\mu \tau)}{2(\text{Im } \tau)^2} - g_s \frac{M_{ij}}{2} F_3^i \cdot F_3^j - \frac{g_s^2}{4} |\tilde{F}_5|^2 \right] \\ &\quad - g_s^2 \frac{\epsilon_{ij}}{8\kappa^2} \int C_4 \wedge F_3^i \wedge F_3^j , \end{aligned} \quad (263)$$

where

$$\tau = C_0 + \frac{i}{g_s e^\phi} , \quad M_{ij} = \frac{1}{\text{Im } \tau} \begin{bmatrix} 1 & -\text{Re } \tau \\ -\text{Re } \tau & |\tau|^2 \end{bmatrix} , \quad F_3^i = \begin{bmatrix} F_3 \\ H_3 \end{bmatrix} . \quad (264)$$

The group  $SL(2, \mathbb{R})$  acts via

$$\tau' = \frac{a\tau + b}{c\tau + d} , \quad (265)$$

$$F_3^{i'} = \Lambda_j^i F_3^j , \quad \Lambda_j^i = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \quad (266)$$

The objects  $\tilde{F}_5$  and the metric remain invariant under the group action.

### A.3 M theory

The eleven dimensional SUGRA action[50] is

$$\frac{1}{2\kappa_{11}^2} \int d^{11}x (-g)^{1/2} R - \frac{1}{4\kappa_{11}^2} \int \left( F_4 \wedge \star F_4 + \frac{1}{3} A_3 \wedge F_4 \wedge F_4 \right) . \quad (267)$$

The field equations are then

$$\begin{aligned} d\star F_4 &= \frac{1}{2} F_4 \wedge F_4 , \\ R_{MN} &= \frac{1}{12} \left( F_M^{PQR} F_{NPQR} - \frac{1}{12} G_{MN} F^{PQRS} F_{PQRS} \right) \end{aligned} \quad (268)$$

supplemented by the Bianchi identity  $dF_4 = 0$ .

## B Anti-de Sitter Space Metrics

We start with  $\mathbb{R}^{2,p+1}$  with line element

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2 . \quad (269)$$

Anti-de Sitter space is a covering space of the hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{i=1}^{p+1} X_i^2 = R^2 . \quad (270)$$

This embedding makes manifest the  $\text{SO}(2, p+1)$  symmetry of  $\text{AdS}_{p+2}$ .

There are three standard ways of coordinatizing this space: planar, spherical, and hyperbolic slicing. For the most part in these lecture notes, we are interested in planar slicing, also called the Poincaré patch, for which the field theory lives on  $\mathbb{R}^{1,3}$ . Spherical slicing is also referred to as global AdS. In this parametrization, the CFT lives on  $\mathbb{R}^1 \times S^p$ . Similarly, in hyperbolic slicing, the CFT lives on a line cross a hyperbola.

### Planar Slicing

$$\begin{aligned} X_0 &= \frac{1}{2}[e^{-\rho_M} + e^{\rho_M}(R^2 + r^2 - t^2)] , \\ X_i &= R e^{\rho_M} r \Omega_i , \\ X_{p+1} &= -\frac{1}{2}[e^{-\rho_M} + e^{\rho_M}(-R^2 + r^2 - t^2)] , \\ X_{p+2} &= R e^{\rho_M} t . \\ ds^2 &= R^2[d\rho_M^2 + e^{2\rho_M}(-dt^2 + dr^2 + r^2 d\Omega^2)] . \end{aligned} \quad (271)$$

To recover the AdS part of the metric (25) in the body of the paper, we make the identification  $e^{\rho_M} = 1/z$ .

### Spherical Slicing

$$\begin{aligned} X_0 &= R \cosh \rho_S \cos \tau_S , \\ X_i &= R \sinh \rho_S \sin \theta \Omega_i , \\ X_{p+1} &= R \sinh \rho_S \cos \theta , \\ X_{p+2} &= R \cosh \rho_S \sin \tau_S . \\ ds^2 &= R^2[d\rho_S^2 - \cosh^2 \rho_S d\tau_S^2 + \sinh^2 \rho_S (du^2 + \sinh^2 u d\Omega^2)] . \end{aligned} \quad (272)$$

In this slicing, one considers the covering space of the hyperbola where  $\tau_s$  is allowed to take any real value.

### Hyperbolic Slicing

$$\begin{aligned} X_0 &= R \cosh \rho_H \cosh u , \\ X_i &= R \cosh \rho_H \sinh u \Omega_i , \\ X_{p+1} &= R \sinh \rho_H \cosh \tau_H , \\ X_{p+2} &= R \sinh \rho_H \sinh \tau_H . \end{aligned}$$

$$ds^2 = R^2[d\rho_H^2 - \sinh^2 \rho_H d\tau_H^2 + \cosh^2 \rho_H (du^2 + \sinh^2 u d\Omega^2)] . \quad (273)$$

## C Supersymmetries of the type IIB D3-brane solution

For a more thorough discussion, see refs. [51, 52]. These methods will reveal the 16 ordinary supercharges but not the superconformal ones. We begin with the dilatino  $\lambda$  and gravitino  $\psi_M$  SUSY transformations in type IIB. Both are complex Weyl spinors that satisfy the chirality constraints  $\gamma^{11}\psi_M = -\psi_M$  and  $\gamma^{11}\lambda = \lambda$ . Because the  $\Phi$ ,  $C$ ,  $B_2$ , and  $C_2$  fields are trivial for the D3-brane solution, the dilatino  $\lambda$  variation is trivially satisfied. The nontrivial constraint comes from the gravitino  $\psi_M$  variation, which reduces to

$$\delta\psi_M = D_M\epsilon + \frac{ig_s}{16 \cdot 5!} \gamma^{M_1 \dots M_5} F_{M_1 \dots M_5} \gamma_M \epsilon , \quad (274)$$

where

$$D_M\epsilon = \partial_M\epsilon - \frac{1}{4}(\omega_{PQ})_M \tilde{\gamma}^P \tilde{\gamma}^Q \epsilon , \quad (275)$$

$\omega_{PQ}$  is the spin connection and while  $\{\gamma^M, \gamma^N\} = -2g^{MN}$ . We have also  $\{\tilde{\gamma}^M, \tilde{\gamma}^N\} = -2\eta^{MN}$ .

Consider now the D3-brane solution

$$ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2} dx^m dx^m , \quad (276)$$

$$F_{\mu\nu\lambda\rho m} = \frac{1}{g_s} \epsilon_{[\mu\nu\lambda\rho} \partial_m] H , \quad F_{mnpqr} = -\frac{1}{g_s} \epsilon_{mnpqrs} \partial^s H . \quad (277)$$

where  $H(x^m)$  is a harmonic function. We introduce the frame one-forms  $e^\mu = H^{-1/4} dx^\mu$  and  $e^m = H^{1/4} dx^m$ . From these and the relation  $de^M + \omega^M_N e^N = 0$ , we deduce the connection one-forms

$$\omega^\mu_n = -\frac{1}{4}(\partial_n \log H) H^{-1/4} e^\mu , \quad \omega^m_n = \frac{1}{4}(\partial_n \log H) H^{-1/4} e^m . \quad (278)$$

The gravitino variation separates into the cases where  $M = \mu$  and  $M = m$ :

$$\delta\psi_\mu = \partial_\mu\epsilon + \frac{1}{8}(\partial_n \log H) \gamma_\mu \gamma^n (1 + \Gamma^4) \epsilon , \quad (279)$$

$$\delta\psi_m = \partial_m\epsilon + \frac{1}{8}(\partial_n \log H) \epsilon - \frac{1}{8}(\partial_n \log H) \gamma_m \gamma^n (1 + \Gamma^4) \epsilon \quad (280)$$

where  $\Gamma^4 = i\tilde{\gamma}^{0123}$ . We choose  $\Gamma^4\epsilon = -\epsilon$  and  $\epsilon = H^{-1/8}\eta$  to ensure that  $\delta\psi_M = 0$ . We can decompose  $\eta = \zeta \otimes \chi$  where  $\zeta$  is a 4d spinor and  $\chi$  is a 6d spinor. We need to choose  $\Gamma^4\zeta = -\zeta$ . Then it follows, since  $\gamma^{11}\epsilon = -\epsilon$ , that  $\Gamma^6\chi = \chi$ . There are two such  $\zeta$ 's and four such  $\chi$ 's for a total of eight complex or 16 real supersymmetries. The remaining 16 superconformal supersymmetries that appear when  $H^{-1} \sim x^m x^m$  are more difficult to see from this approach.

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