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NORMS OF BOUND STATES

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The formula for the norms of the Bethe wave functions in the form of a Jacobian plays an important role in the computation of the correlation functions. In the present paper this formula is generalized to the case of bound states.

1. Introduction

In the present paper we prove that the square of the norm of a wave function, describing bound states, is proportional to a certain Jacobian (see Theorem 1).

The quantum inverse problem method [1] allows us to construct the eigenfunctions of the Hamiltonians of integrable models with the aid of the algebraic Bethe Ansatz. We introduce some notations. We consider the case when the dimension of the monodromy matrix $T(\lambda)$ is 2×2:

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & \mathfrak{D}(\lambda) \end{pmatrix} \tag{1}$$

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The quantities $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $\mathcal{D}(\lambda)$ are quantum operators, depending on the spectral parameter λ, their commutation relations are given with the aid of R-matrices (dimension 4×4)

$$R(\lambda,\mu)(T(\lambda)\otimes I)(I\otimes T(\mu))=(I\otimes T(\mu))(T(\lambda)\otimes I)\,R(\lambda,\mu)\,. \tag{2}$$

The structure of the Bethe Ansatz is the same for the R-matrices of the XXX and XXZ Heisenberg models

$$R(\lambda,\mu)=\left|\begin{array}{cc|cc} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ \hline 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{array}\right| . \tag{3}$$

In the XXZ case:

$$f(\lambda,\mu)=\frac{sh\,(\lambda-\mu+2ih)}{sh\,(\lambda-\mu)}\,; \quad g(\lambda,\mu)=\frac{i\,\sin 2h}{sh\,(\lambda-\mu)}\ . \tag{4}$$

In the XXX case:

$$f(\lambda,\mu)=(\lambda-\mu+i\varkappa)/(\lambda-\mu); \quad g(\lambda,\mu)=i\varkappa/(\lambda-\mu). \tag{5}$$

For the sake of definiteness, all the formulas will be given in the XXZ case. Important roles are played also by the pseudovacuum $|0\rangle$ and the dual pseudovacuum $\langle 0|$. These are vectors in the quantum space with the following properties:

$$\begin{aligned} &C(\lambda)|0\rangle=0; \ A(\lambda)|0\rangle=a(\lambda)|0\rangle; \ \mathcal{D}(\lambda)|0\rangle=d(\lambda)|0\rangle \\ &\langle 0|B(\lambda)=0; \ \langle 0|A(\lambda)=a(\lambda)\langle 0|; \ \langle 0|\mathcal{D}(\lambda)=d(\lambda)\langle 0|. \end{aligned} \tag{6}$$

The vacuum eigenvalues $a(\lambda)$ and $d(\lambda)$ are complex-valued functions. The eigenfunctions of the trace $\tau(\lambda)=A(\lambda)+\mathcal{D}(\lambda)$ of the monodromy matrix have the form

$$\Psi_N(\{\lambda\})=B(\lambda_1)\dots B(\lambda_N)|0\rangle \tag{7}$$

$$\tilde{\Psi}_N(\{\lambda\})=\langle 0|C(\lambda_1)\dots C(\lambda_N) \tag{8}$$

in the case when the parameters $\lambda_1, \dots, \lambda_N$ satisfy the system of transcendental equations (STE)

$$\frac{a(\lambda_n)}{d(\lambda_n)}\prod_{\substack{j=1\\ j\neq n}}^{N}\frac{f(\lambda_n,\lambda_j)}{f(\lambda_j,\lambda_n)}=1\ . \tag{9}$$

The quantities λ_j are called particle speeds. We rewrite the STE in logarithmic form.

For this we introduce the variables φ_K:

$$\varphi_K=i\,\ln\frac{a(\lambda_K)}{d(\lambda_K)}+i\sum_{\substack{j=1\\ j\neq k}}^{N}\ln\frac{f(\lambda_K,\lambda_j)}{f(\lambda_j,\lambda_K)}\ . \tag{10}$$

Now the system (9) has the form:

$$\varphi_k \equiv 0 \pmod{2\pi}. \tag{11}$$

In [2] one has computed the norms of the eigenfunctions (7), (8):

$$\langle 0|\prod_{j=1}^{N}\frac{C(\lambda_j)}{a(\lambda_j)}\prod_{j=1}^{N}\frac{B(\lambda_j)}{d(\lambda_j)}|0\rangle=(\sin 2\eta)^N\left\{\prod_{\substack{j\neq k\\1}}^{N} f(\lambda_j\lambda_k)\right\}\det_N\left(\frac{\partial\varphi_k}{\partial\lambda_j}\right). \tag{12}$$

Here we have set $\langle 0|0\rangle = 1$. The Jacobian has the form

$$\partial\varphi_k/\partial\lambda_j=\delta_{kj}\Big(L(\lambda_k)+\sum_{\ell=1}^{N}X(\lambda_k,\lambda_\ell)\Big)-X(\lambda_k,\lambda_j) \tag{13}$$

$$\begin{aligned}L(\lambda_k)&=i\frac{d}{d\lambda_k}\ln\frac{a(\lambda_k)}{d(\lambda_k)};\ X(\lambda_k,\lambda_\ell)=i\frac{\partial}{\partial\lambda_k}\ln\frac{f(\lambda_k,\lambda_\ell)}{f(\lambda_\ell,\lambda_k)}=\\&=\sin 4\eta/\operatorname{sh}(\lambda_k-\lambda_\ell+2i\eta)\operatorname{sh}(\lambda_k-\lambda_\ell-2i\eta).\end{aligned} \tag{14}$$

We mention now concrete models which are in the considered class. First of all it is the XXZ Heisenberg model [3]. For it we have

$$a(\lambda)=\operatorname{sh}^M(\lambda-i\eta),\quad d(\lambda)=\operatorname{sh}^M(\lambda+i\eta). \tag{15}$$

Here M is the number of nodes of the lattice.

Another example is the sine-Gordon lattice model [4]. For it:

$$a(\lambda)=\left[1+2h\operatorname{ch}(2\lambda+2i\eta)\right]^M,\ d(\lambda)=\left[1+2h\operatorname{ch}(2\lambda-2i\eta)\right]^M. \tag{16}$$

Here h is an arbitrary real parameter. In both cases, $T(\lambda)$ satisfies the involution:

$$T^{+}(\bar\lambda)=\sigma_2 T(\lambda)\sigma_2 \Longrightarrow B^{+}(\bar\lambda)=-C(\lambda). \tag{17}$$

The cross denotes Hermitian conjugation only of the quantum operators. Formula (17) means that in the left-hand side of (12) one has the square of the norm of the eigenfunction in the case when under complex conjugation the collection $\{\lambda_j\}$ goes into itself:

$$\{\lambda_j\}=\{\bar\lambda_j\}. \tag{18}$$

We recall that $[C(\lambda),C(\mu)]=[B(\lambda),B(\mu)]=0$. We note that formula (12) plays a central role at the computation of the correlation functions [5, 6].

2. Bound States

In the above mentioned models one has bound states. Equidistant bound states are formed at the limit when the number of nodes tends to infinity; see [7]. We denote the number of particles, joined in one bound state, by ν. The collection of the allowed values of ν depends on the coupling constant η. The complete classification of the bound states, sometimes called strings, are given in [7] for the XXZ Heisenberg model. In a bound state the particle speeds are distributed in the following manner:

$$\lambda_\alpha^a = \lambda_\alpha + i\eta\,(2a-\nu-1), \quad a=1,\ldots,\nu_\alpha . \tag{19}$$

The quantities λ_α are called string speeds. We shall assume that in the string with speed λ_α there are bound ν_α particles. Usually, the STE (9) are multiplied over all the particles occurring in a given string (with respect to a) and one writes a system for the centers of the strings:

$$r_{\nu_\alpha}(\lambda_\alpha)\prod_{\beta\neq\alpha} S_{\nu_\alpha,\nu_\beta}(\lambda_\alpha,\lambda_\beta)=1 . \tag{20}$$

Here

$$r_{\nu_\alpha}(\lambda_\alpha)=\prod_{a=1}^{\nu_\alpha}\frac{a(\lambda_\alpha^a)}{d(\lambda_\alpha^a)} \tag{21}$$

$$S_{\nu_\alpha,\nu_\beta}(\lambda_\alpha,\lambda_\beta)=\prod_{a=1}^{\nu_\alpha}\prod_{b=1}^{\nu_\beta}\frac{f(\lambda_\alpha^a,\lambda_\beta^b)}{f(\lambda_\beta^b,\lambda_\alpha^a)} . \tag{22}$$

The system (21) can be rewritten also in the logarithmic form:

$$\Phi_\alpha \equiv 0 \;\;(\mathrm{mod}\; 2\pi)$$
$$\Phi_\alpha = i\ln r_{\nu_\alpha}(\lambda_\alpha)+i\sum_{\beta\neq\alpha}\ln S_{\nu_\alpha,\nu_\beta}(\lambda_\alpha,\lambda_\beta) . \tag{23}$$

We consider the Jacobian

$$\frac{\partial\Phi_\alpha}{\partial\lambda_\beta}=\delta_{\alpha\beta}\Big(\mathcal{L}_\alpha+\sum_{\gamma\neq\alpha}X_{\alpha\gamma}\Big)-X_{\alpha\beta} \tag{24}$$

$$\mathcal{L}_\alpha=\sum_{a=1}^{\nu_\alpha}L(\lambda_\alpha^a);\quad X_{\alpha,\beta}=\sum_{a=1}^{\nu_\alpha}\sum_{b=1}^{\nu_\beta}X(\lambda_\alpha^a,\lambda_\beta^b) . \tag{25}$$

For notations, see (13), (14). We consider the eigenfunction in which there are ℓ bound states and in each bound state there are ν_α particles. The total number of elementary particles is equal to N:

$$N = \nu_1+\ldots+\nu_\ell . \tag{26}$$

Below we prove

<u>THEOREM 1.</u> The square of the norm of the wave function, describing the bound states, is proportional to the determinant of the matrix $(\partial\Phi_\alpha/\partial\lambda_\beta)$ [see (24)]; moreover,

$$\left\| \prod_{j=1}^{N} \frac{B(\lambda_j)}{d(\lambda_j)} |0\rangle \right\|^2 = (-1)^{\ell} \cdot 2^{N-\ell} (\sin 2\eta)^{\ell} (\cos 2\eta)^{N-\ell} \prod_{\alpha \neq \beta} F_{\nu_\alpha \nu_\beta}(\lambda_\alpha, \lambda_\beta) \cdot \det_{\ell} (\partial \Phi_\alpha / \partial \lambda_\beta) . \tag{27}$$

Here

$$F_{\nu_\alpha, \nu_\beta}(\lambda_\alpha, \lambda_\beta) = \prod_{a=1}^{\nu_\alpha} \prod_{b=1}^{\nu_\beta} f(\lambda_\alpha^a, \lambda_\beta^b) . \tag{28}$$

3. Thermodynamic Limit

We consider a lattice with M nodes. In the thermodynamic limit $M \to \infty$. For homogeneous lattices

$$i \ln \frac{a(\lambda)}{d(\lambda)} = M\, p_1(\lambda) . \tag{29}$$

For a bound state of ν particles we have

$$i \ln r_\nu(\lambda) = M p_\nu(\lambda) = M \sum_{a=1}^{\nu} p_1(\lambda^a) . \tag{30}$$

For a finite temperature, the number of bound states for each fixed ν tends to infinity in the thermodynamic limit [7]. In connection with this, we change somewhat the notations of the previous section. We rewrite the system of transcendental equations in the form

$$\Phi_\alpha^\nu = 2\pi n_\alpha^\nu . \tag{31}$$

Here n_α^ν is some collection of integers,

$$\Phi_{\alpha_1}^{(\nu_1)} = i \ln r_{\nu_1}(\lambda_{\alpha_1}^{(\nu_1)}) + i \sum_{\nu_2} \sum_{\substack{\alpha_2 \\ (\lambda_{\alpha_1}^{(\nu_1)} \neq \lambda_{\alpha_2}^{(\nu_2)})}} \ln S_{\nu_1, \nu_2}(\lambda_{\alpha_1}^{(\nu_1)}, \lambda_{\alpha_2}^{(\nu_2)}) . \tag{32}$$

The index α enumerates the various bound states with same number of particles.

$$\frac{\partial \Phi_{\alpha_1}^{(\nu_1)}}{\partial \lambda_{\alpha_2}^{(\nu_2)}} = \delta_{\nu_1 \nu_2} \delta_{\alpha_1 \alpha_2} \Big(\mathcal{Z}_{\nu_1}(\lambda_{\alpha_1}) + \sum_{\nu_3, \alpha_3} X_{\nu_1 \nu_3}(\lambda_{\alpha_1}, \lambda_{\alpha_3}) \Big) - X_{\nu_1, \nu_2}(\lambda_{\alpha_1}, \lambda_{\alpha_2}) \tag{33}$$

$$X_{\nu_1, \nu_2}(\lambda_{\alpha_1}, \lambda_{\alpha_2}) = \sum_{a=1}^{\nu_1} \sum_{b=1}^{\nu_2} X(\lambda_{\alpha_1}^a, \lambda_{\alpha_2}^b); \quad \mathcal{Z}_\nu(\lambda) = M p_\nu'(\lambda). \tag{34}$$

In the thermodynamic limit the system of transcendental equations (31) turns into the system of integral equations

$$2\pi \rho_{\nu_1}(\lambda) = p_{\nu_1}'(\lambda) + \sum_{\nu_2} \int X_{\nu_1 \nu_2}(\lambda, \mu)\, \rho_{\nu_2}^{\circ}(\mu)\, d\mu . \tag{35}$$

Here $\rho_\nu^0(\lambda)$ is the density distribution of the bound states according to the speeds, while $\rho_\nu(\lambda)$ is the density of the vacancies for the bound states. The thermodynamic limit of the determinant in (27) [see also (33)] is equal to

$$\det\left(\frac{\partial\Phi_\alpha}{\partial\lambda_\beta}\right)=\left[\prod_\nu\prod_\alpha\left\{2\pi M\rho_\nu(\lambda_\alpha)\right\}\right]\det\left(1-\frac{1}{2\pi}\hat{X}_T\right). \tag{36}$$

The integral operator, occurring under the determinant symbol, acts on the vector function $u_\nu(\lambda)$

$$\left[\left(1-\frac{1}{2\pi}\hat{X}_T\right)\vec{u}\right]^{\nu_1}(\lambda_1)=u^{\nu_1}(\lambda_1)-\frac{1}{2\pi}\sum_{\nu_2}\int d\lambda_2 X_{\nu_1\nu_2}(\lambda_1,\lambda_2)\left[\frac{\rho^0_{\nu_2}(\lambda_2)}{\rho_{\nu_2}(\lambda_2)}\right]u^{(\nu_2)}(\lambda_2)\,d\lambda_2\,. \tag{37}$$

4. Proof of the Formula for the Bound States

In order to obtain from (12) formula (27) it is necessary to solve the indeterminacy. Indeed, in formula (12) the product $\prod f(\lambda_j,\lambda_k)$ tends to zero at the passage to the bound states, while $\det(\partial\varphi_k/\partial\lambda_j)\to\infty$ since $X(\lambda_k,\lambda_\ell)\to\infty$. In order to do this accurately, we consider the determinant of the matrices A_{ij},

$$A_{ij}=\delta_{ij}\left(L_i+\sum_{k=1}^{N}X_{ik}\right)-X_{ij}$$
$$\Delta_N(L_1,\dots,L_N,X_{ij})=\det|A_{ij}|,\quad 1\leqslant i,j\leqslant N. \tag{38}$$

Here $X_{ij}=X_{ji}$ is an arbitrary collection of variables. It is easy to see that A_{ij} does not depend on the diagonal elements of the symmetric matrix X; therefore, for the sake of definiteness, we set

$$X_{ii}=0,\quad 1\leqslant i\leqslant N. \tag{39}$$

We note first of all that Δ_N is a linear function in each of the variables X_{ij}. By virtue of the complete symmetry, it is sufficient to show that Δ_N is a linear function of X_{12} (when all the other variables are fixed). We carry out some transformations of Δ_N. Clearly, X_{12} occurs only in the first two rows and in the first two columns. Adding to the first row of the matrix A_{ij} all the remaining rows, and then to the first column all the remaining columns, we obtain a matrix $B_{ij}^{(N)}$ (the upper index denotes the dimensions of the matrix),

$$B_{ij}^{(N)}=A_{ij}^{(N)}\quad\text{for}\quad 2\leqslant i,j\leqslant N$$
$$B_{11}=\sum_{i=1}^{N}L_i,\quad B_{1j}=B_{j1}=L_j\qquad\text{for } 2\leqslant j\leqslant N \tag{40}$$
$$\det|B_{ij}^{(N)}|=\det|A_{ij}^{(N)}|=\Delta_N(L_1,\dots,L_N;X_{ij}).$$

The matrix B_{ij} depends on X_{12} in a simpler manner. Only the element B_{22} depends on X_{12} (this is a linear function). Obviously,

$$\frac{\partial}{\partial x_{12}} \det |B_{ij}^{(N)}| = \det |\tilde{B}_{ij}^{(N-1)}|$$

$$\tilde{B}_{11}^{(N-1)} = \sum_{i=1}^{N} L_i = (L_1 + L_2) + \sum_{i=3}^{N} L_i \tag{41}$$

$$\tilde{B}_{1j}^{(N-1)} = \tilde{B}_{j,1}^{(N-1)} = L_j \quad \text{for } j \geq 3$$

$$\tilde{B}_{ij}^{(N-1)} = B_{ij}^{(N)} = A_{ij}^{(N)} \quad \text{for } 3 \leq i, j \leq N \tag{42}$$

$$\tilde{B}_{ij}^{(N-1)} = \delta_{ij}\Big(L_i + x_{i1} + x_{i2} + \sum_{k=3}^{N} x_{ik}\Big) - x_{ij}, \quad 3 \leq i, j \leq N .$$

The second row and the second column are missing in the matrix $\tilde{B}^{(N-}$. Obviously, $\tilde{B}^{(N-1)}$ is the matrix $B^{(N)}$ for which the arguments are changed. Thus,

$$\frac{\partial}{\partial x_{12}} \Delta_N (L_1, \ldots, L_N ; x_{ij}) = \det |\tilde{B}_{ij}^{(N-1)}| = \\ = \Delta_{N-1}(L_1 + L_2, L_3, \ldots, L_N ; x_{1j} + x_{2j}, x_{k,\ell}), \quad 3 \leq k, \ \ell \leq N . \tag{43}$$

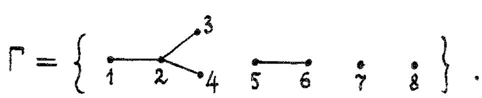
In order to compute the derivative of Δ_N with respect to several x_{ij}, it is convenient to introduce some notations. Let $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$, $i_s < j_s$ $(1 \leq s \leq k)$ be two subsets in $[1, N]^k$. We set $X_{I,J} = \prod_{s=1}^{k} x_{i_s, j_s}$. With the subsets I, J we associate a graph $\Gamma(I, J)$, defined in the following manner: we mark by points the numbers from 1 to N and we join by segments the points i_s and j_s for $1 \leq s \leq k$. Let $\Gamma(I,J) = \bigcup_{\alpha} \Gamma_\alpha$ be the decomposition of the graph $\Gamma(I, J)$ in the union of its connected components. We define $L_\alpha = \sum L_i$, i running over all the vertices of the graph Γ_α,

$$X_{\alpha\beta} = \sum_{i \subset \alpha} \sum_{k \subset \beta} x_{ik} . \tag{44}$$

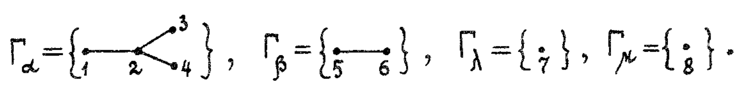
If the graph $\Gamma(I, J)$ does not have cycles, we define

$$A(I, J) = \prod L_\alpha . \tag{44a}$$

Here α runs over all the connected components of the graph $\Gamma(I, J)$. We illustrate the construction of the graph Γ on an example. Let $N = 8$, $k = 4$, $I = \{1, 2, 2, 5\}$, $J = \{2, 3, 4, 6\}$. The graph $\Gamma(I, J)$ has the following form:



The graph Γ has four connected components:



Correspondingly

$$L_\alpha = L_1 + L_2 + L_3 + L_4$$

$$X_{\alpha\beta} = x_{15} + x_{25} + x_{35} + x_{45} + x_{16} + x_{26} + x_{36} + x_{46}$$

$$A(I, J) = (L_1 + L_2 + L_3 + L_4) \cdot (L_5 + L_6) \cdot L_7 \cdot L_8 .$$

We compute now the derivatives of the determinant Δ_N. To this end we set

$$\partial_{ij} = \frac{\partial}{\partial x_{ij}} , \quad \partial_{I,J} = \prod_{s=1}^{k} \partial_{i_s, j_s} . \tag{45}$$

<u>THEOREM 2.</u> If the graph $\Gamma(I, J)$ has cycles, then

$$\partial_{I,J} \Delta_N = 0 . \tag{46}$$

If the graph $\Gamma(I, J)$ does not have cycles, then the number of its connected components is equal to $(N - k)$ and

$$\partial_{I,J} \Delta_N = \Delta_{(N-k)} (L_\alpha ; x_{\alpha\beta}). \tag{47}$$

Here the index α parametrizes the connected components of the graph $\Gamma(I, J)$.

The proof is carried out by induction on k. The base of the induction is formula (43). We carry out the induction step. Let

$$\partial_{I,J} \Delta_N = \Delta_{N-k} (I , J) = \Delta_{I,J} . \tag{48}$$

We compute $\partial_{ij} \Delta_{I,J}$.

The first case: (i, j) lies in a connected component Γ_α. Then the graph $\Gamma(I \cup i ; J \cup j)$ has cycles. On the other hand,

$$\partial_{ij} \Delta_{I,J} = \frac{\partial x_{\alpha\beta}}{\partial x_{ij}} \left(\frac{\partial}{\partial x_{\alpha\beta}} \Delta_{I,J} \right) = 0 \tag{49}$$

since $\frac{\partial x_{\alpha\beta}}{\partial x_{ij}} = 0$ by virtue of the fact that in $x_{\alpha\beta}$ there occur only those $x_{k,m}$ for which k and m lies in different connected components.

The second case: $i \in \Gamma_\alpha , j \in \Gamma_\beta , \ \alpha \neq \beta$. Then the graph $\Gamma(I \cup i , J \cup j)$ has $N - k - 1$ connected components: Γ_α and Γ_β are joined by means of (ij). Making use of formula (43), we obtain:

$$\partial_{ij} \Delta_{I,J} = \Delta_{N+k-1} (L_\alpha , x_{\alpha\beta}) . \tag{50}$$

This concludes the induction proof of Theorem 2.

We obtain now a certain representation for the determinant Δ_N. We make use of this for the linearity of Δ_N with respect to each x_{ij} and also of the formula

$$\Delta_N (L_i , 0) = \prod_{i=1}^{N} L_i . \tag{51}$$

We expand Δ_N in a Taylor series with respect to x_{ij} and obtain:

$$\Delta_N (L_1, \ldots, L_N ; x_{ij}) = \sum_{I,J} A (I, J) x_{IJ}$$
$$A(I,J) = \begin{cases} \prod_\alpha L_i , & \text{if the graph } \Gamma(I,J) \text{ does not have cycles} \\ 0 , & \text{if the graph has cycles.} \end{cases} \tag{52}$$

Here we have made use of Theorem 1. We return now to the norms of the bound states. We consider the partition of the number

$$N = \nu_1 + \ldots + \nu_\ell \ , \quad \nu_i \geqslant 1 \quad \text{(integer)}. \tag{53}$$

We denote by α_j the segment $[\nu_1+\ldots+\nu_j+1, \nu_1+\ldots+\nu_j+\ldots+\nu_{j+\ell}]$, corresponding to one bound state

$$L_{\alpha_j} = \sum_{i\in\alpha_j} L_i \ , \quad X_{\alpha_i,\alpha_j} = \sum_{p\in\alpha_i} \sum_{q\in\alpha_j} x_{pq} \tag{54}$$

$$x_{\alpha_i} = \prod_{\substack{j\in\alpha_i \\ j+1\in\alpha_i}} x_{j,j+1} \ .$$

The symbol $x_{\alpha i} \to \infty$ means that $x_{j,j+1} \to \infty$ for all $j \in \alpha_i$. A consequence of the formula (47) and of the linearity of Δ_N with respect to each x_{ij} is the assertion:

$$\lim_{x_{\alpha_i}\to\infty} \frac{\Delta_N(L_1,\ldots,L_N; x_{ij})}{\prod_i x_{\alpha_i}} = \Delta_\ell(L_{\alpha_1},\ldots,L_{\alpha_\ell}; X_{\alpha_i,\alpha_j}). \tag{55}$$

This formula gives us the possibility to solve the indeterminacy in formula (12) and leads to formula (27).

At the conclusion of the paper we give a certain induction relation for Δ_N which may be use in the future:

$$\begin{aligned} &\Delta_N(L_1,\ldots,L_N; x_{ij}) = \Delta_{N-1}(\tilde{L}_1,\ldots,\tilde{L}_{N-1}, y_{ij})\Big(\sum_{j=1}^N L_j\Big) \\ &y_{ij} = x_{ij} + \frac{L_i L_j}{\left(\sum_{j=1}^N L_j\right)}\ , \quad 1 \leqslant i,\ j \leqslant N-1 \\ &\tilde{L}_i = x_{iN} + \frac{L_i L_j}{\left(\sum_{j=1}^N L_j\right)}\ , \quad 1 \leqslant i \leqslant N-1\ . \end{aligned} \tag{56}$$

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