Analytical Continuation from Positive Integres

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The $\Gamma(z)$ function has simple poles at zero and negative integers $z = 0, -1, -2\dots$

$$\Gamma(z) = \frac{e^{-Cz}}{z} \Pi_{k=1}^{\infty} \frac{e^{z\kappa}}{1 + z/k}$$

Here C is the Euler constant. The function $\Gamma^{-1}(-z)$ vanish at zero and positive integers $z = 0, 1, 2, \dots$ We shall denote this set by $\mathcal{N}^+$. Let us introduce a function $\phi(z)$ which is regular at $\mathcal{N}^+$ but otherwise arbitrary. It can have singularities [poles and branch cuts] at other points on the complex plane. The following function vanish at $z \in \mathcal{N}^+$

$$\phi(z)\Gamma^{-1}(-z) = 0 \quad \text{for} \quad z = 0, 1, 2, 3\dots$$

Remark:
Actually the function $\phi(z)$ can have weak singularities at $\mathcal{N}^+$, weaker then the simple pole.

Consider $f(n) = tr(\rho^n)$. We know it only for $n \in \mathcal{N}^+$. The $f(z)$ is a continuation of $f(n)$ to the complex plane. The continuation is not unique. A function

$$\widetilde{f(z)} = f(z) + \phi(z)\Gamma^{-1}(-z)$$

is another continuation:

$$\widetilde{f(n)} = f(n) = tr(\rho^n) \quad \text{for} \quad n \in \mathcal{N}^+$$

There are infinitely many analytical continuations from positive integers to the complex plane.
If one wants to consider only functions with singularities at infinity, then one can consider ϕ as a polynomial of many variables:

$$\phi(z) = P(z, e^z, e^{e^z}, \dots)$$

There are infinitely many of those.
Remark: Maybe we know $f(n) = tr(\rho^n)$ only for positive integres $n = 1, 2, 3\dots$
Maybe zero is excluded. Then we replace $\Gamma(z)$ by $\Gamma(z+1)$.

Consider different chains:
In AKLT the Renyi entropy does not depend on n , see `https://arxiv.org/pdf/0802.3221.pdf`
In XX spin chain the Renyi entropy has simple pole at $n = 0$, see `https://arxiv.org/pdf/quant-ph/0304108.pdf`
In Fredkin spin chain the Renyi entropy scales (with the size of the block x) differently at different n: $\log x$, $\sqrt{x}$, x. The dependence on n does not factorize from the dependence on x, see `https://arxiv.org/pdf/1806.04049.pdf`