DYNAMICS OF THE LONG JOSEPHSON JUNCTIONS

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Static and dynamic properties of long sandwich-type Josephson junctions have been analyzed. These junctions, both rectangular ("uniform") and non-rectangular ("shaped"), can be described by the one-dimensional equation for the phase difference $\varphi(x,t)$, with the coefficients generally dependent on x. The variation of these coefficients reflects that of effective junction inductance, capacitance, critical current density and injected current density, along the junction length L. If $L \gg \lambda_J$, the equation for $\varphi(x,t)$ can be reduced to a simpler "hydrodynamic-type" equation for the Josephson vortex density. Coefficients of this reduced equation have been found analytically for the limit cases. The static version of the reduced equation has been used for calculating the threshold characteristics of the shaped Josephson junctions, including the amplitude of "sidelobes". The dynamic version of the equation has been used for the description of the I-V curves of the uniform junctions; viscous flux flow, Eck peak and "displaced linear branch" are particularly discussed.

I Introduction

Long Josephson junctions attract much attention, particularly because of their computer applications [1]. Properties of such junctions, both uniform[2] and "shaped"[3], can be described by the well-known second-order differential equation for the phase difference (φ) distribution along the junction length

$$\frac{\partial}{\partial x}\left[\mu^2(x)\frac{\partial\varphi}{\partial x}\right] - \nu^2(x)\frac{\partial^2\varphi}{\partial t^2} = \rho^2(x)j + j_e(x) \qquad (1)$$

Here μ^{-2}, ν^2 and ρ^2 are the junction inductance, capacitance and critical current per unit length, respectively, while j_e describes the possible laterally injected current[4,5]. All the parameters can change considerably along the junction length L.

Equation (1) should be solved numerically in most cases. The difficulties of such a solution, however, grow rapidly at $L/\lambda_J \gg 1$. Hence, there had been a need for a simpler ("reduced") equation which would give an adequate description of long Josephson junction in terms of slowly varying vortex density h(x) rather than in terms of rapidly changing phase $\varphi(x,t)$. A general approach of this kind had been developed earlier [7,9] (see also the review [8]). Concrete reduced equation had been derived, however, only for the case of the uniform junction with small damping[6]. Another attempt[10] to derive such a reduced equation for arbitrary damping had used the phenomenological approach which had led to an equation valid in rare cases only.

The objective of the next section of the present paper is to describe an accurate derivation of the reduced equation for the vortex density distribution along the Josephson junction. In Sec.III, the simplest applications of the equation and the physical meaning of its coefficients are discussed. Derivation of the generalized equation suitable for the transient process analysis is given in Sec.IV. In Sec.V, the reduced equation is used for calculating the threshold characteristics of the shaped Josephson junctions. Dynamic properties of long junctions with both lateral and edge current injection are discussed in Section VI.

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II Derivation of the Reduced Equation

We start from the one-dimensional equation (1) of the Josephson junction, where the normalized current density is assumed to be equal to

$$j = \sin\varphi + \frac{\partial\varphi}{\partial t}, \qquad (2)$$

in accordance with the ordinary Resistively Shunted Junction (RSJ) model. One would see, however, that our derivation can be readily extended to more complex current-phase relationships.

The basic idea of the method lies in the following. In the infinite and uniform Josephson junction, Eq.(1) has a solution

$$\varphi_o = \varphi_o(\theta,h,\omega), \qquad \theta = hx - \omega t, \qquad (3)$$

which represents an infinite and uniform array of the Josephson vortices with constant density h and velocity $u=\omega/h$. Now, if a long junction has slowly varying parameters μ, ν, ρ and j_e, the solution $\varphi(x,t)$ at each point is close to the solution φ_o (3) with slowly varying density h(x) and velocity $u(x)=\omega/h(x)$. Thus, φ can be expressed as

$$\varphi = \varphi_o + \varphi_1, \qquad |\varphi_1| \ll 1, \qquad (4)$$

where $h=\partial\theta/\partial x$ is some smooth function of x, and φ_1 describes the small array deformation due to (small) junction parameter gradients.

According to the above arguments, function φ_o should satisfy equation

$$k^2\frac{d^2\varphi_o}{d\theta^2} = \sin\varphi_o - \omega\frac{d\varphi_o}{d\theta} + j_o, \qquad (5)$$

where k is defined as

$$\rho^2k^2 = \mu^2h^2 - \nu^2\omega^2. \qquad (6)$$

Equation (5) has a required 2π-periodic solution $\varphi_o(\theta)$ only if j_o is some definite function of ω and k,

$$j_o = \frac{\omega}{2\pi}\int_0^{2\pi}\left(\frac{d\varphi_o}{d\theta}\right)^2 d\theta. \qquad (7)$$

Functions $\varphi_o(\theta,k,\omega)$ and $j_o(k,\omega)$ can be easily found numerically and can be treated as some known functions; Fig. 1 shows the plots of j_o (see [11, 12]).

Now, substituting Eq.(4) to the basic equation (1) and taking into account the zero-order and the first-order terms only, we obtain a linear equation for φ_1

$$\rho^2 k^2 \frac{d^2\varphi_1}{d\theta^2} + \omega\frac{d\varphi_1}{d\theta} - \varphi_1\cos\varphi_o = f(\theta,k,\omega), \qquad (8)$$

$$f = -\frac{d(\mu^2h)}{dx}\frac{\partial\varphi_o}{\partial\theta} - 2\mu^2h\frac{d(k^2)}{dx}\frac{\partial^2\varphi_o}{\partial\theta\partial(k^2)} + j_e - \rho^2 j_o.$$

According to the basic idea of the method, φ_1 should be a 2π-periodic function of θ. This is possible only if coefficients in the function $f(\theta,k,\omega)$ satisfy the additional requirement

$$A\frac{d}{dx}(\mu^2h) + 2B\mu^2h\frac{d}{dx}(k^2) = j_e - \rho^2 j_o, \qquad (9)$$

where A and B are functions of k and ω

$$A = I_1/I_o, \qquad B = I_2/I_o,$$

$$I_i = \int_{-\pi}^{+\pi} d\theta \frac{\partial\varphi_o}{\partial\theta} F_i(\theta)\int_0^{2\pi} d\theta'\left[\frac{\partial\varphi_o(\theta-\theta')}{\partial\theta}\right]^{-2}\exp\left(\frac{\omega\theta'}{k^2}\right), \qquad (10)$$

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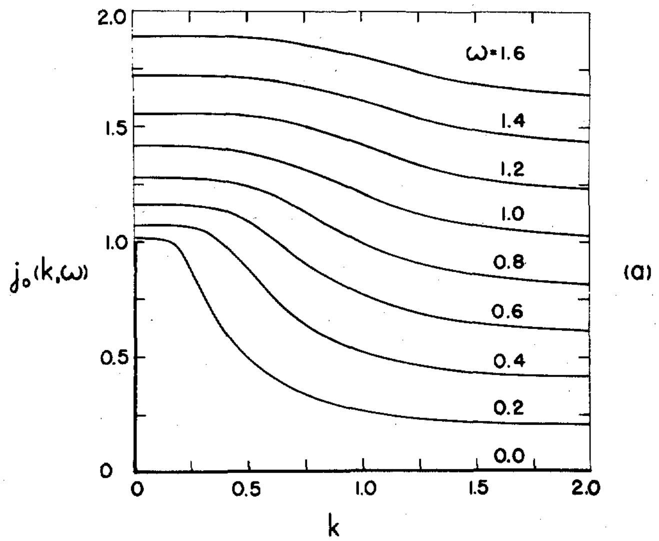

Fig. 1. Coefficient j_o as the function of the effective vortex density at various Josephson frequencies ω. Some important asymptotes:

$$j_o = 0, \qquad \text{at } \omega = 0 \text{ and } k \neq 0;$$

$$j_o = (\omega^2 + 1)^{1/2}, \qquad \text{at } \omega \neq 0 \text{ and } k \to 0;$$

$$j_o = \omega, \qquad \text{at } \omega \gg 1 \text{ or } k \gg 1.$$

with functions F_i being equal to

$$F_o = 1, \quad F_1 = \frac{\partial \varphi_o}{\partial \theta}, \quad F_2 = \frac{\partial^2 \varphi_o}{\partial \theta \partial (k^2)}. \qquad (11)$$

For what follows, the function

$$C = A + 4k^2 B \qquad (12)$$

will be more convenient sometimes.

Equation (9) considered together with the definition (6) of parameter k, is just the reduced equation needed. It describes the variation of the vortex density h along the Josephson junction at the fixed Josephson oscillation frequency ω.

III Physical Sense of the Equation Coefficients

According to Eq.(5), j_o is a normalized density of the laterally injected current j_e which induces the motion of vortex array of density k with velocity $u = \omega/k$ in a uniform infinite junction with high damping (small capacitance). To clarify the sense of the other two coefficients, A and C, let us apply Eq.(9) to two simplest problems concerning a static vortex array, $\omega = 0$. In this case, $\rho k = \mu h$, $j_o = 0$ and the reduced equations takes a simpler form

$$Ak \frac{d(\mu\rho)}{dx} + C\mu\rho \frac{dk}{dx} = j_e. \qquad (13)$$

1. Static Vortex Array in a Uniform Junction

Consider a uniform (μ=const, ρ=const) segment of a long junction, containing a motionless vortex array. From Eq.(13) we get

$$j_e = \text{const} \times C \frac{dh}{dx}. \qquad (14)$$

Thus, factor C shows what lateral current j_e, i. e., what Lorenz force should be applied to the array to counterbalance the vortex density gradient. In other words, C plays the role of the vortex array elasticity modulus. Some important asymptotes for C are

$$C = 64\pi k^{-3} \exp(-2\pi/k), \qquad \text{at } k \to 0;$$
$$C = 1, \qquad \text{at } k \to \infty; \qquad (15)$$

at arbitrary k, C can be expressed through the complete elliptic integrals (ω = 0)

$$C = \frac{4(1-m^2)\, K(m)}{k^2 m^2\, E(m)}, \qquad k = \frac{\pi}{m\, K(m)}. \qquad (16)$$

2. Static Vortex Array in a Non-Uniform Junction

As it follows from Eqs.(1), (5) and (6), variable k has the physical sense of the effective vortex density

$$k = \frac{\lambda_J(x)}{\Phi_o} \frac{d\Phi}{dx}, \qquad \Phi_o = \frac{h}{2e}, \qquad (17)$$

where $d\Phi$ is the magnetic flux enclosed in a junction segment $dx \gg \lambda_J$, and $\lambda_J(x)$ is the local value of the Josephson penetration depth.

According to Eq.(13), the density is constant if

$$j_e = kA \frac{d(\mu\rho)}{dx}. \qquad (18)$$

Hence, factor kA shows what current j_e should be applied to counterbalance the junction parameter gradient.

At ω = 0, the following formulas take place

$$A = \frac{4}{\pi^2} K(m)\, E(m), \qquad \text{at any } k;$$
$$A = 4/\pi k, \qquad \text{at } k \to 0; \qquad (19)$$
$$A = 1, \qquad \text{at } k \to \infty.$$

Note also, that at ω = 0

$$C = d(kA)/dk, \qquad (20)$$

and that at any k and ω

$$A = \partial j_o / \partial \omega. \qquad (21)$$

IV Generalization for the Transient Processes

The method can be readily generalized for the transient processes, where $h=h(x,t)$ and $\omega=\omega(x,t)$ are slowly varying functions of both space and time. Taking into account that

$$\frac{\partial h}{\partial t} = \frac{\partial^2 \theta}{\partial x \partial t} = -\frac{\partial \omega}{\partial x}, \qquad (22)$$

and accepting the additional conditions

$$\int_{-\pi}^{+\pi} \frac{\partial \varphi_o}{\partial \omega} d\theta = \int_{-\pi}^{+\pi} \frac{\partial \varphi_o}{\partial (k^2)} d\theta = 0, \qquad (23)$$

one obtains the equation

$$A \frac{\partial}{\partial x}(\mu^2 h) + 2(\mu^2 h) B \frac{\partial}{\partial x}(k^2) + (\nu^2 A - \rho^2 G) \frac{\partial \omega}{\partial t} + (2\omega\nu^2 B - \rho^2 D) \frac{\partial}{\partial t}(k^2) = j_e - \rho^2 j_o, \qquad (24)$$

where coefficients D, G are defined as

$$D = I_3 / I_o, \qquad G = I_4 / I_o, \qquad (25)$$

while I_i are expressed by Eq.(10) with

$$F_3 = \frac{\partial^2 \varphi_o}{\partial \theta \partial \omega} + \frac{\partial \varphi_o}{\partial (k^2)}, \qquad F_4 = \frac{\partial \varphi_o}{\partial \omega}. \qquad (26)$$

Equation (24) can be used for analysis of rather complex nonstationary processes in the long Josephson junctions. In this paper we, however, will concentrate on the simplest applications of the reduced equation, where $\partial k/\partial t = \partial \omega/\partial t = 0$, and Eq.(9) is valid.

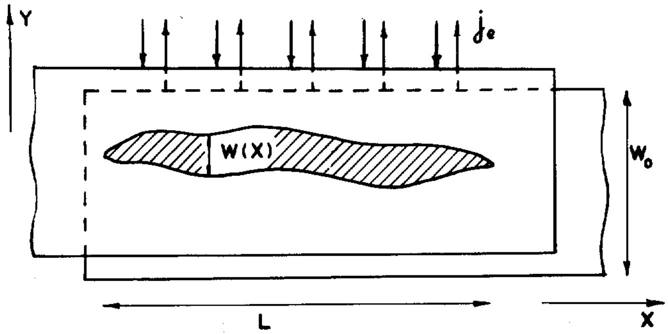

Fig. 2. Schematic view of a long Josephson junction (shaded area) between two thin-film superconducting electrodes. Except the currents flowing along the electrodes (along the x axis), lateral current j_e can be injected into the junction.

V Critical Current of Shaped Josephson Junctions

The "shaped" Josephson junctions have been successfully used in logic gates for the cryogenic computer circuits [1,3]. In comparison with the rectangular-junction gates, their threshold characteristics $(I_G)_{max} = f(I_C)$ can exhibit very small secondary maximums ("side-lobes"), which is favorable for the applications.

Figure 2 shows general view of a shaped junction. If the junction width is small enough [13], one can integrate all variables over the y axis and come to the one-dimensional equation (1) with parameters

$$\mu^2 = \ell + (1-\ell)\rho^2(x) \leqslant 1, \quad \nu^2 = \rho^2(x)\beta, \qquad (27)$$
$$\rho^2 = W(x)/W_o, \quad \ell = (2\lambda+d)/(2\lambda+d_o) \leqslant 1.$$

Physically, ℓ is the ratio of inductance per square of the junction to that of the surrounding overlapping area; d and d_o are the respective electrode spacings, β is the normalized junction capacitance. We measure length in units

$$\lambda_J^o = [\phi_o/2\pi\mu_o j_c(2\lambda+d)]^{1/2}, \qquad (28)$$

the value of λ_J, which a junction of uniform width $W = W_o$ would have.

Assuming the junction to be long, and its parameters (27) to change smoothly along the length L, we can use static version (13) of the reduced equation to find the density h(x) of the static vortex array inside the junction. Taking Eq.(20) into account, we obtain the relation

$$kA\rho\mu = i = \text{const} \qquad (29)$$

for the regions with $j_e = 0$, $h \neq 0$. To find i for the junction with "pointed" edges (Fig. 3a,b)

$$W(x) \to \alpha_0 x, \text{ at } x \to 0,$$
$$W(x) \to \alpha_L(L-x), \text{ at } x \to L, \qquad (30)$$

we should write down the boundary condition for Eq.(1) at x=0

$$\mu^2(0)\,\frac{\partial\varphi}{\partial x}(0) = 2\,\frac{I(0)}{I_1}, \quad I_1 = 2\lambda_J^o W_o j_c, \qquad (31)$$

and the similar condition at x=L. Here I(0) is the current injected into the junction left edge, and I_1 is the critical current of a rectangular junction of width W_o. At a pointed edge (30), $h \to$ const, $\mu \to$ const, $\rho \to 0$ and thus $k \to \infty$. According to Eq.(5), $\varphi_o \to \theta$, and $\partial\varphi/\partial x \to h$, so that using Eq.(19) and Eq.(31) we get

$$i = 2\,\frac{I(0)}{I_1}. \qquad (32)$$

Equations (29) and (32) show that the current I(0) induces vortices at the junction left edge, which penetrate into the junction, the vortex array density decreasing as the junction width W(x) increases. The density becomes zero at some point inside the junction, where h=0, $k \to 0$, $kA \to 4/\pi$, and thus

$$(4/\pi)\rho\mu = 2\,I(0)/I_1. \qquad (33)$$

With growing current I(0), this point shifts deeper inside the junction. When it reaches the widest part of the junction, the static solution becomes impossible, because vortices start to leave the junction through the opposite junction edge (at I(L)=0). For this critical value of I(0), Eq.(33) yields

$$I_o = \frac{2}{\pi}\,I_1\left(\frac{W_{max}}{W_o}\right)^{1/2}\left[\ell + (1-\ell)\frac{W_{max}}{W_o}\right]^{1/2} = \frac{2}{\pi}\,j_c W_{max}\,\lambda_J(x)\Big|_{W_{max}}. \qquad (34)$$

In the Josephson logic gates, I(0) and I(L) are some linear functions of the transport (gate) current I_G and the control current I_C. For example, in the in-line gate (Fig. 3a) placed over the superconducting ground plane,

$$I(0) = I_G + I_C, \quad I(L) = I_C. \qquad (35)$$

By the usual argumentation [2,3], we come to the well-known "saw-tooth" threshold characteristic of the gate (Fig. 3b), but with the new current scale (34) and without any side-lobes: $(I_G)_{max} = 0$ at $|I_C| > I_o$. This result can be easily explained: pointed edges of the junction remove the energy barriers from the edges and make the vortex entrance and exit possible as soon as they are energy-advantageous. This is why the critical current is equal to just the "thermodynamic value" (34) (see, for example, the monograph [2]). The side-lobes have vanished in the zeroth approximation of the method, because they have been the result of the vortex pinning at the edge energy barriers.

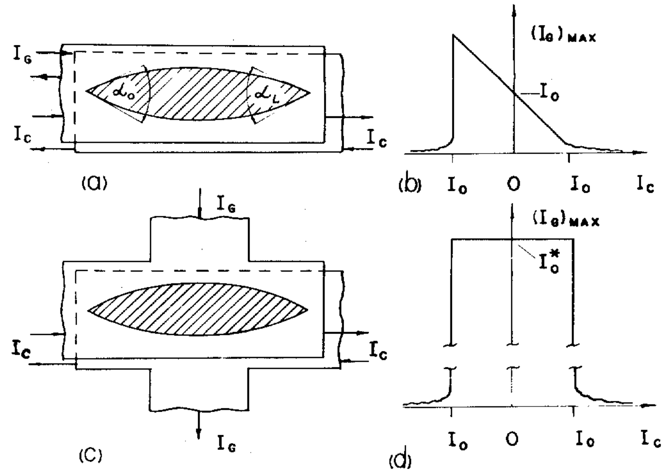

Fig. 3. Logic gates with shaped point-edged Josephson junctions, using edge (a) and lateral (c) gate current injection, and their threshold characteristics (b,d).

To find the (small) amplitude of the side-lobes, one should make the first approximation by writing down the general expression for the gate current

$$I_G = \frac{I_1}{2}\int_0^L \rho^2(x)\,\sin\varphi(x)\,dx, \qquad (36)$$

taking it by parts twice, and neglecting the higher terms with respect to $\lambda_J/L \ll 1$:

$$I_G = \left(\frac{\pi}{4}\right)^3\,\frac{\ell^2 I_o^3}{I_C^2}\left[\sin\varphi(x)\,\frac{d(\rho^2)}{dx}\right]_0^L, \quad |I_C| \geqslant I_o. \qquad (37)$$

At a large junction length, $L \gg \lambda_J$, small variations of I_C result in large changes of $\varphi(0)$, $\varphi(L)$, and thus the critical current rapidly oscillates remaining below the smooth envelope curve [14]

$$(I_G)_{max} = \left(\frac{\pi}{4}\right)^3 \frac{\ell^2 I_o^3}{I_C^2} \frac{\lambda_J^o}{W_o} (\alpha_o + \alpha_L). \quad (38)$$

For experimental "sine-shaped" junctions [3] with $j_c = 1.6$ kA/cm^2, $L \simeq 6\lambda_J^o \simeq 60$ μm, $W_o = 35$ μm, $W_{max} \simeq W_o$, Eq. (34) yields $I_o \simeq 7.3$ mA, the value equal to the experimental one, $I_o \simeq 7.5$ mA, within the limits of uncertanity of L/λ_J^o. For the amplitude of the first side-lobe (at $I_C = 1.2\ I_o$), Eq.(38) yields the experimental result $(I_G)_{max} = 0.7$ mA for known values $\alpha_o \simeq \alpha_L \simeq 1.2$ and a reasonable value $\lambda = 150$ nm.

Thus, the simple analytical theory described above is in quantitative agreement with the experimental data available, even for very moderate L/λ_J ratios.

One can readily apply the same approach to gates with the lateral current injection [4,5], supplied with the pointed edges to suppress side-lobes (Fig. 3c). At $L/\lambda_J \to \infty$, one obtaines the quasi-rectangular threshold curve (Fig. 3d) with the same I_o (34) but the different zero-field critical current [5]

$$(I_G)_{max}\Big|_{I_C=0} = I_o^* = j_c W_{max} L', \quad (39)$$

where L' is the length of the current injection segment. This length can be much larger λ_J^o, in which case $I_o^* \gg I_o$, i.e., the logic gain is large.

VI Dynamic Properties of Long Junctions

Junction properties at finite dc voltage ($\omega \neq 0$) are even more sensitive to the method of the current injection. We shall consider two extreme cases for a long uniform junction ($\mu^2=1$, $\rho^2=1$, $\nu^2=\beta$=const).

1. Lateral Current Injection

Let the long junction be fed with the uniform current j_e (Fig. 2). In this case, the reduced equation (9) has the uniform solution: h=const, $k = |h^2 - \beta\omega^2|^{1/2}$, with h and ω satisfying the condition

$$j_o(|h^2 - \beta\omega^2|^{1/2}, \omega) = j_e. \quad (40)$$

Hence, at a fixed magnetic field the function j_o gives the I-V curve of a long Josephson junction. Figure 4 shows the I-V curves for the low-capacitance (dashed line) and finite-capacitance (solid line) junctions [11]. At small currents, dc voltage ω and hence the vortex velocity $u = \omega/h$ are proportional to j_e (see also [12]). This behavior corresponds to the viscous motion of vortices ("flux flow") with the viscosity factor

$$\eta \equiv \partial j_e/\partial u\Big|_{\omega \to 0} = h\ A(h,0), \quad (41)$$

independent of the junction capacitance.

At larger currents, voltage increases smoothly in the highly damped (low-capacitance) junctions, while in the finite-capacitance junctions the I-V curve shows the "Eck peak" [1]. At the top of the peak, vortex array velocity u is equal to the wave propagation velocity $\bar{c}$ (= $\beta^{-1/2}$ in our units),

$$h_{peak} = \beta^{1/2}\omega. \quad (42)$$

2. Edge Current Injection

In this case, h changes along the junction length and all terms of Eq.(9) should be taken into account. If voltage and/or magnetic field are large, we get

$$dh/dx = -\omega, \quad \text{at } \omega^2 + k^4 \gtrsim 1, \quad (43)$$

merely a linear change of magnetic field h and electrode currents $I = (I_J/2)h$ with x, due to the junction normal conductivity. If dc voltage is lower than unity, a considerable change of the linear law (43) takes place in vicinity of the resonance (42). Numerical solution of

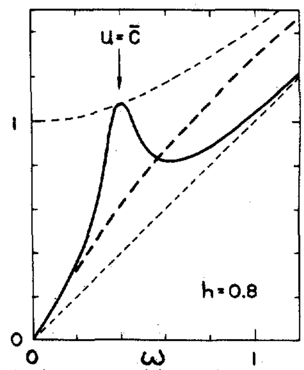

Fig. 4. I-V curves of the long junction with uniform lateral current injection. Dashed line: small capacitance, β=0. Solid line: finite capacitance, β=4. Thin dashed lines: asymptotes, $j_e = (\omega^2+1)^{1/2}$ and $j_e = \omega$.

Eq.(9) shows that this resonant region is closely related with the "displaced linear branch"[16,17] of I-V curve. The detailed discussion of the problem shall be the subject of a separate publication.

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13. Strictly speaking, the condition $W \ll \lambda_J$ should be fulfilled. Numerical calculations [3] have shown, however, that the one-dimensional equation (1) is adequate for description of junctions as wide as $\sim 3\lambda_J$.
14. In fact, Eq.(37) and (38) are valid only if the function $d(\rho^2)/dx$ is continuous inside the junction. If the function has some "jumps" (i.e., the junction has some angles inside, like those of the "diamond-type" junction [3]), another term appears in these equations, corresponding to the vortex array pinning at the angles.
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