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On the equivalence of the discrete nonlinear Schrödinger equation and the discrete isotropic Heisenberg magnet

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Abstract

The equivalence of the discrete isotropic Heisenberg magnet (IHM) model and the discrete nonlinear Schrödinger equation (NLSE) given by Ablowitz and Ladik is shown. This is used to derive the equivalence of their discretization with the one by Izgerin and Korepin. Moreover a doubly discrete IHM is presented that is equivalent to Ablowitz' and Ladiks doubly discrete NLSE. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

The gauge equivalence of the continuous isotropic Heisenberg magnet model and the nonlinear Schrödinger equation is well known [7]. On the other hand there are several discretizations of the nonlinear Schrödinger equation in literature (e.g. [1,11,5,12]). In particular there are two famous versions with continuous time. One introduced by Ablowitz and Ladik [1] (from now on called $dNLSE_{AL}$) and one given by Izgerin and Korepin [11] (from now on referred to as $dNLSE_{IK}$) (see also [7]). The second can be obtained from the discrete (or lattice) isotropic Heisenberg magnet model (dIHM) with slight modification via a gauge transformation [7].

In this paper the gauge equivalence of the dIHM model and the $dNLSE_{AL}$ is shown. In fact this is in complete analogy to the continuous case. The equivalence of the two discretizations of the nonlinear Schrödinger equation is derived from this. An other interesting relation between the discrete Heisenberg spin chain and the $dNLSE_{AL}$ should be mentioned: It can be found in the brilliant paper of Its, Isergin, Korepin and Slavnov [10] where it is shown, that the $dNLSE_{AL}$ arises as the quantum correlation functions of the Heisenberg spin chain.

In addition in Section 3 a doubly discrete (with discrete time) version of the IHM model is given that links in the same way with the doubly discrete NLSE introduced by Ablowitz and Ladik in [2]. It first appeared in a somewhat implicit form in [4,12].

In [8] the author explains the geometric background of the interplay between IHM model and NLSE (see also [3,6]) From the geometric point of

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view the $dNLSE_{AL}$ seems to be the more natural choice.

In the following we will identify $\mathbb{R}^3$ with su(2) that is the span of $\mathfrak{i}, \mathfrak{j}$, and $\mathfrak{k}$ where

$$\mathfrak{i} = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathfrak{j} = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$\mathfrak{k} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

2. Equivalence of the discrete Heisenberg magnetic model and the nonlinear Schrödinger equation

The dIHM model and the $dNLSE_{AL}$ are well known [1,7,3,13]. In this section it is shown that – as in the smooth case – both models are gauge equivalent. This equivalence seem to appear first in [9] without any reference to the dIHM model. We start by giving the discretizations.

The $dNLSE_{AL}$ has the form

$$-i\dot{\Psi}_k = \Psi_{k+1} - 2\Psi_k + \Psi_{k-1} + |\Psi_k|^2 \times (\Psi_{k+1} + \Psi_{k-1}) \tag{1}$$

It has the following zero curvature representation (see [1,13])

$${}_k = \hat{M}_{k+1}\hat{L}_k - \hat{L}_k\hat{M}_k \tag{2}$$

with $\hat{L}_k$ and $\hat{M}_k$ of the form

$$\hat{L}_k(\mu) = \begin{pmatrix} \mu & \Psi_k \\ -\overline{\Psi}_k & \mu^{-1} \end{pmatrix}$$

$$\hat{M}_k(\mu) = \begin{pmatrix} \mu^2 i - i + i\Psi_k\overline{\Psi}_{k-1} & \mu i\Psi_k - \mu^{-1} i\Psi_{k-1} \\ -\mu i\overline{\Psi}_{k-1} + \mu^{-1} i\overline{\Psi}_k & -\mu^{-2} i + i - i\overline{\Psi}_k\Psi_{k-1} \end{pmatrix} \tag{3}$$

where the overbar denotes complex conjugation. Aiming to the forthcoming theorem we gauge this Lax pair with $\begin{pmatrix} \sqrt{\mu} & 0 \\ 0 & \sqrt{\mu}^{-1} \end{pmatrix}$ and get

$$L_k(\mu) = \begin{pmatrix} 1 & \Psi_k \\ -\overline{\Psi}_k & 1 \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

$$M_k(\mu) = \begin{pmatrix} i\Psi_k\overline{\Psi}_{k-1} & i\Psi_k - i\Psi_{k-1} \\ -i\overline{\Psi}_{k-1} + i\overline{\Psi}_k & -i\overline{\Psi}_k\Psi_{k-1} \end{pmatrix} + \begin{pmatrix} 1 & \Psi_{k-1} \\ -\overline{\Psi}_{k-1} & 1 \end{pmatrix} \times \begin{pmatrix} i(\mu^2 - 1) & 0 \\ 0 & -i(\mu^{-2} - 1) \end{pmatrix} \tag{4}$$

We now turn our attention for a moment to the discrete isotropic Heisenberg magnet model. It is given by the following evolution equation

$$\dot{S}_k = 2\frac{S_{k+1} \times S_k}{1 + \langle S_{k+1}, S_k \rangle} - 2\frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} \tag{5}$$

with the S_k being unit vectors in $\mathbb{R}^3$. Its zero curvature representation is given by

$$\dot{U}_k = V_{k+1}U_k - U_kV_k \tag{6}$$

with U_k and V_k of the form

$$U_k = \mathbb{I} + \lambda S_k$$

$$V_k = -\frac{1}{1+\lambda^2} \times \left(2\lambda^2 \frac{S_k + S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} + 2\lambda \frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} \right) \tag{7}$$

if one identifies the $\mathbb{R}^3$ with su(2) in the usual way. Now we are prepared to state

Theorem 1. *The discrete nonlinear Schrödinger equation $dNLSE_{AL}$ (1) and the discrete isotropic Heisenberg magnet model dIHM (5) are gauge equivalent.*

Proof. We use the notation introduced above. Let $\mathscr{F}$ be a solution to the linear problem

$$\mathscr{F}_{k+1} = L_k(1)\mathscr{F}_k,$$

$$\dot{\mathscr{F}}_k = \hat{M}_k(1)\mathscr{F}_k := \left(M_k(1) + \mathscr{F}_k c \mathscr{F}_k^{-1} \right)\mathscr{F}_k \tag{8}$$

with a constant vector c. Since $\hat{M}_{k+1}(1)L_k(1) - L_k(1)\hat{M}_k(1) = M_{k+1}(1)L_k(1) - L_k(1)M_k(1) = \dot{L}_k(1)$

the zero curvature condition stays valid and the system is solvable. The additional term $\mathcal{F}_k c \mathcal{F}_k^{-1}$ will give rise to an additional rotation around c in the dIHM model. The importance of this possibility will be clarified in the next section. Moreover define

$$S_k := \mathcal{F}_k^{-1} \mathfrak{i} \mathcal{F}_k \tag{9}$$

Note that this implies that

$$\frac{|S_k \times S_{k+1}|}{1 + \langle S_k, S_{k+1} \rangle} = |\Psi_k| \tag{10}$$

In other words: $|\Psi_k| = \tan(\frac{\phi_k}{2})$ with $\phi_k = \angle(S_k, S_{k+1})$. We will show, that the S_k solve the dIHM model (if $c = 0$). To do so we use $\mathcal{F}^{-1}$ as a gauge field:

$$L_k^{\mathcal{F}^{-1}}(\mu) := \mathcal{F}_{k+1}^{-1} L_k(\mu) \mathcal{F}_k = \mathcal{F}_k^{-1} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \mathcal{F}_k$$

If one writes $\mu = \sqrt{\frac{1+i\lambda}{1-i\lambda}} = \frac{1+i\lambda}{\sqrt{1+\lambda^2}}$ one gets $\mu^{-1} = \frac{1-i\lambda}{\sqrt{1+\lambda^2}}$ and one can conclude that

$$L_k^{\mathcal{F}^{-1}}(\lambda) = \mathcal{F}_k^{-1} \frac{\mathbb{I} + \mathfrak{i}\lambda}{\sqrt{1+\lambda^2}} \mathcal{F}_k = \frac{1}{\sqrt{1+\lambda^2}} (\mathbb{I} + \lambda S_k) \tag{11}$$

This clearly coincides with $U_k(\lambda)$ up to the irrelevant normalization factor $\frac{1}{\sqrt{1+\lambda^2}}$. On the other hand one gets for the gauge transform of $M_k(\mu)$

$$\begin{aligned} M_k^{\mathcal{F}^{-1}}(\mu) &:= \mathcal{F}_k^{-1} M_k(\mu) \mathcal{F}_k - \mathcal{F}_k^{-1} \dot{\mathcal{F}}_k \\ &= \mathcal{F}_k^{-1} \left(M_k(\mu) - M_k(1) - \mathcal{F}_k c \mathcal{F}_k^{-1} \right) \mathcal{F}_k \\ &= \mathcal{F}_k^{-1} L_{k-1}(1) \mathcal{F}_k \mathcal{F}_k^{-1} \\ &\quad \times \begin{pmatrix} i(\mu^2 - 1) & 0 \\ 0 & -i(\mu^{-2} - 1) \end{pmatrix} \mathcal{F}_k - c \end{aligned}$$

But with above substitution for μ one gets

$$\begin{pmatrix} i(\mu^2 - 1) & 0 \\ 0 & -i(\mu^{-2} - 1) \end{pmatrix} = -2 \frac{\lambda \mathbb{I} + \lambda^2 \mathfrak{i}}{1 + \lambda^2} \tag{12}$$

and since $\mathcal{F}_k^{-1} L_{k-1}(1) \mathcal{F}_k = \mathcal{F}_{k-1}^{-1} L_{k-1}(1) \mathcal{F}_{k-1}$ we get

$$\begin{aligned} &\mathcal{F}_k^{-1} L_{k-1}(1) \mathcal{F}_k \\ &\quad = \mathbb{I} + \mathcal{F}_{k-1}^{-1} \left(\operatorname{Im}(\Psi_{k-1}) \mathfrak{j} - \operatorname{Re}(\Psi_{k-1}) \mathfrak{k} \right) \mathcal{F}_{k-1} \\ &\quad = \mathbb{I} + \mathcal{F}_k^{-1} \left(\operatorname{Im}(\Psi_{k-1}) \mathfrak{j} - \operatorname{Re}(\Psi_{k-1}) \mathfrak{k} \right) \mathcal{F}_k \end{aligned}$$

Remember that $S_k = \mathcal{F}_k^{-1} \mathfrak{i} \mathcal{F}_k$ and $S_{k-1} = \mathcal{F}_{k-1}^{-1} \mathfrak{i} \mathcal{F}_{k-1}$. Using Eq. (10) and the fact that $\mathfrak{i}$ and $\operatorname{Im}(\Psi_{k-1})\mathfrak{j} - \operatorname{Re}(\Psi_{k-1})\mathfrak{k}$ anti-commute we conclude [1]

$$\mathcal{F}_k^{-1} L_{k-1}(1) \mathcal{F}_k = \mathbb{I} + \frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} \tag{13}$$

Combining this and Eq. (12) one obtains for the gauge transform of M_k

$$\begin{aligned} M_k^{\mathcal{F}^{-1}}(\lambda) &= -2 \left(\mathbb{I} + \frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} \right) \frac{\lambda \mathbb{I} + \lambda^2 S_k}{1 + \lambda^2} \\ &\quad - c = \frac{-2}{1+\lambda^2} \left(\lambda \mathbb{I} + \lambda \frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} \right. \\ &\quad \left. + \lambda^2 \left(S_k + \frac{(S_k \times S_{k-1}) S_k}{1 + \langle S_k, S_{k-1} \rangle} \right) \right) - c \\ &= \frac{-2\lambda}{1+\lambda^2} \mathbb{I} - \frac{2}{1+\lambda^2} \left(\lambda \frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} \right. \\ &\quad \left. + \lambda^2 \frac{S_k + S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} \right) - c \\ &= \frac{-2\lambda}{1+\lambda^2} \mathbb{I} + V_k(\lambda) - c \end{aligned} \tag{14}$$

Since the first term is a multiple of the identity and independent of k it cancels in the zero curvature condition and therefore can be dropped. This gives the desired result if $c = 0$. □

[1] to fix the sign of the second term one needs to look at the sign of the scalar product

$$\left\langle \mathcal{F}_k^{-1} \left(\operatorname{Im}(\Psi_{k-1}) \mathfrak{j} - \operatorname{Re}(\Psi_{k-1}) \mathfrak{k} \right) \mathcal{F}_k, \frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1} \rangle} \right\rangle.$$

2.1. Equivalence of the two discrete nonlinear Schrödinger equations

There has been another discretization of the nonlinear Schrödinger equation in the literature [11,7]. It can be derived from a slightly modified dIHM model by a gauge transformation. Since we showed that the $\mathrm{dNLSE_{AL}}$ introduced by Ablowitz and Ladik is gauge equivalent to the dIHM it is a corollary of the last theorem that the two discretizations of the NLSE are in fact equivalent.

The method of getting the variables of this other discretization is basically a stereographic projection of the variables S_k from the dIHM [7]: One defines

$$\chi_k = \chi(S_k) = \sqrt{2}\,(-1)^k \times \frac{2(S_k + \mathfrak{i}) - |S_k + \mathfrak{i}|^2\,\mathfrak{i}}{\sqrt{|S_k + \mathfrak{i}|^4 + |2(S_k + \mathfrak{i}) - |S_k + \mathfrak{i}|^2\,\mathfrak{i}|^2}} \tag{15}$$

or

$$S_k = \left(1 - |\chi_k|^2\right)\mathfrak{i} + \mathrm{Im}\left(\sqrt{2}\,(-1)^k \chi_k \sqrt{1 - \frac{|\chi_k|^2}{2}}\right) \times \mathfrak{j} - \mathrm{Re}\left(\sqrt{2}\,(-1)^k \chi_k \sqrt{1 - \frac{|\chi_k|^2}{2}}\right)\mathfrak{k} \tag{16}$$

If one modifies the evolution (5) by adding a rotation around $\mathfrak{i}$

$$\dot{S}_k = 2\frac{S_{k+1} \times S_k}{1 + \langle S_{k+1}, S_k\rangle} - 2\frac{S_k \times S_{k-1}}{1 + \langle S_k, S_{k-1}\rangle} - 4S_k \times \mathfrak{i} \tag{17}$$

writing this in terms of the new variables χ_k gives rise to the following evolution equation ($\mathrm{dNLSE_{IK}}$):

$$-i\dot{\chi}_k = 4\chi_k + \frac{P_{k,k+1}}{Q_{k,k+1}} + \frac{P_{k,k-1}}{Q_{k,k-1}} \tag{18}$$

where

$$P_{n,m} = -\Bigg(\chi_n + \chi_m \sqrt{1 - \frac{|\chi_n|^2}{2}}\sqrt{1 - \frac{|\chi_m|^2}{2}} - \chi_n|\chi_m|^2 - \frac{1}{4}\left(|\chi_n|^2\chi_m + \chi_n^2\bar{\chi}_m\right) \times \sqrt{1 - \frac{|\chi_m|^2}{2}} \Big/ \sqrt{1 - \frac{|\chi_n|^2}{2}}\Bigg)$$

and

$$Q_{n,m} = 1 - \frac{1}{2}\Bigg(|\chi_n|^2 + |\chi_m|^2 + (\chi_n\bar{\chi}_m + \bar{\chi}_n\chi_m) \times \sqrt{1 - \frac{|\chi_n|^2}{2}}\sqrt{1 - \frac{|\chi_m|^2}{2}} - |\chi_n|^2|\chi_m|^2\Bigg).$$

This evolution clearly possesses a zero curvature condition $\dot{U}_k = \hat{V}_{k+1}U_k - U_k\hat{V}_k$ with

$$\hat{V}_k(\lambda) = V_k(\lambda) - 2\mathfrak{i} \tag{19}$$

since one can view S_k as a function of χ_k via Eq. (16).

Theorem 2. *The* $\mathrm{dNLSE_{IK}}$ *(18) and the* $\mathrm{dNLSE_{AL}}$ *(1) are gauge equivalent.*

Proof. This is already covered by the proof of Theorem 1. □

Since the S_k are given by $S_k = \mathscr{F}_k^{-1}\,\mathfrak{i}\,\mathscr{F}_k$ the χ_k are functions of the Ψ_k and vice versa, but these maps are nonlocal.

3. A doubly discrete IHM model and the doubly discrete NLSE

In the following we will construct a discrete time evolution for the variables S_k that – applied twice – can be viewed as a doubly discrete IHM model. In fact it will turn out that this system is equivalent to the doubly discrete NLSE introduced by Ablowitz and Ladik [2]. We start by defining the zero curvature representation.

$$U_k(\lambda) = \mathbb{I} + \lambda S_k, \quad V_k(\lambda) = \mathbb{I} + \lambda(r\mathbb{I} + v_k) \tag{20}$$

with $r \in \mathbb{R}$. The v_k (as well as the S_k) are vectors in $\mathbb{R}^3$ (again written as complex 2 by 2 matrix). The zero curvature condition $\tilde{L}_k V_k = V_{k+1} L_k$ should hold for all λ giving $v_k + \tilde{S}_k = S_k + v_{k+1}$ and $r(\tilde{S}_k - S_k) = v_{k+1} S_k - \tilde{S}_k v_k$. (Here and in the forthcoming we use $\tilde{}$ to denote the time shift.) One can solve this for v_{k+1} or $\tilde{S}_k$ getting

$$v_{k+1} = (S_k - v_k - r)\, v_k (S_k - v_k - r)^{-1},$$
$$\tilde{S}_k = (S_k - v_k - r)\, S_k (S_k - v_k - r)^{-1} \tag{21}$$

This can be interpreted in the following way: Since $S_k, v_{k+1}, -\tilde{S}_k$, and $-v_k$ sum up to zero they can be viewed as a quadrilateral in $\mathbb{R}^3$. But Eq. (21) says that v_{k+1} and $\tilde{S}_k$ are rotations [2] of v_k and S_k around $S_k - v_k$. So the resulting quadrilateral is a parallelogram that is folded along one diagonal. See [8] to get a more elaborate investigation of the underlying geometry.

Eq. (21) is still a transformation [3] and no evolution since one has to fix an initial v_0. But in the case of periodic S_k one can find in general two fix points of the transport of v_0 once around the period and thus single out certain solutions. If on the other hand one has rapidly decreasing boundary conditions one can extract solutions by the condition that $\tilde{S}_k \to \pm S_k$ for $k \to \infty$ and $k \to -\infty$. But instead of going into this we will show, that doing this transformation twice is equivalent to Ablowitz' and Ladiks system.

Let us recall their results.

Theorem 3. (Ablowitz and Ladik 77) *Given the matrices*

$$L_k(\mu) = \begin{pmatrix} 1 & \Psi_k \\ -\overline{\Psi}_k & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

and $V_k(\mu)$ with the following μ-dependency:

$$V_k(\mu) = \mu^{-2} V_k^{(-2)} + V_k^{(0)} + \mu^2 V_k^{(2)}$$

with $V_k^{(-2)}$ being upper and $V_k^{(2)}$ being lower triangular. Then the zero curvature condition $V_{k+1}(\mu) L_k(\mu) = \tilde{L}_k(\mu) V_k(\mu)$ gives the following equations:

$$\begin{aligned}
&\left(\tilde{\Psi}_k - \Psi_k\right)/i \\
&\quad = \alpha_+ \Psi_{k+1} - \alpha_0 \Psi_k + \overline{\alpha}_0 \tilde{\Psi}_k - \overline{\alpha}_+ \tilde{\Psi}_{k-1} \\
&\qquad + \left(\alpha_+ \Psi_k \mathscr{A}_{k+1} - \overline{\alpha}_+ \tilde{\Psi}_k \overline{\mathscr{A}_k}\right) \\
&\qquad + \left(-\overline{\alpha}_- \tilde{\Psi}_{k+1} + \alpha_- \Psi_{k-1}\right) \\
&\qquad \times \left(1 + |\tilde{\Psi}_k|^2\right) \Lambda_k
\end{aligned}$$

$$\mathscr{A}_{k+1} - \mathscr{A}_k = \tilde{\Psi}_{\tilde{\bar{k}}k-1} - \Psi_{k+1}\overline{\Psi}_k$$

$$\Lambda_{k+1}\left(1 + |\Psi_k|^2\right) = \Lambda_k\left(1 + |\tilde{\Psi}_k|^2\right) \tag{22}$$

with constants α_+, α_0 and α_-.

In the case of periodic or rapidly decreasing boundary conditions the natural conditions $\mathscr{A}_k \to 0$, and $\Lambda_k \to 1$ for $k \to \pm\infty$ give formulas for $\mathscr{A}_k$ and Λ_k:

$$\mathscr{A}_k = \Psi_k \overline{\Psi}_{k-1} + \sum_{j=j_0}^{k-1} \left(\Psi_j \overline{\Psi}_{j-1} - \tilde{\Psi}_{\tilde{\bar{j}}j-1}\right)$$

$$\Lambda_k = \prod_{j=j_0}^{k-1} \frac{1 + |\tilde{\Psi}_j|^2}{1 + |\Psi_j|^2}$$

with $j_0 = 0$ in the periodic case and $j_0 = -\infty$ in case of rapidly decreasing boundary conditions.

Note that this is not the most general version of their result. One can make Ψ and $\overline{\Psi}$ independent variables which results in slightly more complicated equations but the given reduction to the NLSE case is sufficient for our purpose.

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[2] Any rotation of a vector v in $\mathbb{R}^3 = \mathrm{su}(2)$ can be written asconjugation with a matrix σ of the form $\sigma = \cos(\frac{\phi}{2})\mathbb{I} + \sin(\frac{\phi}{2})a$where ϕ is the rotation angle and a the rotation axis with $|a| = 1$.

[3] In fact it is the Bäcklund transformation for the dIHM model!

Theorem 4. *The system obtained by applying the above transformation twice is equivalent to the doubly discrete Ablowitz Ladik system in Theorem 3.*

Proof. The method is more or less the same as in the singly discrete case although this time we start from the other side:

Start with a solution S_k of the ddIHM model. Choose $\mathscr{F}_k$ such that

$$\mathscr{F}_k^{-1}\mathfrak{i}\mathscr{F}_k = S_k, \quad \left[\left(\mathscr{F}_{k+1}^{-1}\mathfrak{j}\mathscr{F}_{k+1}\right),\left(\mathscr{F}_k^{-1}\mathfrak{j}\mathscr{F}_k\right)\right] \times \| [S_{k+1}, S_k] \tag{23}$$

This is always possible since the first equation leaves a gauge freedom of rotating around $\mathfrak{i}$. Moreover define $L_k(1) = \mathscr{F}_{k+1}\mathscr{F}_k^{-1}$ and normalize $\mathscr{F}_k$ in such a way that $L_k(1)$ takes the form

$$L_k(1) = \mathbb{I} + A_k$$

Eq. (23) ensure that $A_k \in \operatorname{span}(\mathfrak{j},\mathfrak{k})$ and thus can be written $A_k = \operatorname{Re}(\Psi_k)\mathfrak{k} - \operatorname{Im}(\Psi_k)\mathfrak{j}$ for some complex Ψ_k. Equipped with this we can gauge a normalized version of $M_k(\lambda)$ with $\mathscr{F}_k$ and get

$$M_k^{\mathscr{F}} = \frac{1}{\sqrt{1+\lambda^2}}\mathscr{F}_{k+1}M_k(\lambda)\mathscr{F}_k^{-1} = L_k(1)\frac{\mathbb{I}+\lambda\mathfrak{i}}{\sqrt{1+\lambda^2}} = \begin{pmatrix} 1 & \Psi_k \\ -\overline{\Psi}_k & 1 \end{pmatrix}\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \tag{24}$$

if we write $\mu = \frac{1+i\lambda}{\sqrt{1+\lambda^2}}$ as before. On the other hand we get for an – again renormalized – $N_k(\lambda)$

$$\begin{aligned} N_k^{\mathscr{F}} &= \frac{1+\mu^2}{\mu}\tilde{\mathscr{F}}_k N_k(\lambda)\mathscr{F}_k^{-1} \\ &= \left(\frac{1}{\mu}+\mu\right)\tilde{\mathscr{F}}_k\mathscr{F}_k^{-1} \\ &\quad + \left(\frac{1}{\mu}-\mu\right)\tilde{\mathscr{F}}_k(r+v_k)\mathscr{F}_k^{-1} \\ &= \mu^{-1}V_k^- + \mu V_k^+ \end{aligned} \tag{25}$$

But the zero curvature condition $\tilde{L}_k(\mu)N_k^{\mathscr{F}}(\mu) = N_{k+1}^{\mathscr{F}_k}(\mu)L_k(\mu)$ yields that V_k^+ must be lower and V_k^- upper triangular. Thus $N_k^{\mathscr{F}}(\mu)N_k^{\mathscr{F}}(\mu)$ has the μ-dependency as required in Theorem 3. □

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