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The Mass Spectrum and the S Matrix of the Massive Thirring Model in the Repulsive Case

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Abstract. The repulsive case of the quantum version of the massive Thirring model is considered. It is shown that there is a rich particle spectrum in the theory. The S matrix of fermions proves to be a discontinuous function of the coupling constant. These effects are the result of the qualitative change of the physical vacuum in the limit of the strong repulsion $g \to -\pi$.

1. Introduction

The massive Thirring model abounds in interesting results. In [1] the classical equation was shown to be integrable. In quantum version there is no multiple production, N particle S matrix is the product of two particle S matrices due to the infinite number of the integrals of motion [2]. It was shown in [3] that in the quantum case the massive Thirring model is equivalent to the sine-Gordon model, the same problem was considered in [4]. Classical integrability of the sine-Gordon model was established in [5]. The quasiclassical mass spectrum and the S matrix of the sine-Gordon model were calculated in [6, 7]. Quantum version of the massive Thirring model is investigated quite well in the attractive case $g>0$. The mass spectrum and the scattering matrix were calculated in [8–10], respectively. A direct way to exact quantum results for the sine-Gordon model is provided by the quantum inverse scattering method [11, 12]. In particular in [11] the value of the generating function for the integrals of motion on the physical particle state was calculated.

In present paper it is shown that in the repulsive case of the massive Thirring model $g<0$ the mass spectrum and the S matrix are of essentially new form. We study only the zero charge sector of the theory. The main idea is to use the Bethe ansatz, i.e. all eigenfunctions of the quantum Hamiltonian of the massive Thirring model [13] and the method of constructing of physical states developed in [14]. The technical aspect of this paper is a direct matrix generalization of [15, 16].

The plan of the paper is the following: In Sect. 2 the main results are described. In Sect. 3 we describe all the eigenfunctions of the quantum Hamiltonian. In Sect.

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4 the physical vacuum is constructed as a Dirac sea. In Sect. 5 the mass spectrum of the excitations are calculated. In Sect. 6 the S matrix of the physical particles is calculated. Section 7 is conclusion.

2. Main Results

First of all let us remind the reader the known results in the attractive case. The Lagrangian of the massive Thirring model is

$$\mathcal{L}=\int dx\{i\tilde{\psi}\gamma^\mu\partial_\mu\psi-m_0\tilde{\psi}\psi-\tfrac{1}{2}g(\tilde{\psi}\gamma^\mu\psi)^2\}$$
$$\gamma^0=\sigma_1,\gamma^1=i\sigma_2,\tilde{\psi}=\psi^*\gamma^0\,. \tag{1}$$

Here σ_j are Pauli matrices, $\psi(x)$ is a two component quantum, fermion field with the anticommutation relations:

$$\{\psi^*_\alpha(x,t),\psi_\beta(y,t)\}=\delta^\alpha_\beta\cdot\delta(x-y)\,;\alpha,\beta=1,2\,. \tag{2}$$

We consider this model in two dimensions, one space and one time. This model is a relativistic one. The massive Thirring model is equivalent to the sine-Gordon model:

$$\mathcal{L}=\int dx\{\tfrac{1}{2}(\partial_\mu\varphi)^2-(1/\gamma)M_0^2(1-\cos(\gamma^{1/2}\varphi))\}\,. \tag{3}$$

Here $\varphi(x)$ is a quantum, one component, boson field. The form of the connection between the coupling constants depends on the renormalization scheme [3, 9]. The quantization by means of the Bethe ansatz leads to the coincidence of mass spectra in both models if (see [16] for example):

$$g=\pi-\gamma/4\,. \tag{4}$$

It is also convenient to use values ω and μ:

$$\omega=\gamma/8=(\pi-g)/2,\mu=\pi-\gamma/8=(\pi+g)/2\,. \tag{5}$$

Notice that the fermion in (1) is equivalent to the soliton in (3).

To describe the mass spectrum in the attractive case it is convenient to divide the whole interval $0<g<\pi(0<\omega<\pi/2)$ into an infinite number of segments

$$\pi/(q+2)<\omega<\pi/(q+1),q=1,2,\ldots\,. \tag{6}$$

The mass spectrum on these segments consists of fermion and antifermion of mass M_f and q neutral particles with masses

$$M_n=2M_f\sin(n\gamma'/16),n=1,\ldots,q$$
$$\gamma'=8\pi\omega/(\pi-\omega)\,. \tag{7}$$

Neutral particles are bound states of fermion and antifermion. The fermion-antifermion S matrix is equal to Zamolodchikov's one:

$$S_{f\tilde{f}}(\theta|\omega)=S_Z(\theta|\omega),\theta=\theta_f-\theta_{\tilde{f}}\,. \tag{8}$$

Here θ is the rapidity of the particle (fermion or antifermion). The velocity v is equal to $v=\tanh\theta$. Zamolodchikov's S matrix has its simplest form in the fixed parity basis [17]:

$$S_Z^{\pm}(\theta|\omega)=U_{\pm}(\theta|\omega)\,S(\theta|\omega)\,,$$
$$U_{+}(\theta|\omega)=\frac{\sinh 4\pi(i\pi+\theta)/\gamma'}{\sinh 4\pi(i\pi-\theta)/\gamma'},\quad U(\theta|\omega)=-\frac{\cosh 4\pi(\theta+i\pi)\gamma'}{\cosh 4\pi(\theta-i\pi)/\gamma'},\tag{9}$$
$$S(\theta|\omega)=\exp\left\{-\int_0^{\infty}\frac{dx}{x}\;\frac{\sinh((4\pi x/\gamma')-x/2)\sinh(8i\theta x/\gamma')}{\sinh(x/2)\cosh(4\pi x/\gamma')}\right\}.$$

This mass spectrum and S matrix may be simply calculated by means of Bethe ansatz see [15, 16], respectively. It might seem plausible that formulae (7) and (8) are correct in the repulsive case also. However this is true only in the interval $-\pi/3<g<0$. In the interval $-\pi<g<-\pi/3$ the answers are essentially changed.

We show in this paper that the mass spectrum and the S matrix in the repulsive case $-\pi<g<0(\pi/2<\omega<\pi,\ 0<\mu<\pi/2)$ is constructed as follows. To describe the mass spectrum it is convenient to divide this interval into an infinite number of segments

$$\pi/(q+2)<\mu<\pi/(q+1),\ q=1,2,\ldots.\tag{10}$$

The mass spectrum on this segment consists of fermion and antifermion with mass M_f

$$M_f=M\left\{\frac{\sin(q-1)\mu}{\sin\mu}+\frac{\sin q\mu}{\sin\mu}\tan\frac{\pi}{2}\left(\frac{\pi}{\mu}-q-1\right)\right\}\tag{11}$$

and $(q-1)$ neutral particles with masses:

$$M_n=2M\sin(n\mu)/\tan\mu,\ n=1,\ldots,q-1\,.\tag{12}$$

The scattering matrix is constructed as follows. The S matrix for two neutral particles is:

$$S_n^{\ell}(\theta)=\delta_{n-1}^{\ell}\left(\frac{i\exp(\theta)+1}{\exp(\theta)+i}\right),\ \theta=\theta_n-\theta_{\ell},\ \ell=1,\ldots,q-2\,.\tag{13}$$

The neutral particle-fermion (antifermion) S matrix is:

$$S_f^{\ell}(\theta)=\delta_{q-1}^{\ell}\left(\frac{i\exp(\theta)+1}{\exp(\theta)+i}\right),\ \theta=\theta_f-\theta_{\ell},\ \ell=1,\ldots,q-1\,.\tag{14}$$

The fermion-antifermion S matrix is equal to Zamolodchikov's S matrix (9) with a modified (renormalized) coupling constant $\omega\to\omega_q$:

$$S_{f\bar{f}}(\theta|\omega)=S_Z(\theta|\omega_q)\,.\tag{15}$$

The renormalized coupling constant ω_q is q-dependent (10):

$$\omega_q=\pi\frac{\pi-\mu q}{\pi-\mu(q-1)}.\tag{16}$$

We shall note that the S matrix (13)–(15) has no poles on the physical sheet. So the particles (12) are not bound states. We shall also note, that while μ change through segment (10), ω_q (16) change through segment

$$\pi/2<\omega_q<2\pi/3\,.\tag{17}$$

3. Bethe Ansatz

The eigenfunctions of the quantum Hamiltonian of the massive Thirring model are [13]:

$$\Psi = \int d^\ell x \chi^{\alpha_1 \ldots \alpha_\ell}(x_1, \ldots, x_\ell | \beta_1, \ldots, \beta_\ell)\, \psi^*_{\alpha_1}(x_1) \cdot \ldots \cdot \psi^*_{\alpha_\ell}(x_\ell)|0\rangle ,$$

$$\chi^{\alpha_1 \ldots \alpha_\ell}(x_1, \ldots, x_\ell | \beta_1, \ldots, \beta_\ell) = \prod_{k=1}^{\ell} \chi^{\alpha_k}(x_k|\beta_k) \prod_{j>n} \exp\left\{\frac{i}{2}\varepsilon(x_j - x_n)\, \Phi(\beta_j - \beta_n)\right\},$$

$$\chi^\alpha(x|\beta) = \begin{pmatrix} \exp\{-\beta/2\} \\ \exp\{\beta/2\} \end{pmatrix} \exp\{im_0 x \sinh\beta\}, \psi_\alpha(x)|0\rangle = 0 . \tag{18}$$

Here $\varepsilon(x)$ is the sign-function, β – is the pseudoparticle rapidity, $\exp\{i\Phi(\beta_j - \beta_k)\}$ is the two pseudoparticle S matrix:

$$\exp\{i\Phi(\beta)\} = -\exp\{-2i\mu\} \frac{\exp(\beta) - \exp\{2i\mu\}}{\exp(\beta) - \exp\{-2i\mu\}} . \tag{19}$$

Note that the coupling constant dependence of the Bethe ansatz in [15] is different from (18), (60). The reason is the problem of definition of the product of two generalized functions $\varepsilon(x)$ and $\delta(x)$. This problem was solved in a correct way in [13]. In spite of this difference the final answers are essentially the same in both cases [15, 16]. The scattering matrix of N-pseudoparticles is equal to the product of two-pseudoparticle S matrices. The energy and momenta of the wave function (18) are

$$E = m_0 \sum_{k=1}^{\ell} \cosh\beta_k, P = m_0 \sum_{k=1}^{\ell} \sinh\beta_k . \tag{20}$$

If $\operatorname{Im}\beta = 0$ the energy of a pseudoparticle is positive, if $\operatorname{Im}\beta = \pi$ the energy is negative.

A bound state may be formed by n-pseudoparticles if the values of

$$\sin(\mu p)\sin(\mu(n-p)) \quad \text{at} \quad p = 1, 2, \ldots, n-1 \tag{21}$$

are all of the same sign [16]. To analyse the bound states let us divide the repulsive interval $0 < \mu < \pi/2$ into segments

$$\pi/(q+2) < \mu < \pi/(q+1), q = 1, 2, \ldots . \tag{22}$$

It is evident that in the segment $\pi/(q+2) < \mu < \pi/(q+1)$ we have $(q+1)$ bound states of n pseudoparticles for

$$n = 1, 2, \ldots, q+1 . \tag{23}$$

There also exist other bound states with some n in the region

$$n \geqq q+2 . \tag{24}$$

The n pseudoparticles which form a bound state (23) have complex rapidities:

$$\beta^n_\ell = B + i\pi + i\mu(n - 1 - 2\ell) \bmod (2\pi i)$$
$$\ell = 0, 1, \ldots, n-1, \operatorname{Im} B = 0 . \tag{25}$$

The energy of each bound state (23) is negative

$$E_n = m_n \cosh B, P_n = m_n \sinh B, m_n = -m_0 \frac{\sin(\mu n)}{\sin \mu}. \tag{26}$$

The scattering matrix of n-th bound state (23) with rapidity $\beta_n^1 (B = \beta_n^1)$ on ℓ-th bound state (23) with rapidity $\beta_\ell^2 (B = \beta_\ell^2)$ is equal to the product of the constituent pseudoparticle S matrices (19), (25):

$$s_\ell^n(\beta_n^1 - \beta_\ell^2) = \exp\{i\Phi_\ell^n(\beta_n^1 - \beta_\ell^2)\}, \tag{27}$$

$$\Phi_\ell^n(\beta) = \sum_{j=0}^{n-1} \sum_{p=0}^{\ell-1} \Phi(\beta + i\mu(2j - 2p + \ell - n)). \tag{28}$$

4. Construction of the Physical Vacuum

The physical vacuum is a state of minimal energy. It is shown in [8, 15] that in the attractive case the vacuum is the Fock space vector with all pseudoparticle states of negative energy filled in. In the repulsive case this is true only up to the point $g = -\pi/3$ (on the interval $-\pi/3 < g < 0$). In the region $-\pi < g < -\pi/3$ the situation is essentially different. The old vacuum is unstable. If we insert a bound state of two pseudoparticles in such "vacuum", the energy of this state will be lower than the energy of the vacuum. (One can also show that the perturbation series is divergent in the point $g = -\pi/3$.) We'll show that each segment (22) requires the construction of its own vacuum. On the interval (22) the vacuum is constructed not only from the elementary pseudoparticles, but also from the bound states of pseudoparticles. A bound state of n-pseudoparticles is present in vacuum with $n = 1, 2, \ldots, q$ see (23). So the vacuum is a q-component condensate. Moreover, all permitted states of these composite particles must be filled in. Let us calculate β_j^n the real parts of the permitted values of the rapidities of the n-th bound state in the vacuum. To regularize the calculations let us put the system in the periodical box of length L and make a cut in the rapidity $|\beta_j^n| < \Lambda$. The periodicity conditions for the vacuum wave function are [see (18), (26)]:

$$m_0 L \frac{\sin(\mu n)}{\sin \mu} \sinh \beta_j^n = 2\pi j_n + \sum_{\ell=1}^{q} \sum_{\substack{k \\ \beta_k^n \neq \beta_j^n}} \Phi_n^l(\beta_j^n - \beta_k^l) \quad n = 1, 2, \ldots, q,$$

$$j_n = -N_n, -N_n + 1, \ldots, N_n - 1, N_n. \tag{29}$$

Here $\Phi_n^l(\beta)$ is the scattering phase of n-th bound state on ℓ-th bound state (28), N_n is the maximal admissible integer $\beta_{N_n}^n < \Lambda$. In the limit $L \to \infty$ we transform this equation into an integral one like in [14]:

$$m_0 \frac{\sin \mu n}{\sin \mu} \cosh \beta = 2\pi \varrho_n(\beta) + \sum_{\ell=1}^{q} \int_{-\Lambda}^{\Lambda} \Phi_l'^n(\beta - \alpha) \varrho_l(\alpha)\, d\alpha$$

$$\varrho_n(\beta_j^n) = 1/(L(\beta_{j+1}^n - \beta_j^n)). \tag{30}$$

We want to solve this equation in the limit $\Lambda \to \infty$. But we must remember that m_0 depends on Λ so as to make finite the energy of the observable particles. We can calculate dependence m_0 on Λ in independent way. In Sect. 5 it is shown that to calculate the excitation energy we do not need an explicite knowledge of the functions $\varrho_n(\beta)$ (30). Analizing the dependence of the physical particle energy on Λ (38)–(41) one can see that the Λ-dependence of m_0 is as follows $m_0 \sim \exp\{(\pi-2\mu)\Lambda/2\mu\}$. Now we solve (30) by means of Fourier transform, as in [11]:

$$2\pi\delta_n^l + \Phi_n'^l(k) = \frac{4\pi\cosh(k\mu)}{\sinh(k\pi)\sinh(k\mu)}\begin{cases}\sinh(k\mu\ell)\sinh(k\pi-k\mu n)\, n\geqq \ell\\ \sinh(k\mu n)\sinh(k\pi-k\mu\ell)\, \ell\geqq n\end{cases}$$

$$\Phi_n'^l(k) = \int_{-\infty}^{\infty} e^{i\beta k}\Phi_n'^l(\beta)\,d\beta = \Phi_l'^n(k). \tag{31}$$

The functions $\varrho_l(\beta)$ are determined by the zero of the quantity $\det(2\pi\delta_l^n + \Phi_l'^n(k))$ which is nearest to the real axis [16, 11, 15]. This quantity is equal to

$$\det(2\pi\delta_n^l + \Phi_n'^l(K)) = \frac{(4\pi)^q}{\sinh(k\pi)}\sinh(k\pi - k\mu q)\cosh^q(k\mu) \tag{32}$$

and the nearest zero on the segment (22) *is*

$$k_0 = i\pi/(2\mu). \tag{33}$$

So one can show that in the limit $\Lambda\to\infty$ the solution of (30) is

$$\varrho_n(\beta) = 4\mu\sin(n\mu)\cos\mu\cosh(\pi\beta/2\mu)M/\pi(\pi+2\mu)\sin\mu$$

$$\varrho_q(\beta) = \frac{2\mu\cdot M\cosh(\pi\beta/2\mu)}{\pi(\pi+2\mu)}$$

$$\cdot\left[\frac{\sin(q-1)\mu}{\sin\mu} + \frac{\sin q\mu}{\sin\mu}\tan\frac{\pi}{2}\left(\frac{\pi}{\mu}-q-1\right)\right]$$

$$n = 1, \ldots, q-1, m_0 = (\pi-2\mu)M\exp\{(\pi-2\mu)\Lambda/2\mu\}. \tag{34}$$

It is interesting to note that one can solve (30) without reference to Sect. 5, i.e. we can elucidate the Λ dependence of m_0 directly from (30). Let us demand the vacuum to be stable in the limit $\Lambda\mapsto\infty$. In other words m_0 must depend on Λ in such a way that $\varrho_n(\beta)$ are Λ independent in this limit. In this way we can obtain (34) also. We see that the mass renormalization formula is the same as in the attractive case [16]. In the next section we shall prove that all excitations with vacuum charge have positive energies.

5. Excitations

Let us make a hole in the n-th component of vacuum with rapidity β_n. The permitted values of the condensate pseudoparticle rapidities will be changed. Let us denote their real parts by $\tilde{\beta}_j^n$. The periodicity conditions for the wave function of such configuration are:

$$m_0 L \frac{\sin b\mu}{\sin\mu} \sinh \tilde{\beta}_j^b = 2\pi j_b + \sum_{l=1}^{q} \sum_k{}' \Phi_l^b(\tilde{\beta}_j^b - \tilde{\beta}_k^l) - \Phi_n^b(\tilde{\beta}_j^b - \beta_n). \tag{35}$$

In the limit $L \to \infty$ these equations turn into integral equations [14, 15]:

$$\Phi_a'^n(\beta - \beta_n) = 2\pi F_a'(\beta|n) + \sum_{b=1}^{q} \int_{-\infty}^{\infty} \Phi_b'^a(\beta - \alpha) F_b'(\alpha|n) \, d\alpha. \tag{36}$$

Here

$$F_l(\beta_j^l|n) \equiv (\beta_j^l - \tilde{\beta}_j^l)/(\beta_{j+1}^l - \beta_j^l) \equiv f_l(\beta_j^l - \beta_n|n). \tag{37}$$

Let us solve this equation by means of Fourier transformation. To do this we must invert the matrix (31). One can show that the matrix inverse to (31) is a Jacobi matrix (three diagonal one). So the nonzero components of the solution are

$$f_b'(k|1) = \delta_2^b (2\cosh k\mu)^{-1}$$

$$f_b'(k|l) = (\delta_{l-1}^b + \delta_{l+1}^b)(2\cosh k\mu)^{-1}, \qquad l = 2, 3, \ldots, q-1, \tag{38}$$

$$f_b'(k|q) = \delta_{q-1}^b (2\cosh k\mu)^{-1} + \delta_q^b \frac{\sinh(k\pi - k\mu(q+1))}{2\cosh(k\mu)\sinh(k\pi - k\mu q)}. \tag{39}$$

Here

$$f_b'(k|a) = \int_{-\infty}^{\infty} \exp\{i\beta k\} f_b'(\beta|a) \, d\beta = f_a'(k|b). \tag{40}$$

Let us calculate the observable values of energy, momentum and charge of these hole configurations. The observable value of these quantities is the difference between the value of these quantities on the hole configuration and the value of these quantities on the vacuum. By means of (20), (26) one can show that [14, 15]:

$$\begin{aligned} E_n &= m_0 \frac{\sin\mu n}{\sin\mu} \cosh\beta_n - m_0 \sum_{b=1}^{q} \frac{\sin\mu b}{\sin\mu} \int_{-\Lambda}^{\Lambda} \cosh\beta F_b'(\beta|n) \, d\beta \\ P_n &= m_0 \frac{\sin\mu n}{\sin\mu} \sinh\beta_n - m_0 \sum_{b=1}^{q} \frac{\sin\mu b}{\sin\mu} \int_{-\Lambda}^{\Lambda} \sinh\beta F_b'(\beta|n) \, d\beta. \end{aligned} \tag{41}$$

We must calculate the charge of these configurations carefully. While we make a hole in the vacuum some condensate pseudoparticles go out of the cutoff. The number of ℓ-th bound states pushed out over Λ or $-\Lambda$ equal respectively to [16]:

$$\begin{aligned} \Delta N(\Lambda) &= -F_l(\Lambda|n) \\ \Delta N(-\Lambda) &= F_l(-\Lambda|n). \end{aligned} \tag{42}$$

So the whole change of the quantity of the condensate pseudoparticles (the observable charge) is

$$Q_n = -n + \sum_{b=1}^{q} b \int_{-\Lambda}^{\Lambda} F_b'(\beta|n) \, d\beta. \tag{43}$$

By means of these formulae, and formulae (37)–(40) and (34) one can show that P_n, E_n, Q_n and masses M_n are equal to:

$$E_n = M_n \cosh\theta_n, P_n = M_n \sinh\theta_n$$
$$M_l = 2M\sin(\mu\ell)/\tan\mu > 0, Q_l = 0, \ell = 1, \ldots, q-1, \tag{44}$$

$$M_q = M\left\{\frac{\sin(q-1)\mu}{\sin\mu} + \frac{\sin q\mu}{\sin\mu}\tan\frac{\pi}{2}\left(\frac{\pi}{\mu} - q - 1\right)\right\} > 0$$
$$Q_q = -\pi/2(\pi - q\mu) < 0. \tag{45}$$

Here θ is the observable rapidity

$$\theta = \pi\beta/(2\mu). \tag{46}$$

We see that the holes in the first $(q-1)$ components represent neutral particles. The hole in the last q-th component represent a charged particle. The situation is just the same as in the attractive case.

We'll analyse all other excitations. Let us put an elementary pseudoparticle with positive energy in the vacuum. In this situation we also can construct a function $f_n(\beta|p)$ like (37). In the same way we can calculate the Fourier transformation of $f'_n(\beta|p)$ denoted by $f'_n(k|p)$:

$$f'_n(k|p) = \delta^n_q \sinh(\mu k)/\sinh(k\pi - k\mu q), \tag{47}$$

and E_p, P_p, Q_p

$$Q_p = \pi/(\pi - \mu q) > 0, E_p = 0, P_p = 0. \tag{48}$$

The pole of (47) is situated farther from the real axis than (33) so this exsitation has no energy. Now let us put in the vacuum a bound state of $(q+1)$ pseudoparticles (23). In this case we have

$$f'_n(k|q+1) = -\delta^n_q \cdot \left(\frac{\sinh(k\pi - k\mu(q+1))}{\sinh(k\pi - k\mu q)}\right), \tag{49}$$

$$E_{q+1} = 0, P_{q+1} = 0, Q_{q+1} = \pi/(\pi - \mu q) > 0. \tag{50}$$

At last let us put in the vacuum any other bound state (24). One can show that in this case

$$E_n = 0, P_n = 0, Q_n > Q_p. \tag{51}$$

We study only the zero charge sector of the theory. So the states (45), (48), (50), (51) are not interesting. We shall regard only the configurations of holes and bound states with zero charge.

It is important to note that if we put several bound states in the vacuum and make several holes, no new effects will take place. In this case the value of the observables is an algebraic sum of their values on the individual holes and particles, due to the linearity of all equations. For example let us make two holes in

n-th and ℓ-th component of vacuum with rapidities β_n^1 and β_l^2. One can show that the corresponding function

$$F_m(\beta_j^m) = (\beta_j^m - \tilde{\beta}_j^m)/(\beta_{j+1}^m - \beta_j^m) \tag{52}$$

is equal to the sum (37)

$$F_m(\beta) = f_m(\beta - \beta_n^1|n) + f_m(\beta - \beta_l^2|\ell). \tag{53}$$

So by means of (44), (45), (48), (50), and (51) we prove that every excitation in the sector with vacuum charge has positive energy. Thus the vacuum is constructed correctly. We also have shown that the formulae (11), (12) give us the mass spectrum of the theory in the interval (10).

The last problem is to construct the scattering state of a fermion on an antifermion. We shall do this in the usual way [8, 15, 16]. Fermion-antifermion state is represented by two holes in the q-th component of vacuum (45) with rapidities β_1 and β_2 plus a "binding element". This "binding element" is the elementary pseudoparticle with positive energy (48) or a bound state of $(q+1)$ pseudoparticles (50) with rapidity $(\beta_1+\beta_2)/2$ (this depends on the parity of the fermion-antifermion state). The total charge of this state is zero.

6. *S* Matrix

Let us calculate the scattering matrix of the physical particles. The construction of physical particle states shows us that we must calculate the scattering phase of two (dressed) holes and the scattering phase of a hole and a "binding element". Bethe ansatz shows that the S matrix is diagonal. When we discuss the scattering of two pseudoparticles inserted in the condensate we must remember that the firstly inserted pseudoparticle is scattered not only by the secondly inserted pseudoparticle but also by all condensate pseudoparticles. So one can show that the scattering matrix of a hole in the n-th component with rapidity β_n^1 on a hole in the ℓ-th component with rapidity β_l^2 is the product of three factors [16]:

$$S_l^n(\beta_n^1 - \beta_l^2) = \exp\{i\Phi_l^n(\beta_n^1 - \beta_l^2)\} \cdot S_2^{-1}(\beta_n^1, \beta_l^2) \cdot S_1(\beta_n^1). \tag{54}$$

The first factor is a bare pseudoparticle S matrix (27), (28). The second factor S_2 is constructed as follows. Let us consider a two hole wave function. The permitted values of rapidities in the condensate are given by (52), (53). S_2 is the scattering matrix of the bound state of n-pseudoparticles with rapidity $\beta(\beta \to \beta_n^1)$ on all pseudoparticles in the condensate. The third factor S_1 is defined similarly. Let us consider the one hole (in n-th component with rapidity β_n^1) wave function. The permitted values of the rapidities in the condensate are given by (37)–(40). S_1 is the scattering matrix of the bound state of n pseudoparticles with same rapidity $\beta(\beta \to \beta_n^1)$ on all pseudoparticles in the condensate in this case. So we have [16]:

$$-i \ln S_l^n(\beta) = \Phi_l^n(\beta) - \sum_{c=1}^{q} \int_{-\infty}^{\infty} \Phi_c^n(\beta - \alpha) f_c'(\alpha|\ell) d\alpha. \tag{55}$$

By means of (36), (37) we have

$$(\ln S_l^n(\beta))' = 2\pi i f_n'(\beta|\ell). \tag{56}$$

In the same way we shall calculate the scattering matrix of a hole and on elementary pseudoparticle (47):

$$(\ln S_p^n(\beta))' = 2\pi i f'_n(\beta|p). \tag{57}$$

The scattering matrix of the n-th hole and a bound state of $(q+1)$ pseudoparticles is (49)

$$(\ln S_{q+1}^n(\beta))' = 2\pi i f'_n(\beta|q+1). \tag{58}$$

Formulae (57), (58) give us the S matrix of a hole and a "binding element".

Let us calculate now the scattering matrix of two neutral particles (44). Formulae (56), (38) lead to the answer (13).

Let us calculate the neutral particle fermion S matrix. As we are in the neutral sector we can only calculate the three body S matrix which describes the scattering of a neutral particle on the fermion-antifermion scattering configuration. Fortunately, formulae (57), (47), and (58), (49) show that the neutral particle – "binding element" S matrix is equal to unity. So the three body S matrix is a product of two factors. Each factor is a neutral particle fermion (antifermion) S matrix. This S matrix is the q-th hole ℓ-th hole ($\ell = 1, \ldots, q-1$) S matrix. By means of (56) and (38) or (39) we obtain (14).

At last let us calculate the fermion-antifermion S matrix. This S matrix is a product of two factors. The first factor is a q-th hole q-th hole S matrix. The second factor is a q-th hole – "binding element" S matrix, which depends on the parity of the fermion-antifermion state. In the case of positive parity the "binding element" is a bound state of $(q+1)$ pseudoparticles. In the case of negative parity it is an elementary pseudoparticle. Direct calculation of fermion-antifermion S matrix leads us to the answer (15), (9), (16) on the interval (10). The "binding element" – q-th hole S matrix gives us the factor $U_\pm(\theta)$ in (9), (15) by means of (57), (47), and (58), (49). The two hole S matrix gives us the factor $S(\theta)$ in (9), (15) by means of (56), (39).

Bethe ansatz shows that the n-particle S matrix is a product of two-particle S matrices.

7. Conclusions

We have seen that the dynamics of the massive Thirring model is quite nontrivial in the limit $g \to -\pi$.

All described effects have quite clear physical meaning. Really, one can show that

$$H_T(-g) = -\mathbf{u} H_T(g) \mathbf{u}^{-1}. \tag{59}$$

This is usual γ_5 transformation.

Here $H_T(g)$ is the quantum Hamiltonian of the massive Thirring model and $\mathbf{u}$ is unitary operator. First of all we see from (59), (4), and (3) that

$$H_T(-\pi+0) = -\mathbf{u} \int dx \{\tfrac{1}{2}(\varphi_t)^2 + \tfrac{1}{2}(\varphi_x)^2 + \tfrac{1}{2}M_0^2\varphi^2\} \mathbf{u}^{-1}.$$

So the massive Thirring model is meaningless at $g=-\pi+0$ (the Hamiltonian unbounded from below). This explains the very complicated construction which we used to give sense to the theory in the limit $g\to-\pi$. Everything becomes clear if we remember that in the limit $g\to\pi$ there is a rich particle spectrum (7). They appear in the points $\omega=\pi/n(g=\pi-2\pi/n)$ (6). The formula (59) shows that the same particles appear in the points $\mu=\pi/n(g=-\pi+2\pi/n)$ but with negative energies. These particles must therefore form a condensate.

Finally, we shall note that one can write Bethe ansatz (18) in the form

$$\chi^{\alpha_1\ldots\alpha_l}(X_1,\ldots,X_l|\beta_1,\ldots,\beta_l)=\prod_{k=1}^{l}\chi^{\alpha_k}(X_k|\beta_k)\cdot\prod_{j>n}[1-i\tan(g/2)\,\varepsilon(X_j-X_n)\tanh((\beta_j-\beta_n)/2)]\,. \tag{60}$$

So we see that there will be no new effects at any other value of the coupling constant in the massive Thirring model due to the periodic dependence of the Bethe ansatz on the coupling constant.

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