DIFFERENTIAL EQUATIONS
FOR
QUANTUM CORRELATION FUNCTIONS

A.R. Its, A.G. Izergin, V.E. Korepin, N.A. Slavnov

CMA-R26-89

Centre for Mathematical Analysis
Australian National University
GPO Box 4
Canberra ACT 2601
Australia

DIFFERENTIAL EQUATIONS FOR QUANTUM CORRELATION FUNCTIONS

A.R.Its, A.G.Izergin, V.E.Korepin, N.A.Slavnov

Abstract

The quantum nonlinear Schrödinger equation (one dimensional Bose gas) is considered. Classification of representations of Yangians with highest weight vector permits us to represent correlation function as a determinant of a Fredholm integral operator. This integral operator can be treated as the Gelfand-Levitan operator for some new differential equation. These differential equations are written down in the paper. They generalize the fifth Painlève transcendent, which describe equal time, zero temperature correlation function of an impenetrable Bose gas. These differential equations drive the quantum correlation functions of the Bose gas. The Riemann problem, associated with these differential equations permits us to calculate asymptotics of quantum correlation functions. Quantum correlation function (Fredholm determinant) plays the role of τ functions of these new differential equations. For the impenetrable Bose gas space and time dependent correlation function is equal to τ function of the nonlinear Schrödinger equation itself. For a penetrable Bose gas (finite coupling constant c) the correlator is τ-function of an integro-differentiation equation.

CONTENTS

0. Introduction

The nonlinear Schrödinger (NS) equation is very well known :

$$
\begin{aligned}
i\partial_t\psi &= -\partial_x^2\psi + 2c\psi^+\psi\psi \\
i\partial_t\psi^+ &= \partial_x^2\psi^+ - 2c\psi^+\psi^+\psi
\end{aligned}
\tag{0.1}
$$

The Hamiltonian of the model is equal to

$$
H = \int_{-\infty}^{\infty} dx(\partial_x\psi^+\partial_x\psi + c\psi^+\psi^+\psi - h\psi^+\psi)
\tag{0.2}
$$

where h denotes the chemical potential. The operator for the number of particles Q and momentum P are equal to

$$
\begin{aligned}
Q &= \int_{-\infty}^{\infty} dx\psi^+(x)\psi(x) \\
P &= -\frac{i}{2}\int_{-\infty}^{\infty} dx(\psi^+(x)\partial_x\psi - (\partial_x\psi^+(x))\psi(x)) ,
\end{aligned}
\tag{0.3}
$$

In the quantum case the commutation relations are

$$
[\psi(x,t),\psi^+(y,t)] = \delta(x-y), [\psi(x,t),\psi(y,t)] = 0 .
\tag{0.4}
$$

The bare vacuum is defined by

$$
\psi(x)|0\rangle = 0 ; \quad \langle 0|\psi^+(x) = 0
\tag{0.5}
$$

Another name of the model is the one-dimensional Bose gas. The model is exactly solvable [1]. We shall consider a ground state $|\Omega\rangle$ of the model with finite density D. In this case momenta of the particles fill a one-dimensional Fermi sphere $[-q,q]$ (q is the Fermi momenta). Thermodynamics of the model at temperature $T>0$ was constructed in [2]. We shall be interested in thermodynamics only for an impenetrable Bose gas ($c=\infty$). In this case the spectrum of the Hamiltonian is equivalent to a free fermion one. In the state of thermodynamic equilibrium the distribution of particles in the momentum space are given by the Fermi weight

$$
\rho(\lambda) = \frac{1}{2\pi}\left(1 + e^{(\lambda^2-h)/T}\right)^{-1}
\tag{0.6}
$$

In this case the density D is equal to

$$
D = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{1+e^{(\lambda^2-h)/T}} ; \quad c=\infty
\tag{0.7}
$$

We shall consider different correlation functions in the model. First we shall consider the impenetrable Bose gas ($c = \infty$). In this case we shall calculate the field correlator

$$\langle \psi(x_2,t_2)\psi^+(x_1,t_1)\rangle \equiv \frac{\langle \Omega|\psi(x_2,t_2)\psi^+(x_1,t_1)|\Omega\rangle}{\langle \Omega|\Omega\rangle} \tag{0.8}$$

This is the zero temperature case. Recall that $|\Omega\rangle$ is the ground state with finite density D and

$$\psi(x,t) = e^{iHt}\psi(x,0)e^{-iHt} \ . \tag{0.9}$$

We shall calculate also finite-temperature correlation functions

$$\langle \psi(x_2,t_2)\psi^+(x_1,t_1)\rangle_T = \frac{tr(e^{-H/T}\psi(x_2,t_2)\psi^+(x_1,t_1))}{tr(e^{-H/T})} \ . \tag{0.10}$$

We shall consider also many point correlators. It should be noted that the zero-temperature equal time correlator was calculated in [3-5]. It was proved that it is associated with the fifth Painlève transcendent. There exist models similar to the impenetrable Bose gas (Ising model, XY model) where different correlators were calculated [6-9]. For a penetrable Bose gas ($0 < c < \infty$) the problem of correlators is especially interesting, because the model is not equivalent to free fermions, nevertheless we succeed in calculating correlators in this case (see section 4, 8).

Let us define the simplest correlation function which we calculate here for finite c. In the ground state of the model particles move in the Brownian way. There exist some probability

$$P(x) \tag{0.11}$$

that on the interval $[-x,x]$ there will be no particles (empty place). This probability is the simplest correlation function and we shall calculate it here.

Calculation of correlation functions consists of two steps. First we represent correlation function as a Fredholm determinant (minor) of some integral operator. (These integral operators are of very special form, they form infinite dimensional group, see part II.) Then we treat this integral operator as a Gelfand-Levitan [10] operator of some new differential equation and derive this differential equation. It follows that correlation functions (Fredholm determinant) play the role of τ-functions [11] for this new differential equation. A Riemann problem [11] can be associated with this differential equation, which enables us to calculate the asymptotics of correlation functions.

The plan of the paper is as follows. In part I we represent correlation functions as Fredholm determinants. In section 1 we present the Lenard formula. In section 2 we derive the Fredholm determinant for time dependent correlation functions. In section 3 we present generalization to the case of many fields. We use the Quantum Inverse Scattering Method (QISM) [12-16] in section 4 where we derive a Fredholm determinant for finite coupling constant. The algebraic part of the derivation is associated with enumeration of representations of Yangians with highest weight vector [17-21].

In part II, starting from Fredholm determinants we derive differential equations which drive correlation functions. Correlation functions (Fredholm determinants of integral operators) are equal to τ-functions of these differential equations, and integral operators play the role of Gelfand-Levitan operators. This approach is close to [22-25][42,43]. First we consider the impenetrable Bose gas $c = \infty$. In section 5 we construct a differential equation for the equal time temperature correlator. This is the case most close to the fifth Painlève transcendent [5,26]. We get a kind of Painlève equation with partial derivatives. This is a new integrable equation. The place of this equation in the general scheme is discussed in [27]. All ideas are explained in this case. All other cases are treated similarly. In section 6 we consider the time and space dependent correlator (0.8) (zero temperature case). In this case the fifth Painlève equation is deformed into a nonlinear Schrödinger equation (not selfajoint). In section 7 time-temperature correlators (0.10) are considered. We derive a three-dimensional completely integrable system of differential equations (based on NS equation) for this correlation function. Maybe it is interesting from the point of view of integrable differential equations. The most interesting results are in section 8, where we derive an integro-differential equation for correlation function at finite coupling constant c.

In part III we consider asymptotics of correlation functions. All differential equations derived in Part II are associated with some Riemann problem. This permits us to calculate the long distance asymptotics of correlation functions. In section 9 we do this for the example of the equal time finite temperature correlator. In section 10 we present some formulas of current $\psi^{+}_{(x)}\psi_{(x)}$ correlators, obtained together with N.M. Bogoliubov. Many point space and time dependent finite temperature correlators of currents of the impenetrable Bose gas can be represented as a determinant of matrix.

I. Correlation Functions as Fredholm determinants

In this part we shall represent different correlation functions as Fredholm determinants (minor) of some integral operators. We shall begin with the impenetrable Bose gas $c = \infty$. In the section 4 of this part we shall derive a Fredholm determinant for finite $c (0 < c < \infty)$. The derivation is similar to the derivation of the determinant formula for norms of Bethe wave function [16].

1. Lenard formula

First let us consider equal time finite temperature field correlators

$$\langle \psi^{+}(z)\psi(-z)\rangle_T = \frac{tr(e^{-H/T}\psi^{+}(z)\psi(-z))}{tr(e^{-H/T})} \tag{1.1}$$

by z here we denote distance. Let us make a rescaling and introduce new variables

$$\beta = \frac{h}{T}\,; \quad x = z\sqrt{T}\,. \tag{1.2}$$

The reason is that correlator depends only on two variables x and β

$$\langle \psi^+(z)\psi(-z)\rangle_T = \sqrt{T} g(x,\beta)\,. \tag{1.3}$$

To define g let us introduce an integral operator K, it acts on the real axis. The kernel $K(\lambda,\mu)$ is equal to :

$$K(\lambda,\mu) = \sqrt{\vartheta(\lambda)}\frac{\sin x(\lambda-\mu)}{(\lambda-\mu)}\sqrt{\vartheta(\mu)}\,. \tag{1.4}$$

Here $\vartheta(\lambda)$ is Fermi weight

$$\vartheta(\lambda) = (1+\exp\{\lambda^2-\beta\})^{-1}\,. \tag{1.5}$$

It is convenient to describe properties of the operator K in terms of functions

$$f_i(\lambda,x,\beta) \qquad\qquad (i=\pm)$$

(the dependence on x and β will be suppressed). The functions $f_\pm(\lambda)$ are defined as solutions of linear integral equations :

$$f_i(\lambda) - \gamma\int_{-\infty}^{\infty} K(\lambda,\mu)f_i(\mu)d\mu = e_i(\lambda)\,; \quad i=\pm\,. \tag{1.6}$$

Here γ is a real parameter and the functions $e_i(\lambda)$ are given as

$$e_\pm(\lambda) = \sqrt{\vartheta(\lambda)}\exp\{\pm i\lambda x\}\,. \tag{1.7}$$

Of primary importance will be the matrix of "potentials" $B_{ik}(x,\beta)$:

$$B_{ik}(x,\beta) = \gamma\int_{-\infty}^{\infty} e_i(\lambda)f_k(\lambda)d\lambda\,. \tag{1.8}$$

It is a real symmetric matrix with two independent matrix elements B_{++} and B_{--}

$$\overline{B_{++}} = B_{++} = B_{--}\,; \quad \overline{B_{+-}} = B_{+-} = B_{-+}\,. \tag{1.9}$$

Now all the notation necessary to express correlators in terms of the operator K has been introduced. The field correlator $\langle \psi^+(z)\psi(-z)\rangle_T = \sqrt{T}g(x,\beta)$ can be represented as

$$g(x,\beta) = \frac{1}{4}B_{++}(x,\beta)\det(1-\gamma K)\big/_{\gamma=2/\pi}\,. \tag{1.10}$$

The last factor here is a Fredholm determinant. In the paper [27] it is proved that formula (1.10) can be transformed to Lenard formula by means of Fourier transformation. Correlators of many fields can be represented in the similar way [27]. It is interesting to mention that $\det(1-\gamma K)$ has meaning as correlator at any complex value of parameter γ. Let us consider

$$\langle \exp(\alpha Q(z))\rangle_T = \frac{tr(e^{-H/T}e^{\alpha Q(z)})}{tr(e^{-H/T}} \, . \tag{1.11}$$

Here α is an arbitrary complex parameter and $Q(z)$ is the operator of the number of particles on the interval $[-z,z]$

$$Q(z) = \int_{-z}^{z} \psi^+(y)\psi(y)dy \, . \tag{1.12}$$

In [28], [29] it is shown that

$$\begin{aligned} \langle e^{\alpha Q(z)}\rangle_T &= \det(1-\gamma K) \\ \gamma &= \frac{1-e^\alpha}{\pi} \end{aligned} \quad . \tag{1.13}$$

Especially interesting is the value $e^\alpha = 0$, $\gamma = \frac{1}{\pi}$ it describes the probability (0.11) that on the interval $[-z,z]$ there will be no particles of the Bose gas. This case $\gamma = \frac{1}{\pi}$ also describes the level spacing probability distribution function which appears in the theory of random matrices [30, 5]. There is an interesting connection between formulae (1.10) and (1.13). From [31] we know the fermionization formula :

$$\psi_B(x) = \psi_\varphi(x) \exp\left\{ i\pi \int_{-\infty}^{x} \psi_\varphi^+(y)\psi_\varphi(y)dy \right\} \, . \tag{1.14}$$

In the l.h.s. we have the canonical Bose field (of impenetrable Bose gas), ψ_φ is canonical Fermi field. So (1.14) shows that for the field correlator (1.1) $\alpha = i\pi$. Formula (1.13) $\gamma = \frac{1-e^\alpha}{\pi}$ leads to the correct value of $\gamma = \frac{2}{\pi}$ for field correlator (1.10). One should mention that for zero temperature case $T=0$ the operator in (1.4) turns in

$$K_0(\lambda,\mu) = \frac{\sin x(\lambda-\mu)}{\lambda-\mu} \, .$$

It acts on the interval $[-q,q]$. The probability of empty place (0.11) is equal to

$$P(\lambda) = \det\left(1 - \frac{1}{\pi}K_0\right) \, . \tag{1.15}$$

In section 4 we shall generalize this representation to finite coupling constant. In section 5 we shall derive differential equation which drive equal time finite temperature correlation function, based on representation (1.3) and (1.10).

2. Time dependent correlator

First let us consider the zero temperature $T = 0$ case. In this case the ground state $|\Omega\rangle$ is a Fermi sphere $[-q, q]$, with Fermi momentum q

$$\langle \psi^+_{(2)} \psi_{(1)} \rangle = \frac{\langle \Omega | \psi^+_{(2)} \psi_{(1)} | \Omega \rangle}{\langle \Omega | \Omega \rangle} .$$

In the paper [32] it was proved that :

$$\langle \psi(x_2, t_2) \psi^+(x_1, t_1) \rangle = e^{iht_{21}} \left(\frac{1}{2\pi} G(t_{12}, x_{12}) + \frac{\partial}{\partial \alpha} \right) \det(1 + V_0) /_{\alpha=0} \cdot \qquad (2.1)$$

Here

$$t_{12} = t_1 - t_2 , \quad x_{12} = x_1 - x_2 ,$$

$$G(t_{12}, x_{12}) = \int_{-\infty}^{\infty} d\mu e^{it_{12}\mu^2 - ix_{12}\mu} . \qquad (2.2)$$

The last factor in (2.1) is a Fredholm determinant with the kernel

$$V_0(\lambda, \mu) = \left[\frac{E(\lambda) - E(\mu)}{(\lambda - \mu)\pi^2} - \frac{\alpha}{2\pi^3} E(\lambda).E(\mu) \right] \times e^{\frac{i}{2} t_{21}(\lambda^2 + \mu^2)} \times e^{-\frac{i}{2} x_{21}(\lambda + \mu)} . \qquad (2.3)$$

This integral operator acts on the interval $[-q, q](\lambda, \mu \in [-q, q])$. The function $E(\lambda)$ is defined as follows :

$$E(\lambda) = \fint_{-\infty}^{\infty} d\mu \frac{\exp\{it_{12}\mu^2 - ix_{12}\mu\}}{\mu - \lambda} . \qquad (2.4)$$

Now let us discuss temperature - time correlators

$$\langle \psi(x_2, t_2) \psi^+(x_1, t_1) \rangle_T = \frac{tr(e^{-H/T} \psi(x_2, t_2) \psi^+(x_1, t_1))}{tr(e^{-H/T})} . \qquad (2.5)$$

They are given by a similar formula

$$\langle \psi(x_2, t_2) \psi^+(x_1, t_1) \rangle_T = e^{iht_{21}} \left(\frac{1}{2\pi} G(t_{12}, x_{12}) + \frac{\partial}{\partial \alpha} \right) \det(1 + V_T) /_{\alpha=0} \cdot \qquad (2.6)$$

Now the integral operator V_T acts on the whole real axis and the kernel $V_T(\lambda, \mu)$ can be expressed in terms of V_0

$$V_T(\lambda, \mu) = \sqrt{\vartheta(\lambda)} V_0(\lambda, \mu) \sqrt{\vartheta(\mu)} . \qquad (2.7)$$

Here $\vartheta(\lambda) = (1 + \exp\{\lambda^2 - \beta\})^{-1}$. Another interesting object is the Euclidean time correlator. Let us denote $\tau = it$

$$\psi(x, \tau) = e^{H\tau} \psi(x, 0) e^{-H\tau} \qquad (2.8)$$

$$\langle\psi(x_2,\tau_2)\psi^+(x_1,\tau_1)\rangle = \frac{tr(e^{-H/T}\psi(x_2,\tau_2)\psi^+(x_1,\tau_1))}{tr(e^{-H/T})} . \tag{2.9}$$

Let us write down the Fredholm determinant formula for it

$$\langle\psi(x_2,\tau_2)\psi^+(x_1,\tau_1)\rangle_T = e^{h\tau_{21}}\left(\frac{1}{2\pi}\tilde{G}(\tau_{12},x_{12}) + \frac{\partial}{\partial\alpha}\right)\det(1+\tilde{V}_T)/_{\alpha=0} . \tag{2.10}$$

Here $\tilde{V}_T$ is an integral operator on the whole real axis, with the kernel

$$\tilde{V}_T(\lambda,\mu) = \frac{e^{(\tau_{21}(\lambda^2+\mu^2)-ix_{21}(\lambda-\mu))/2}}{\sqrt{1+e^{(\lambda^2-h)/T}}\sqrt{1+e^{(\mu^2-h)/T}}} \times \left\{\frac{\tilde{E}(\lambda)-\tilde{E}(\mu)}{\pi^2(\lambda-\mu)} - \frac{\alpha}{2\pi^3}\tilde{E}(\lambda).\tilde{E}(\mu)\right\} . \tag{2.11}$$

Here $x_{12} = x_1 - x_2$; $\quad 0 \le \tau_{21} = \tau_2 - \tau_1 \le \frac{1}{T}$; $\quad \tau_{12} = -\tau_{21}$

$$\begin{aligned} \tilde{G}(\tau_{12},x_{12}) &= \int_{-\infty}^{\infty} d\mu e^{-\tau_{21}\mu^2 - ix_{12}\mu} \\ \tilde{E}(\mu) &= \fint_{-\infty}^{\infty} \frac{d\lambda}{\lambda-\mu} e^{-\tau_{21}\lambda^2 - ix_{12}\lambda} \end{aligned} . \tag{2.12}$$

In sections 6, 7 we shall derive differential equations, based on these Fredolm determinants.

3. Manypoint field correlators.

In the paper [33] many field correlator is represented as the Fredholm minor :

$$\begin{aligned} \langle\prod_{k=1}^{N}\psi(x_{2k},t_{2k})\psi^+(x_{2k-1},t_{2k-1})\rangle &= e^{ih\sum_{k=1}^{N}(t_{2k}-t_{2k-1})} \\ \cdot\prod_{k=1}^{N}&\left(\frac{1}{2\pi}G(t_{2k-1}-t_{2k},x_{2k-1}-x_{2k}) + \frac{\partial}{\partial\alpha_k}\right)\det\mathcal{K}/_{\alpha_j=0}. \end{aligned} \tag{3.1}$$

Here $\mathcal{K}$ is a Fredholm integral operator acting on the interval $[-q,q]$. The kernel $\mathcal{K}(\lambda_0,\lambda_N)$ of this operator is equal to

$$\begin{aligned} \mathcal{K}(\lambda_0,\lambda_N) = \int_{-\infty}^{\infty} d\lambda_1\ldots d\lambda_{N-1}\prod_{m=1}^{N}\Big[\delta(\lambda_m-\lambda_{m-1})+ \\ + K_m(\lambda_m,\lambda_{m-1})\Big]\cdot\exp\Big(i\{t_{2m}\lambda_m^2 - t_{2m-1}\lambda_{m-1}^2 + \\ + x_{2m-1}\lambda_{m-1} - x_{2m}\lambda_m\}\Big). \end{aligned} \tag{3.2}$$

Here

$$K_m(\lambda,\mu) = \frac{E_m(\lambda)-E_m(\mu)}{\pi^2(\lambda-\mu)} - \frac{\alpha_m}{2\pi^2}E_m(\lambda)\cdot E_m(\mu) \tag{3.3}$$

and

$$E_m(\lambda) = \fint_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\exp\left\{i\mu^2(t_{2m-1}-t_{2m}) - i\mu(x_{2m-1}-x_{2m})\right\} . \tag{3.4}$$

These formulae represent the correlation function at zero temperature. Integral operator $\mathcal{K}$ can be represented in the form

$$\mathcal{K}(\lambda, \mu) = I + V(\lambda, \mu) . \tag{3.5}$$

The kernel of V has canonical form

$$V(\lambda, \mu) = \left(\frac{1}{\lambda - \mu}\right) \sum_{j=1}^{2N} e_j(\lambda) \tilde{e}_j(\mu) . \tag{3.6}$$

This will permit us to derive differential equations which drive correlation function (3.1). To represent the finite temperature correlation function as a Fredholm minor one should replace in (3.1) the operator $\mathcal{K}$ by an operator acting on the whole real axis, its kernel being equal to :

$$\mathcal{K}_T(\lambda, \mu) = I + V_T(\lambda, \mu) \tag{3.7}$$

and

$$V_T(\lambda, \mu) = \sqrt{\vartheta(\lambda)} V(\lambda, \mu) \sqrt{\vartheta(\mu)} .$$

The kernel $V(\lambda, \mu)$ can be taken from (3.5).

4. Finite coupling constant correlator

We shall discuss the penetrable Bose gas (finite coupling constant $0 < c < \infty$) in terms of the Quantum Inverse Scattering Method. In terms of this method, the model can be associated with the monodromy matrix

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{4.1}$$

It is matrix 2×2 with matrix elements being quantum operators (functions of ψ, ψ^+). Their commutation relations are given by the R-matrix

$$R(\lambda, \mu) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) R(\lambda, \mu) . \tag{4.2}$$

The R-matrix is a 4×4 matrix

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu, \lambda) & & & \\ & g(\mu, \lambda) & 1 & \\ & 1 & g(\mu, \lambda) & \\ & & & f(\mu, \lambda) \end{pmatrix} . \tag{4.3}$$

Here

$$g(\mu, \lambda) = \frac{ic}{\mu - \lambda} ; \quad f(\mu, \lambda) = \frac{\mu - \lambda + ic}{\mu - \lambda} \tag{4.4}$$

The bare vacuum $|0\rangle$ plays the role of the highest weight vector :

$$C(\lambda)|0\rangle = 0 \; ; \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle \; ; \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle \; . \tag{4.5}$$

The complex valued functions $a(\lambda)$ and $d(\lambda)$ for NS model are equal to $a(\lambda) = \exp\{\frac{-i\lambda L}{2}\}$, $d(\lambda) = \exp\{\frac{i\lambda L}{2}\}$ (L is the length of periodical box). For the finite coupling constant case $0 < c < \infty$, the ground state of the Bose gas with finite density D (for zero temperature) is still a one-dimensional Fermi sphere $[-q, q]$ (q is the Fermi momentum). Detailed explanations are given for example in [34] (see also references there)). For calculation of correlation functions it is necessary to enumerate all $T(\lambda)$ (4.1) with commutation relations given by (4.2), (4.3) and with highest weight vector (4.5). This is connected with the problem of enumeration of representation of Yangians (associated with $s\ell_2$) with highest weight vector [18]. Solution of this problem can be taken from [21]: there are no restrictions on the functions $a(\lambda)$ and $d(\lambda)$ - they are arbitrary functions. To solve the recursion relations for correlation functions in terms of determinants one should use variational derivatives $\frac{\delta}{\delta a(\lambda)}$, $\frac{\delta}{\delta d(\lambda)}$ with respect to arbitrary functions $a(\lambda)$ and $d(\lambda)$. They act from one irreducible representation of $Y(s\ell_2)$ with highest weight vector into another. This leads to the appearance of dual quanum fields

$$\hat{\varphi}(\lambda) = \hat{a}(\lambda) + \int_{-q}^{q} d\nu \left(\ell n \frac{c^2}{c^2 + (\lambda - \nu)^2} \right) \hat{a}^+(\nu) \tag{4.6}$$

acting in auxiliary Fock space. Here $\hat{a}$ and $\hat{a}^+$ are canonical Bose fields

$$[a(\lambda), a^+(\mu)] = \delta(\lambda - \mu) \; ; \quad [a(\lambda), a(\mu)] = [a^+(\lambda), a^+(\mu)] = 0 \tag{4.7}$$

in auxiliary Fock space, and $|0\rangle$ in Fock vacuum of the new space :

$$\hat{a}(\lambda)|0\rangle = 0 \; ; \quad \langle 0|\hat{a}^+(\lambda) = 0 \; . \tag{4.8}$$

It is remarkable that the dual fields (4.6) commute :

$$[\hat{\varphi}(\lambda), \hat{\varphi}(\mu)] = 0 \; , \tag{4.9}$$

so they can be treated as classical functions. This construction permits us to represent correlation functions as Fredholm determinants [28],[29]. Let us recall that the simplest correlation function is the probability $P(x)$ (0.11) that on the interval $[-x, x]$ there will be no particles. Let us write it down as a Fredholm determinant using the dual quantum fields $\varphi(\lambda)$ (4.6)

$$P(x) = \frac{\langle 0|\det(I + V)|0\rangle}{\det(I - \frac{1}{2\pi}K)} \; . \tag{4.10}$$

Both integral operators V and K act on the interval $[-q,q]$. The kernel of $K(\lambda,\mu)$ is equal to

$$K(\lambda,\mu)=\frac{2c}{c^2+(\lambda-\mu)^2}\,. \tag{4.11}$$

The kernel of $V(\lambda_1,\lambda_2)$ is expressed in terms of the dual fields :

$$V(\lambda_1,\lambda_2)=\frac{-c}{2\pi}\left\{\frac{e^{ix\lambda_1+\frac{1}{2}\varphi(\lambda_1)}\cdot e^{-ix\lambda_2-\frac{1}{2}\varphi(\lambda_2)}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_2+ic)}+\frac{e^{ix\lambda_2+\frac{1}{2}\varphi(\lambda_2)}\cdot e^{-ix\lambda_1-\frac{1}{2}\varphi(\lambda_1)}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_1+ic)}\right\} \tag{4.12}$$

This is the answer. Remember that in the limit $c\to\infty$ (q-fixed)

$$\begin{gathered}\hat{\varphi}(\lambda)\to 0\,;\quad K\to 0\,;\quad c\to\infty\\ V(\lambda_1,\lambda_2)\to-\frac{1}{\pi}\frac{\sin x(\lambda_1-\lambda_2)}{\lambda_1-\lambda_2}\,.\end{gathered} \tag{4.13}$$

This is in accordance with (1.15). Let us rewrite $V(\lambda_1,\lambda_2)$ in the form

$$V(\lambda_1,\lambda_2)=i\int_0^\infty dse^{-sc}\left\{\frac{e_+(\lambda_1)e_-(\lambda_2)-e_+(\lambda_2)e_-(\lambda_1)}{\lambda_1-\lambda_2}\right\} \tag{4.14}$$

$$e_\pm(\lambda)=\sqrt{\frac{c}{2\pi}}\exp\left\{\pm(ix\lambda+is\lambda+\frac{1}{2}\varphi(\lambda))\right\}\,. \tag{4.15}$$

Now the integral operator V for finite coupling constant is represented in a form similar to $c=\infty$ but we have an additional variable s.

II. Differential equations for correlation functions

Now we shall use Fredholm determinants (of integral operators obtained in part I) to derive differential equations, which drive correlation functions,. One should note that all Fredholm integral operators which occur have very special kernels $V(\lambda_1,\lambda_2)$. The product $(\lambda_1-\lambda_2)V(\lambda_1,\lambda_2)$ is equal to the sum of one-dimensional projectors

$$(\lambda_1-\lambda_2)V(\lambda_1,\lambda_2)=\sum_{j=1}^N e_j(\lambda_1)E_j(\lambda_2) \tag{II.0}$$

(for finite coupling constant the sum is replaced by integration). Functions $e_j(\lambda),E_j(\lambda)$ should also obey the constraint $\sum_{j=1}^N e_j(\lambda)E_j(\lambda)=0$.Such an integral operator one can call "completely integrable". The product of two such operators has the same form. Let

$$\begin{aligned}V^{(1)}(\lambda_1,\lambda_2)&=\sum_{j=1}^N\frac{e_j^{(1)}(\lambda_1)E_j^{(1)}(\lambda_2)}{\lambda_1-\lambda_2}\\ V^{(2)}(\lambda_1,\lambda_2)&=\sum_{k=1}^M\frac{e_k^{(2)}(\lambda_1)\cdot E_k^{(2)}(\lambda_2)}{\lambda_1-\lambda_2}\,.\end{aligned} \tag{II.1}$$

The product of two such operators is equal to

$$\int_{-q}^{q} V^{(1)}(\lambda_1,\nu)V^{(2)}(\nu,\lambda_2)d\nu = V^{(3)}(\lambda_1,\lambda_2) =$$
$$= \frac{1}{\lambda_1-\lambda_2}\left\{\sum_{j=1}^{N} e_j^{(1)}(\lambda_1)E_j^{(3)}(\lambda_2) + \sum_{K=1}^{M} e_K^{(3)}(\lambda_1)E_K^{(2)}(\lambda_2)\right\} . \qquad (II.2)$$

Here

$$E_j^{(3)}(\lambda) = \sum_{K=1}^{M} E_K^{(2)}(\lambda)\int_{-q}^{q}\frac{d\nu}{\nu-\lambda}E_j^{(1)}(\nu)e_K^{(2)}(\nu)$$
$$e_j^{(3)}(\lambda) = \sum_{j=1}^{N} e_j^{(1)}(\lambda)\int_{-q}^{q}\frac{d\nu}{\lambda-\nu}E_j^{(1)}(\nu)e_K^{(2)}(\nu) . \qquad (II.3)$$

This is a very important property, which shows that the resolvent of this operator can be constructed universally, as follows. Let us take kernel operator (II.0). We introduce the resolvent R in the following way

$$(I+V)(I-R) = I \; ; \quad (I+V)R = V . \qquad (II.4)$$

Let us write down the integral equation for the kernel of the resolvent $R(\lambda_1,\lambda_2)$:

$$R(\lambda_1,\lambda_2) + \int_{-q}^{q} V(\lambda_1,\nu)R(\nu,\lambda_2)d\nu = V(\lambda_1,\lambda_2) . \qquad (II.5)$$

To write down $R(\lambda_1,\lambda_2)$ in explicit form let us introduce functions$f_j^L(\lambda)$ and $f_j^R(\lambda)$ by the following integral equations :

$$f_j^R(\lambda) + \int_{-q}^{q} V(\lambda,\mu)f_j^R(\mu)d\mu = e_j(\lambda) \qquad (II.6)$$

$$f_j^L(\lambda) + \int_{-q}^{q} V(\mu,\lambda)f_j^L(\mu)d\mu = E_j(\lambda) . \qquad (II.7)$$

Then one can prove the theorem that

$$R(\lambda_1,\lambda_2) = \frac{\sum_{j=1}^{N} f_j^L(\lambda_1)f_j^R(\lambda_2)}{\lambda_1-\lambda_2} . \qquad (II.8)$$

Formulae (II.2)-(II.8) show that operators $(1+V)$ (here V is given by (II.0) or (II.1) form infinite dimensional group. This permits us to derive differential equations (from integral one), its Lax representation and the associated Riemann problem. We shall do this in more detail for equal time finite-temperature correlators in section 5. All other cases are treated in a similar way. We shall construct differential equations for which these "completely integrable"integral operators play the role of Gelfand-Levitan operators. One should mention that direct quantization of the Gelfand-Levitan equation was proposed in [35]. It will be interesting to understand the connections between these two approaches.

5. Finite-temperature equal time correlator

Let us come back to section 1. The finite-temperature equal time correlators of an impenetrable Bose gas were described as Fredholm determinants $\det(I-\gamma K)$ of an integral operator K (on the real axis)

$$K(\lambda,\mu)=\sqrt{\vartheta(\lambda)}\frac{\sin x(\lambda-\mu)}{\lambda-\mu}\sqrt{\vartheta(\mu)}=\frac{e_+(\lambda)e_-(\mu)-e_-(\lambda)e_+(\mu)}{2i(\lambda-\mu)}=K(\mu,\lambda)\ . \tag{5.1}$$

Here

$$e_\pm(\lambda)=\sqrt{\vartheta(\lambda)}e^{\pm i\lambda x} \tag{5.2}$$

and

$$\vartheta(\lambda)=(1+\exp(\lambda^2-\beta))^{-1}\ .$$

Functions $f_\pm(\lambda)$ are constructed by (1.6)

$$f_\pm(\lambda)-\gamma\int_{-\infty}^{\infty}K(\lambda,\mu)f_\pm(\mu)d\mu=e_\pm(\lambda)\ . \tag{5.3}$$

Let us define the inverse operator as follows

$$(I-\gamma K)(I+\gamma R)=I\ ;\quad (I-\gamma K)R=K\ . \tag{5.4}$$

The kernel of the resolvent R is equal to

$$R(\lambda,\mu)=\frac{f_+(\lambda)f_-(\mu)-f_-(\lambda)f_+(\mu)}{2i(\lambda-\mu)}=R(\mu,\lambda)\ . \tag{5.5}$$

To construct a nonlinear differential equation for B_{++} (see (1.8)), first let us construct a Lax representation of this equation. Let us differentiate (5.3) with respect to x, the result can be written in the vector form :

$$\frac{\partial}{\partial x}\vec{f}=(i\lambda\sigma_3+Q)\vec{f}\ . \tag{5.6}$$

Here $\vec{f}$ is a two-component vector-function

$$\vec{f}(\lambda)=\begin{pmatrix} f+(\lambda) \\ f_-(\lambda)\end{pmatrix} \tag{5.7}$$

and Q is 2×2 matrix

$$Q=\begin{pmatrix} 0 & B_{++} \\ B_{++} & 0\end{pmatrix}\ . \tag{5.8}$$

Here we used that $\partial_x K(\lambda,\mu)$ is sum of one-dimensional projectors

$$2\partial_x K(\lambda,\mu) = e_+(\lambda)e_-(\mu) + e_-(\lambda)e_+(\mu) .$$

We shall consider (5.6) as the L operator; to construct the M opertor we apply $(2\lambda\partial_\beta + \partial_\lambda)$ to the equation (5.3). This gives the M operator

$$(2\lambda\partial_\beta + \partial_\lambda)\vec{f} = (ix\sigma_3 - i\partial_\beta V)\vec{f} \tag{5.9}$$

Here

$$V = \begin{pmatrix} B_{+-} & -B_{++} \\ B_{++} & -B_{+-} \end{pmatrix} \tag{5.10}$$

and we used $(2\lambda\partial_\beta + \partial_\lambda)\vartheta(\lambda) = 0$. Write now the compatibility conditions for the system (5.6) and (5.9)

$$[\partial_x - i\lambda\sigma_3 - Q, 2\lambda\partial_\beta + \partial_\lambda - i\lambda\sigma_3 + i\partial_\beta V] = 0 \tag{5.11}$$

at any λ. We arrive at the equations

$$\begin{aligned} \partial_x B_{+-} &= B_{++}^2 \\ \partial_\beta B_{++}^2 &= 1 + \partial_x\left(\frac{\partial_x\partial_\beta B_{++}}{2B_{++}}\right) . \end{aligned} \tag{5.12}$$

Here ∂_β is a derivative with respect to our independent variable $\beta = \frac{h}{T}$. Let us denote by $\sigma(x,\beta)$

$$\sigma(x,\beta) = \ell n \det(I - \gamma K) . \tag{5.13}$$

One can show [27] that $\sigma|_{x=0} = 0$

$$\partial_x^2\sigma = -B_{++}^2 , \quad \partial_x\sigma = -B_{+-} \tag{5.14}$$

and σ itself satisfies the differential equation

$$(\partial_\beta\partial_x^2\sigma)^2 = -4(\partial_x^2\sigma)[2x\partial_\beta\partial_x\sigma + (\partial_\beta\partial_x\sigma)^2 - 2\partial_\beta\sigma] . \tag{5.15}$$

The initial data for this equation can be extracted from the Fredholm determinant itself

$$\begin{aligned} &\lim_{\beta\to-\infty} B_{++} = \lim_{\beta\to-\infty} \sigma = 0 \\ &B_{++} = \gamma\int\vartheta(\lambda)d\lambda + x\gamma^2\left(\int\vartheta(\lambda)d\lambda\right)^2 + O(x^2) \\ &x \to 0 \\ &\sigma = -x\gamma\int\vartheta(\lambda) - \frac{x^2}{2}\left(\int\vartheta(\lambda)d\lambda\right)^2 + O(x^3) . \end{aligned} \tag{5.16}$$

This fixes the solution uniquely [27] and describes completely the correlation function (1.10), (1.3). It is interesting to mention that at $T = 0$ the value $\sigma = \ell n \det(I - \gamma K)$ depends only on the product of variables x and $\sqrt{\beta}$; $\tau = x\sqrt{\beta} = x\sqrt{h}$. Equation (5.15) is conveniently rewritten for a function

$$\sigma_0(\tau) = \tau \frac{\partial}{\partial \tau} \ell n \det(I - \gamma K)$$

One has

$$(\tau\sigma_0'')^2 = -4(\tau\sigma_0' - \sigma_0)(4\tau\sigma_0' + (\sigma_0')^2 - 4\sigma_0) \,. \tag{5.17}$$

This is the famous fifth Painlève transcendent of [5]. To conclude this section, let us present a matrix Riemann problem, associated with these equations [27]. Consider the 2×2 matrix $\chi(\lambda)$ function of λ.

1) it is holomorphic at $Im(\lambda) > 0$ and $Im(\lambda) < 0$
2) $\chi(\infty) = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
3) $\det \chi(\lambda) \neq 0 \; \forall \lambda$
4) the boundary values $\chi^{\pm}(\lambda)$ at the real axis $\lambda \in R$ are related

$$\chi^-(\lambda) = \chi^+(\lambda) G(\lambda) \, ; \quad Im\lambda = 0 \tag{5.18}$$

$$G(\lambda) = I + \pi\gamma \begin{pmatrix} e_+(\lambda)e_-(\lambda) & -e_+^2(\lambda) \\ e_-^2(\lambda) & -e_+(\lambda)e_-(\lambda) \end{pmatrix} \tag{5.19}$$

This Riemann problem permits one to calculate the long distance asymptotics of correlators. A Riemann problem can be associated with each of the differential equations constructed below. Let us mention also that differential equations for many point correlators are constructed in [27].

6. Time dependent correlator

Here we shall take the time correlator of section 2 and derive a differential equation which drives this correlator. This will be the classical nonlinear Schrödinger equation. So we shall study $\langle \psi(x_2, t_2)\psi^+(x_1, t_1)\rangle$, $(c = \infty)$ for Euclidean time

$$t_1 - t_2 = 2i\tau \, ; \quad \tau > 0 \, ; \quad x_1 - x_2 = 2x \, ; \quad x > 0 \, . \tag{6.1}$$

Let us evaluate the derivative in (2.1)

$$\langle \psi(2)\psi^+(1)\rangle \equiv \langle \psi(x_2, t_2)\psi^+(x_1, t_1)\rangle = \frac{-1}{2\pi} e^{2\tau q^2} b_{++} \det(I + V_0) \, . \tag{6.2}$$

Here V_0 is the integral operator on the interval $[-q, q]$ (for b_{++} see (6.10)):

$$f_{\pm}(\lambda) + \int_{-1}^{q} V_0(\lambda, \mu) f_{\pm}(\mu) d\mu = e_{\pm}(\lambda) \, . \tag{6.3}$$

The kernel $V_0(\lambda,\mu)$ can be represented in the form :

$$V_0(\lambda,\mu) = \frac{e_+(\lambda)e_-(\mu) - e_+(\mu)e_-(\lambda)}{\lambda-\mu} \; . \tag{6.4}$$

Here

$$\begin{aligned} e_-(\lambda) &= \frac{1}{\pi}e^{\tau\lambda^2+ix\lambda} \; ; \; e_+(\lambda) = e_-(\lambda)\cdot E(\lambda) \\ E(\lambda) &= \fint \frac{d\nu}{\nu-\lambda}e^{-2\tau\nu^2-2ix\nu} \end{aligned} \; . \tag{6.5}$$

The resolvent R is defined as usual

$$(I+V)(I-R) = I \; ; \quad (I+V)R = V \; . \tag{6.6}$$

Its kernel can be represented in terms of functions $f_\pm$ (6.3)

$$R(\lambda,\mu) = \frac{f_+(\lambda)f_-(\mu) - f_+(\mu)f_-(\lambda)}{\lambda-\mu} \; . \tag{6.7}$$

It is important to introduce potentials

$$B_{jk} = \int_{-q}^{q} d\mu e_j(\mu)f_k(\mu) \; ; \quad j,k = \pm \tag{6.8}$$

$$C_{jk} = \int_{-q}^{q} d\mu \; \mu e_j(\mu)f_k(\mu) \; . \tag{6.9}$$

Let us introduce also b_{++}

$$\begin{aligned} b_{++} &= B_{++} - G \\ G &= \int_{-\infty}^{\infty} d\mu e^{-2\tau\mu^2-2ix\mu} \end{aligned} \; . \tag{6.10}$$

Let us mention now some symmetry properties : First of all V is symmetric :

$$V(\lambda,\mu) = V(\mu,\lambda) \; ; \quad R(\lambda,\mu) = R(\mu,\lambda) \tag{6.11}$$

$$B_{jk} = B_{kj} \; . \tag{6.12}$$

Second, there exists a complex involution

$$\bar{e}_-(\lambda) = e_-(-\lambda) \; ; \quad \bar{e}_+(\lambda) = -e_+(-\lambda) \; ; \quad \bar{f}_\pm(-\lambda) = \mp f_\pm(\lambda) \tag{6.13}$$

$$\overline{V(\lambda,\mu)} = V(-\lambda,-\mu) \; ; \quad \overline{R(\lambda,\mu)} = R(-\lambda,-\mu) \; . \tag{6.14}$$

So B_{++}, b_{++} and B_{--} are real

$$\overline{B}_{++} = B_{++} \; ; \overline{b}_{++} = b_{++} \; ; \overline{B}_{--} = B_{--} \tag{6.15}$$

but B_{+-} is pure imaginary

$$\overline{B}_{+-} = -B_{+-} \, . \tag{6.16}$$

Third, we mention also that

$$\begin{aligned} &\overline{C}_{++} = -C_{++}; \quad \overline{C}_{--} = -C_{--}; \quad \overline{C}_{+-} = C_{+-}; \\ &\overline{C}_{-+} = C_{-+}; \quad C_{-+} = C_{+-} + B_{++}B_{--} - B_{+-}^2 \, . \end{aligned} \tag{6.17}$$

6. a. Lax representation

In this subsection we shall derive a Lax representation, for a nonlinear differential equation, which will drive the correlator (6.2). First we take the solutions of (6.3) $f_{\pm}(\lambda)$ and define vector-function

$$\vec{f}(\lambda) \equiv \begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix}$$

It is easy to differentiate (6.3) with respect to x, because $\frac{\partial}{\partial x}V_0(\lambda,\mu)$ is equal to the sum of one-dimensional projectors (similar to section 5). The result is the L opertor :

$$\left(\frac{\partial}{\partial x} + i\lambda\sigma_3 - 2iQ\right)\vec{f} = 0 \tag{6.18}$$

Here

$$Q = \begin{pmatrix} 0 & b_{++} \\ B_{--} & 0 \end{pmatrix} \tag{6.19}$$

see (6.10), (6.8). In a similar way one can differentiate (6.3) with respect to time τ. This gives the M operator :

$$\left(\frac{\partial}{\partial \tau} + \lambda^2\sigma_3 - 2\lambda Q - V\right)\vec{f} = 0 \, . \tag{6.20}$$

Here

$$V = \begin{pmatrix} 2b_{++}B_{--} & i\partial_x b_{++} \\ -i\partial_x B_{--} & -2B_{--}b_{++} \end{pmatrix} \, . \tag{6.21}$$

Compatibility condition for (6.18) and (6.20) leads to :

$$\left[\frac{\partial}{\partial x} + i\lambda\sigma_3 - 2iQ; \frac{\partial}{\partial \tau} + \lambda^2\sigma_3 - 2\lambda Q - V\right] = 0 \, . \tag{6.22}$$

This gives the nonlinear Schrödinger equation

$$\begin{cases} \frac{\partial}{\partial \tau}B_{--} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}B_{--} - 4B_{--}^2 b_{++} \\ \frac{\partial}{\partial \tau}b_{++} = \frac{1}{2}\frac{\partial^2}{\partial x^2}b_{++} + 4b_{++}^2 B_{--} \end{cases} \tag{6.23}$$

Initial data $\tau = 0$ for these equations can be extracted from the description [5] of equal time correlator $\langle\psi^+(x)\psi(-x)\rangle$. This also gies the expression for C^{++} and C_{--}

$$\begin{cases} C_{++} = \frac{i}{2}\partial_x B_{++} + B_{++}\cdot B_{+-} - 2GB_{+-} \\ C_{--} = -\frac{i}{2}\partial_x B_{--} - B_{+-}B_{--} \end{cases} \tag{6.24}$$

Direct differentiation of (6.8) (using (6.3)) gives also

$$\frac{\partial}{\partial x}B_{+-} = 2iB_{--}b_{++} \; . \tag{6.25}$$

This permits us to express matrix V (6.21) in terms of the matrix U

$$U = \begin{pmatrix} -B_{+-} & b_{++} \\ -B_{--} & B_{+-} \end{pmatrix} \tag{6.26}$$

$$V = i\partial x U \; . \tag{6.27}$$

6. b. Expression for the correlation function in terms of NS equation.

The field correlator (6.2) is equal to the product of two factors b_{++} and $\det(I + V_0)$. One can get a differential equation for b_{++} by substituting one of eq. (6.23) into another

$$2\frac{\partial}{\partial\tau}\left(\frac{2\dot{b}_{++} - b''_{++}}{b^2_{++}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{2\dot{b}_{++} - b''_{++}}{b^2_{++}}\right) + b_{++}\left(\frac{2\dot{b}_{++} - b''_{++}}{b^2_{++}}\right)^2 = 0 \; . \tag{6.28}$$

To characterize $\det(I + V_a)$ let us introduce the notation

$$\sigma = \ell n \det(I + V_0) \; . \tag{6.29}$$

At $\tau = 0$, function $\sigma(x)$ was described in [5]. Derivatives of $\sigma(x,\tau)$ can be expressed in terms of solutions of the NS equation (6.23)

$$\frac{\partial\sigma}{\partial x} = -2iB_{+-} \; ; \quad \frac{\partial^2\sigma}{\partial x^2} = 4b_{++}B_{--} \tag{6.30}$$

$$\frac{\partial\sigma}{\partial x} = -2GB_{--} - 2(C_{+-} + C_{-+}) \; . \tag{6.31}$$

Straightforward calculations involving (6.3), (6.8), (6.9) show also that

$$\begin{gathered} \frac{\partial}{\partial x}(C_{+-} + C_{-+}) = (B_{++} - 2G)\frac{\partial B_{--}}{\partial x} - B_{--}B'_{++} \\ i\frac{\partial B_{+-}}{\partial\tau} = b_{++}\frac{\partial B_{--}}{\partial x} - B_{--}\frac{\partial b_{++}}{\partial x} \; . \end{gathered} \tag{6.32}$$

This set of equations completely describes the correlator $\langle\psi(x_2,t_2)\psi^+(x_1,t_1)\rangle$. The associated Riemann problem can be constructed similar to section 5. This permits us to evaluate asymptotics.

7. Finite temperature time correlator.

Here we shall consider the temperature time correlator of an impenetrable Bose gas $c=\infty, \langle\psi(x_2,t_2)\psi^+(x_1,t_1)\rangle_T$. The Fredholm determinant representation can be extracted from section 2. Let us consider Euclidean time

$$t_1-t_2=2i\tau\ ;\quad \tau>0\ ;\quad x_1-x_2=2x\ ;\quad x>0\ . \tag{7.1}$$

Here we shall use the notation of section 6. We shall use also the variable $\beta=\frac{h}{T}$, the same as in (1.2) and section 5, and the Fermi weight

$$\vartheta(\lambda)=(1+\exp\{\lambda^2-\beta\})^{-1}\ . \tag{7.2}$$

So

$$\langle\psi(x_2,t_2)\psi^+(x_1,t_1)\rangle_T=\frac{-1}{2\pi}e^{2\tau\beta}b_{++}\det(I+V_T)\ . \tag{7.3}$$

Here V_T is an integral operator on the whole real axis

$$f_\pm(\lambda)+\int_{-\infty}^{\infty}V_T(\lambda,\mu)f_\pm(\mu)d\mu=e_\pm(\lambda) \tag{7.4}$$

with the kernel:

$$V_T(\lambda,\mu)=\frac{e_+(\lambda)e_-(\mu)-e_+(\mu)e_-(\lambda)}{(\lambda-\mu)}=V_T(\mu\lambda) \tag{7.5}$$

and

$$e_-(\lambda)=\frac{1}{\pi}\sqrt{\vartheta(\lambda)}e^{\tau\lambda^2+ix\lambda}\ ;\ e_+(\lambda)=e_-(\lambda)E(\lambda) \tag{7.6}$$

$$E(\lambda)=⨍_{-\infty}^{\infty}\frac{d\nu}{\nu-\lambda}e^{-2\tau\nu^2-2ix\nu}\ ;\ G=\int_{-\infty}^{\infty}d\mu e^{-2\tau\mu^2-2ix\mu}\ . \tag{7.7}$$

The functions $f_\pm(\lambda)$ are defined by (7.4) and the kernel of the resolvent is given by (6.7). The potentials

$$B_{ik}=\int_{-\infty}^{\infty}e_i(\mu)f_k(\mu)d\mu\ ;\ C_{ik}=\int_{-\infty}^{\infty}d\mu e_i(\mu)f_k(\mu)\mu \tag{7.8}$$

now depend on three variables $x,\tau\beta$. The symmetry properties (6.11)-(6.17) are still valid.

7. a. Lax representation.

Let us define a vector-function, as usual

$$\vec{f}(\lambda)=\begin{pmatrix}f_+(\lambda)\\ f_-(\lambda)\end{pmatrix}\ . \tag{7.9}$$

Differentiate (7.4) with respect to x, τ; also apply the operator $(2\lambda\partial_\beta + \partial_\lambda)$ (as in (5.9)). In this way we get three operators

$$\hat{L}(\lambda)\vec{f} = 0\ ; \quad \hat{M}(\lambda)\vec{f} = 0\ ; \quad \hat{N}(\lambda)\vec{f} = 0\ . \tag{7.10}$$

The operators L and M are the same as in (6.18) and (6.20)

$$\hat{L}(\lambda) = \frac{\partial}{\partial x} + i\lambda\sigma_3 - 2iQ \tag{7.11}$$

$$\hat{M}(\lambda) = \frac{\partial}{\partial \tau} + \lambda^2\sigma_3 - 2\lambda Q - V\ . \tag{7.12}$$

Formulae (6.19), (6.21), (6.27) and (6.26) are still valid. The operator $\hat{N}$ is similar to (5.9), but now we have

$$\hat{N}(\lambda) = 2\lambda\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\lambda} + 2\lambda\tau\sigma_3 + ix\sigma_3 - 4\tau Q - 2\partial_\beta U\ . \tag{7.13}$$

Here

$$Q = \begin{pmatrix} 0 & b_{++} \\ B_{--} & 0 \end{pmatrix}\ ; \quad U = \begin{pmatrix} -B_{+-} & b_{++} \\ -B_{--} & B_{+-} \end{pmatrix} \tag{7.14}$$

7. b. Three dimensional completely integrable differential equation.

All three operators L, M, N should commute at arbitrary value of spectral parameter $\lambda, [L(\lambda), M(\lambda)] = 0, [L(\lambda, N(\lambda)] = 0$ and $[M(\lambda), N(\lambda)] = 0$. Now let us write down nonlinear differential equations for the potentials B_{ik} and C_{ik}. First of all (6.23)-(6.25) are still valid. To write down a complete set of equations we introduce the notation :

$$g_- = e^{-2\tau\beta}B_{--}\ ; \quad g_+ = e^{2\tau\beta}b_{++} \tag{7.15}$$

$$n(x,\tau,\beta) \equiv g_+g_-\ ; \quad p(x,\tau,\beta) \equiv g_-\partial_x g_+ - g_+\partial_x g_-\ . \tag{7.16}$$

First of all the NS equations are valid

$$\begin{array}{lll} \text{(i)} & \partial_\tau g_+ & = 2_\beta g_+ + \frac{1}{2}\partial_x^2 g_+ + 4g_+^2 g_- \\ \text{(ii)} & \partial_\tau g_- & = -2_\beta g_- - \frac{1}{2}\partial_x^2 g_- - 4g_-^2 g_+ \\ \text{(iii)} & 2\dot{n} & = p' \end{array} \tag{7.17}$$

The first equation containing the β derivative looks like

$$\frac{\partial_\beta\partial_x g_+}{g_+} = \frac{\partial_\beta\partial_x g_-}{g_-} \equiv \varphi(x,\tau,\beta)\ . \tag{7.18}$$

Here we define a new function $\varphi(x,\tau,\beta)$. The two remaining equations look like

$$\begin{aligned} &\partial_\tau\varphi + 4\partial_\beta p = 0 \\ &\partial_x\varphi + 2 + 8\partial_\beta n = 0 \end{aligned} \tag{7.19}$$

Initial data $\tau = 0$ for this equations can be extracted from equal time temperature correlator $\langle\psi^+(x)\psi(-x)\rangle_T$, see sections 1, 5. So (7.17)-(7.19) are a complete set of equations for b_{++} and B_{--}. Other potentials are defined in terms of these two. The derivatives of B_{+-} are equal to :

$$\begin{aligned} \partial_x B_{+-} &= 2in \ , \partial_\tau B_{+-} = ip \\ \partial_\beta B_{+-} &= -\frac{i}{2}x - \frac{i}{4}\varphi \end{aligned} \tag{7.20}$$

The potentials C_{ik} are still given by the expression (6.24), (6.32).

7. c. Expression for correlation function.

The correlator (7.3) is equal to the product of g_{++} and e^σ

$$\sigma = \ell n \det(I + V_T) \ . \tag{7.21}$$

It is easy to get an equation involving only g_+ because g_- can be expressed, from(7.17)(i)

$$8g_- = \frac{2\dot{g}_+ - 4_\beta g_+ + \partial_x^2 g_+}{g_+^2} \ . \tag{7.22}$$

One can substitute this expression in all other equations of (7.17)-(7.19) to get a complete set of equations for g_+. At $\tau = 0, \sigma$ was described at section 5, its derivatives can be expressed in terms of the potentials B_{ik}, C_{ik}

$$\frac{\partial\sigma}{\partial x} = -2iB_{+-}; \frac{\partial\sigma}{\partial\tau} = -2GB_{--} - 2(C_{+-} + C_{-+}) \tag{7.23}$$

$$\begin{aligned} \frac{\partial\sigma}{\partial\beta} = &-2\tau\partial_\beta(C_{+-} + C_{-+}) - 2ix\partial_\beta B_{+-} - 2\tau B_{--}\partial_\beta B_{--} + \\ &+ 2\tau(B_{++} - 2G)\partial_\beta B_{--} + 2(\partial_\beta B_{++})(\partial_\beta B_{--}) - 2(\partial_\beta B_{+-})^2 \end{aligned} \tag{7.24}$$

This completely defines the correlation function. The corresponding Riemann problem (similar to section 5) also can be constructed.

8. Integro-differential equation for the finite coupling correlator.

Here we shall derive trhe integro-differential equation which drives the simplest correlator at finite coupling $0 < c < \infty$. This is probability of absence of particles on the interval $[-x, x]$ [see (4.10, (0.11)] ;

$$P(x) = \frac{\langle 0|\det(I + V)|0\rangle}{\det(I - \frac{1}{2\pi}K)} \ . \tag{8.1}$$

Only the numerator depend on the distance x. So we shall investigate here

$$\det(I + V) \ . \tag{8.2}$$

The integral operator V acts on the interval $[-q, q]$, its kernel $V(\lambda_1, \lambda_2)$ is given by (4.12). it depends on the dual quantum field $\varphi(\lambda)$ (4.6), but it commutes at different values of spectral parameters $[\varphi(\lambda), \varphi(\mu)] = 0$ so we shall treat $\varphi(\lambda)$ and $V(\lambda_1, \lambda_2)$ (4.12) as classical unknown functions. Let us use (4.14) as representations for

$$V(\lambda_1, \lambda_2) = i \int_0^\infty ds(e^{-sc}) \frac{e_+(\lambda_1|s)e_-(\lambda_2|s) - e_+(\lambda_2|s)e_-(\lambda_1|s)}{\lambda_1 - \lambda_2}$$
$$e_\pm(\lambda|s) = \sqrt{\frac{c}{2\pi}} \exp\left\{\pm(ix\lambda + is\lambda + \frac{1}{2}\varphi(x))\right\} . \tag{8.3}$$

One should emphasize that this V_∞ has the canonical form (II.0) with $\sum_{j=1}^N$ replaced by the integral $\int_0^\infty ds$, so the formulae of previous section can be generalized. Let us define the resolvent in the standard way

$$(I+V)(I-R) = I \qquad (I+V)R = V$$
$$R(\lambda_1, \lambda_2) + \int_{-q}^{q} V(\lambda_1, \lambda_3)R(\lambda_3, \lambda_2)d\lambda_3 = V(\lambda_1, \lambda_2) . \tag{8.4}$$

It is symmetric

$$V(\lambda_1, \lambda_2) = V(\lambda_2, \lambda_1) \; ; \quad R(\lambda_1, \lambda_2) = R(\lambda_2, \lambda_1) . \tag{8.5}$$

To write down an explicit formulae for R, we introduce functions $f_\pm(\lambda|s)$:

$$f_\pm(\lambda|s) + \int_{-q}^{q} V(\lambda, \mu) f_\pm(\mu|s) d\mu = e_\pm(\lambda|s) . \tag{8.6}$$

The kernel of the resolvent can be represented in the form :

$$R(\lambda, \mu) = i(\lambda - \mu)^{-1} \int_0^\infty dse^{-sc}\{f_+(\lambda|s)f_-(\mu|s) - f_-(\lambda|s)f_+(\mu|s)\} . \tag{8.7}$$

Now define potentials $B_{ik}(s,t)$

$$B_{ik}(s,t) = \int_{-q}^{q} d\mu e_i(\mu|s) f_k(\mu|t) \, , \quad i, k = \pm \tag{8.8}$$

not it is the kernel of matrix integral operator. It is symmetric

$$B_{ik}(s,t) = B_{ki}(t,s) . \tag{8.9}$$

Differentiate (8.6) with respect to x, to construct the L operator (it is an integral operator):

$$\frac{\partial}{\partial x} \vec{f}(\lambda|t) = i\lambda\sigma_3 \vec{f}(\lambda|t) + 2\int_0^\infty dse^{-sc} Q(t,s) \vec{f}(\lambda|s) . \tag{8.10}$$

Here we introduce the 2 component vector $\vec{f}$ and matrix Q

$$\vec{f}(\lambda|s) = \begin{pmatrix} f_+(\lambda|s) \\ f_-(\lambda|s) \end{pmatrix} ; \quad Q(t|s) = \begin{pmatrix} 0 & B_{++}(s,t) \\ B_{--}(t,s) & 0 \end{pmatrix} ; \tag{8.11}$$

$$Q(t,s) = Q(s,t) .$$

Now let us differentiate (8.6) with respect to q. We find

$$\frac{\partial}{\partial q}\vec{f}(\lambda|t) + i\int_0^\infty dse^{-sc}\mathcal{U}(t,s)\vec{f}(\lambda|s) = 0, \tag{8.12}$$

$$\mathcal{U}(t,s) = \frac{A_+(s,t)}{\lambda - q} + \frac{A_-(s,t)}{\lambda + q} \tag{8.13}$$

Here

$$A_+(s,t) = \begin{pmatrix} f_-(q|s)f_+(q|t) & -f_+(q|s)f_+(q|t) \\ f_-(q|s)f_-(q|t) & -f_+(q|s)f_-(q|t), \end{pmatrix}, \tag{8.14}$$

$$i\int_0^\infty dse^{-sc}A_+(t,s)A_+(s,u) = 0,$$

$$A_-(s,t) = \begin{pmatrix} f_-(-q|s)f_+(-q|t) & -f_+(-q|s)f_+(-q|t) \\ f_-(-q|s)f_-(-q|t) & -f_+(-q|s)f_-(-q|t) . \end{pmatrix} \tag{8.15}$$

Finally we write the compatibility condition for equations (8.10) and (8.12). In this way we get

$$\begin{aligned} &\frac{\partial}{\partial q}B_{++}(s,t) = f_+(q|s)f_+(q|t) + f_+(-q|s)f_+(-q|t), \\ &\frac{\partial}{\partial q}B_{--}(s,t) = f_-(q|s)f_-(q|t) + f_-(-q|s)f_-(-q|t) \end{aligned} , \tag{8.16}$$

$$\begin{aligned} \frac{\partial}{\partial x}f_+(q|t) &= iqf_+(q|t) + 2\int_0^\infty dse^{-sc}B_{++}(t,s)f_-(q|s), \\ \frac{\partial}{\partial x}f_-(q|t) &= -iqf_-(q|t) + 2\int_0^\infty dse^{-sc}B_{--}(t,s)f_+(q|s), \\ \frac{\partial}{\partial x}f_+(-q|t) &= -iqf_+(-q|t) + 2\int_0^\infty dse^{-sc}B_{++}(t,s)f_-(-q/s), \\ \frac{\partial}{\partial x}f_-(-q|t) &= iqf_-(-q|t) + 2\int_0^\infty dse^{-sc}B_{--}(t,s)f_+(-q|s) . \end{aligned} \tag{8.17}$$

This is a complete set of equations for 6 unknown functions ;

$$B_{++}(s,t) \; ; \; B_{--}(s,t) \; ; \; f_+(q|s) \; ; \; f_-(q|s) \; ; \; f_+(-q|s) \; ; \; f_-(-q|s) .$$

The associated Riemann problem can be constructed in a way similar to section 5. So (8.16) and (8.17) is our integro-differential equation, which drives the correlation function at finite coupling constant.

8. a. Correlation function,.

To study the space dependence of $P(x)$ (8.1), we shall express $\det(I+V)$ (8.2) in terms of solutions of (8.16) and (8.17). First of all we have as usual

$$\frac{\partial}{\partial x}\ell n \det(I+V) = -2\int_0^\infty dse^{-sc}B_{+-}(s,s) \tag{8.18}$$

Differentiating (8.6) and (8.8) with respect to x and q we obtain

$$\frac{\partial}{\partial x}B_{+-}(s,s) = 2\int_0^\infty dte^{-ct}B_{++}(s,t)B_{--}(t,s) \tag{8.19}$$

$$\frac{\partial}{\partial q}B_{+-}(s,s) = f_+(q|s)f_-(q|s) + f_+(-q|s)f_-(-q|s)\ . \tag{8.20}$$

Even more important is the derivative with respect to q. (We know that $\ell n \det(I+V) = 0$ at $q=0$)

$$\frac{\partial}{\partial q}\ell n \det(I+V) = R(q,q) + R(-q,-q)\ . \tag{8.21}$$

These are diagonal values of the resolvent on the edges. From (8.7) we have

$$R(q,q) = i\int_0^\infty dse^{-sc}\left[f_-(q|s)\partial_\lambda f_+(\lambda|s)\big|_{\lambda=q} - f_+(q|s)\partial_\lambda f_-(\lambda|s)\big|_{\lambda=q}\right]\ . \tag{8.22}$$

Here r.h.s. can be calculated from (8.12) (8.13) at $\lambda = q$.

$$\begin{aligned} R(q,q) &= i\int_0^\infty dse^{-sc}[f_-(q|s)\partial_q f_+(q|s) - f_+(q|s)\partial_q f_-(q|s)]+ \\ &\quad + \frac{1}{2q}\left\{\int_0^\infty dse^{-sc}[f_+(q|s)f_-(-q|s) - f_-(q|s)f_+(-q|s)]\right\}^2 \\ R(-q,-q) &= i\int_0^\infty dse^{-sc}[f_+(-q|s)\partial_q f_-(-q|s) - f_-(-q|s)\partial_q f_+(-q|s)]+ \\ &\quad + \frac{1}{2q}\left\{\int_0^\infty dse^{-sc}[f_+(q|s)f_-(-q|s) - f_-(q|s)f_+(-q|s)]\right\}^2 \end{aligned} \tag{8.23}$$

In this way we have expressed $\frac{\partial}{\partial q}\ell n \det(I+V)$ in terms of the solution of the system (8.17). To finish the program one should evaluate the long distance asymptotics of $\det(I+V)$ at $x\to\infty$ and make normal ordering with respect to quantum operators $\hat{a}(\lambda), \hat{a}^+(\mu)$ entering $\varphi(\lambda)$ (4.6) (that is, we must calculate the expectation value with respect to the Fock vacuum $|0\rangle$ in dual Fock space).

8. a. The operator Riemann-Hilbert problem

In this section we present the Riemann-Hilbert problem which corresponds to the integrable non-linear system (8.16), (8.17). The latter is a system of integral-differential equations, which is why the corresponding Riemann-Hilbert problem will be an integral-operator-value problem. One can say that the constructions of this section are the natural generalizations of the techniques of [5,27] to the infinitely-dimensional case.

Consider the integral-operator-valued function at real λ ($\mathbf{R}$ is real axis):

$$G(\lambda) = I + \theta(q^2 - \lambda^2)g(\lambda) \ , \qquad \lambda \in \mathbf{R} \ ,$$

where the matrix-kernel $g(\lambda|s,s')$ of the integral operator $g(\lambda)$ is given by:

$$g(\lambda|s,s') = +2\pi \begin{pmatrix} e_+(\lambda|s)e_-(\lambda|s'), & -e_+(\lambda|s)e_+(\lambda|s') \\ e_-(\lambda|s)e_-(\lambda|s'), & -e_-(\lambda|s)e_+(\lambda|s') \end{pmatrix}$$

$$e_\pm(\lambda|s) = \sqrt{\frac{c}{2\pi}} \exp\{\pm i\lambda(x+s) \pm \frac{1}{2}\varphi(\lambda)\}$$

All our operators are integral operators in the $L_2 \subset \mathbf{R}^+, e^{-sc}$. We propose that the operator $G(\lambda)$ as the "conjugation matrix"for an infinite dimension Riemann-Hilbert problem which is the problem to construct the integral operator-valued function $\chi(\lambda)$ with the following properties:

1. $\chi(\lambda)$ is analytic for $Im\lambda > 0$ or $Im\lambda < 0$.
2. $\chi^-(\lambda) = \chi^+(\lambda)G(\lambda), Im(\lambda) = 0$, and $\chi^\pm$ are the boundary values of function $\chi(\lambda)$ as $\lambda \in \mathbf{R} \pm i0$.
3. $\chi(\lambda) \to I$ as $\lambda \to \infty$.

In terms of the corresponding kernels the properties 1-3 can be rewritten in the following way:

P1. $\chi(\lambda|s,s')$ is an analytic function of λ in the half planes $Im\lambda > 0$ or $Im\lambda < 0$ for all s, s'.

P2. $\chi^-(\lambda|s,s') = \int_0^\infty ds'' e^{-s''c}\chi^+(\lambda|s,s'')G(\lambda|s'',s') = \chi^+(\lambda|s,s') + 2\pi\theta(q^2 - \lambda^2) \times \times \int_0^\infty e^{-s''c} ds''\chi^+(\lambda|s,s'')g(\lambda|s''s')$.

P3. $\chi(\lambda|s,s') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(s-s')e^{s'c} + \lambda^{-1}\psi_1(s,s') + \dots$, as $\lambda \to \infty$.

Suppose that the solution of the Riemann-Hilbert problem 1-3 exists and unique. Then we shall show that the function

$$\psi(\lambda) = \chi(\lambda)E(\lambda),$$

where

$$E(\lambda|s,s') = \delta(s-s')e^{i\lambda(x+s')\sigma_3}e^{s'c},$$

satisfies the integral-operator-value linear system in the form of (8.10),(8.12). We begin with the x-equation. Applying the standard reasoning (see,m for example [27]) based on Liouville's theorem and on the x independence of the conjugation integral operator,

$$G_0(\lambda) = E^{-1}(\lambda)G(\lambda)E(\lambda) \ ,$$

we have the equality,

$$\partial_x \psi(\lambda)\psi^{-1}(\lambda) = i\lambda I_0 + U_0 \ , \tag{8.24}$$

where $I_0(s,s') = \delta(s-s')e^{s'c}\sigma_3$ and U_0 is independent of λ. Rewriting (8.24) in the form,

$$\partial_x \psi(\lambda|s,s') = i\lambda\sigma_3\psi(\lambda|s,s') + \int_0^\infty e^{-s''c}U_0(s,s'')\psi(\lambda|s'',s')ds'' \ . \tag{8.25}$$

We see that

$$B_{ik}(s,s') = \int_{-q}^{q} e_i(\lambda|s)f_k(\lambda|s')d\lambda, \quad U_0(s,s') = i[\psi_1(s,s'),\sigma_3] \ , \tag{8.26}$$

where $\psi_1(s,s')$ is the first coefficient in the series P3. So we obtained the first equation of the system (8.10). Note that we also have the equalities

$$\begin{aligned} B_{++}(s,s') &= -i[\psi_1(s,s')]_{12} \ , \\ B_{--}(s,s') &= +i[\psi_1(s,s')]_{21} \end{aligned} \tag{8.27}$$

$$U_0(s,s') = 2\begin{pmatrix} 0 & +B_{++} \\ +B_{--} & 0 \end{pmatrix}(s,s') \ .$$

Let us now study the q-derivative of the function ψ. To do this it is convenient to consider the integral operator valued function $\psi_0(\lambda)$ with the kernel,

$$\psi_0(\lambda|s,s') = \left[\delta(s-s')e^{s'c} - \frac{ic}{2\pi}\ell n\frac{\lambda+q}{\lambda-q}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} e^{-\varphi(\lambda)/2} & 0 \\ -e^{-\varphi(\lambda)/2} & e^{\varphi(\lambda)/2} \end{pmatrix} \ .$$

It is easy to check that for all real λ the following conjugation condition is satisfies.

$$\psi_0^-(\lambda) = \psi_0^+(\lambda)G_0(\lambda) \ .$$

Then we can represent the integral operator $\psi(\lambda)$ in the neighbourhood of the interval $[-q,q]$ in the form,

$$\psi(\lambda) = \hat{\psi}(\lambda)\psi_0(\lambda) \ , \tag{8.28}$$

where $\hat{\psi}(\lambda)$ is singlevalued and analytic in that neighbourhood. Note that the representation (8.28) is just the infinite dimension analogue of the basic formulae of the article [27]. Just as in this article we conclude from (8.28) that,

$$i\partial_q\psi(\lambda)\psi^{-1}(\lambda) = \frac{A_+}{\lambda-q} + \frac{A_-}{\lambda+q} \ , \tag{8.29}$$

where

$$A_\pm = i\lim_{\lambda\to\pm q}(\lambda\mp q)\hat{\psi}(\lambda)\partial_q\psi_0(\lambda)\psi_0^{-1}(\lambda)\hat{\psi}^{-1}(\lambda) \ . \tag{8.30}$$

Note that the kernels of the integral operators $\psi_0^{-1}(\lambda)$ and $\psi^{-1}(\lambda)$ are described by

$$\psi_0^{-1}(\lambda|s,s') = \begin{pmatrix} e^{\varphi(\lambda)/2} & 0 \\ e^{-\varphi(\lambda)/2} & e^{-\varphi(\lambda)/2} \end{pmatrix} \cdot \cdot \left[\delta(s-s')e^{s'c} + \frac{ic}{2\pi}\ell n\left(\frac{\lambda+q}{\lambda-q}\right)\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \tag{8.31}$$

$$\psi^{-1}(\lambda|s,s') = \begin{pmatrix} \psi_{22}(\lambda|s',s) & -\psi_{12}(\lambda|s',s) \\ -\psi_{21}(\lambda|s',s) & \psi_{11}(\lambda|s',s) \end{pmatrix} \tag{8.32}$$

The first of these equalities is obvious, the second one follows from the identity,

$$G^{-1}(\lambda) = I - \theta(q^2-\lambda^2)g(\lambda) \ .$$

This identity is a direct corollary of the important relation

$$g^2(\lambda) = 0 \ .$$

Note also that from (8.31) we have

$$[\partial_q\psi_0(\lambda)\psi_0^{-1}(\lambda)](s,s') = -\frac{ic}{2\pi}\left(\frac{1}{\lambda-q}+\frac{1}{\lambda+q}\right)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{8.33}$$

Returning to the integral operators A_+ we obtain from (8.30-8.33) that

$$\begin{aligned} A_\pm(s,s') &= \frac{c}{2\pi}\int_0^\infty\int_0^\infty ds''ds'''e^{-(s''+s''')c}\hat{\psi}(\pm q|s,s'')\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\hat{\psi}^{-1}(\pm q|s'''s') \\ &= \frac{c}{2\pi}\int_0^\infty ds''e^{-s'''c}\hat{\psi}(\pm q|s,s'')\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\int_0^\infty ds''e^{-s''c}\hat{\psi}^{-1}(\pm q|s'',s') \\ &= \frac{c}{2\pi}\int_0^\infty ds''e^{-s''c}\psi(\pm q|s,s'')\begin{pmatrix} e^{\varphi(\pm q)/2} & 0 \\ e^{-\varphi(\pm q)/2} & 0 \end{pmatrix}\times \\ &\quad\times\begin{pmatrix} -e^{-\varphi(\pm q)/2} & e^{\varphi(\pm q)/2} \\ 0 & 0 \end{pmatrix}\cdot\int_0^2 ds''e^{-s''c}\psi^{-1}(\pm q|s'',s') = \\ &= \begin{pmatrix} f_+(\pm q|s)f_-(\pm q|s'), & -f_+(\pm q|s)f_+(\pm q|s') \\ f_-(\pm q|s)f_-(\pm q|s'), & -f_-(\pm q|s)f_+(\pm q|s') \end{pmatrix} , \end{aligned}$$

where we set:

$$\begin{aligned} f_+(\lambda|s) &= \int_0^\infty ds'e^{-s'c}[\chi_{11}(\lambda|s,s')e_+(\lambda|s') + \chi_{12}(\lambda|s,s')e_-(\lambda|s')] \\ f_-(\lambda|s) &= \int_0^\infty ds'e^{-s'c}[\chi_{21}(\lambda|s,s')e_+(\lambda|s') + \chi_{22}(\lambda|s,s')e_-(\lambda|s')] \end{aligned} \tag{8.34}$$

By this formulae the reconstruction of the Lax representation (8.12) from the Riemann-Hilbert problem 1-3 is finished.

Our last task is to show that the formulae (8.27) and (8.34) give us just the same functions $B_{++}(s,s'), B_{--}(s,s')$ and $f_+(\lambda|s)$ as we had in the previous section. To do this let us consider the singular integral equation equivalent to the problem 1-3. It has the form

$$\chi^+(\lambda) = I - 1/2\pi i \int_{-q}^{q} \frac{d\mu}{\mu-\lambda-i0}\chi^+(\mu)g(\mu) \tag{8.35}$$

Set

$$\Phi^+(\lambda) = \chi^+(\lambda)T(\lambda) \ , \tag{8.36}$$

where

$$T(\lambda|s,s') = \tau(\lambda|s)\delta(s-s')e^{+s'c} \ ,$$

$$\tau(\lambda|s) = \begin{pmatrix} 1 & e_+(\lambda|s) \\ 0 & e_-(\lambda|s) \end{pmatrix} .$$

Using the function $\Phi^+(\lambda)$ equation (8.35) can be rewritten in the following way :

$$\Phi^+(\lambda) = T(\lambda) - 1/2\pi i \int_{-q}^{q} \frac{d\mu}{\mu-\lambda-i0}\Phi^+(\mu)T^{-1}(\mu)g(\mu)T(\lambda) \ .$$

Note that

$$(T^{-1}(\mu)g(\mu)T(\lambda))(\mu,\lambda|s,s') = \tau^{-1}(\mu|s)g(\mu|s,s')\tau(\lambda|s')$$

$$= +2\pi \begin{pmatrix} 0\ , & 0 \\ e_-(\mu|s'), & e_-(\mu|s')e_+(\lambda|s') - e_+(\mu|s')e_-(\lambda|s') \end{pmatrix} .$$

So, for the matrix elements $\Phi^+_{ij}(\lambda|s,s')$ we have:

$$\begin{aligned}
\Phi^+_{11}(\lambda|s,s') &= \delta(s-s')e^{s'c} + i\int_{-q}^{q} \frac{d\mu}{\mu-\lambda-i0}\int_0^\infty ds'' e^{-s''c}\Phi^+_{12}(\mu|s,s'')e_-(\mu|s'), \\
\Phi^+_{12}(\lambda|s,s') &= e_+(\lambda|s)\delta(s-s')e^{s'c} + \\
&\quad + i\int_{-q}^{q}\frac{d\mu}{\mu-\lambda}\int_0^\infty ds'' e^{-s''c}\Phi^+_{12}(\mu|s,s'')\times \\
&\quad \times [e_-(\mu|s')e_+(\lambda|s') - e_+(\mu|s')e_-(\lambda|s')], \\
\Phi^+_{21}(\lambda|s,s') &= +i\int_{-q}^{q}\frac{d\mu}{\mu-\lambda-i0}\int_0^\infty ds'' e^{-s''c}\Phi^+_{22}(\mu|s,s'')e_-(\mu|s'), \\
\Phi^+_{22}(\lambda|s,s') &= e_-(\lambda|s)\delta(s-s')\exp(s'c) + \\
&\quad + i\int_{-q}^{q}\frac{d\mu}{\mu-\lambda}\int_0^\infty ds'' e^{-s''c}\Phi^+_{22}(\mu|s,s'')\times \\
&\quad \times [e_-(\mu|s')e_+(\lambda|s') - e_+(\mu|s')e_-(\lambda|s')]
\end{aligned} \tag{8.37}$$

Introduce the functions

$$f_{ij}(\lambda|s) = \int_0^\infty ds' e^{-s'c}\Phi^+_{ij}(\lambda|s,s') \ .$$

Then we obtain from (8.37) that

$$f_{12}(\lambda|s) = e_+(\lambda|s) - \int_{-q}^{q} d\mu V(\lambda,\mu) f_{12}(\mu|s) ,$$
$$f_{22}(\lambda|s) = e_-(\lambda|s) - \int_{-q}^{q} d\mu V(\lambda,\mu) f_{22}(\mu|s) .$$

were

$$V(\lambda,\mu) = +i \int_0^\infty ds e^{-sc} \frac{e_+(\lambda|s)e_-(\mu|s) - e_+(\mu|s)e_-(\lambda|s)}{\lambda-\mu} .$$

So we come to our basic integral operator (8.3). In particular we have the following equalities

$$\begin{aligned} f_+(\lambda|s) &= f_{12}(\lambda|s) , \\ f_-(\lambda|s) &= f_{22}(\lambda|s) . \end{aligned} \tag{8.38}$$

Taking into account the definition (8.36) of the function $\Phi^+(\lambda)$ we see that (8.38) coincides with (8.34).

To prove (8.27) note that due to (8.35) the matrix coefficient $\psi_1(s,s')$ can be expressed in the following way :

$$\psi_1(s,s') = 1/2\pi i \int_{-q}^{q} d\mu \int_0^\infty ds'' e^{-s''c} \chi^+(\mu|s,s'') g(\mu|s'',s') ds'' .$$

Therefore for the corresponding matrix elements we have

$$\begin{aligned} [\psi_1(s,s')]_{11} &= -i \int_{-q}^{q} d\mu f_+(\mu|s) e_-(\mu|s') = -iB_{-+}(s',s) \\ [\psi_1(s,s')]_{12} &= +i \int_{-q}^{q} d\mu f_+(\mu|s) e_+(\mu|s') = +iB_{++}(s',s) \\ [\psi_1(s,s')]_{21} &= -i \int_{-q}^{q} d\mu f_-(\mu|s) e_-(\mu|s') = +iB_{--}(s',s) \\ [\psi_1(s,s')]_{22} &= +i \int_{-q}^{q} d\mu f_-(\mu|s) e_+(\mu|s') = +iB_{+-}(s',s) \end{aligned} . \tag{8.39}$$

Note also that the symmetric property

$$B_{ik}(s,s') = B_{ki}(s',s)$$

is the direct corollary of the formula (8.32)

Now we finish the discussion of the Riemann-Hilbert problem which corresponds to the case $0 < c < \infty$. In another paper we will use this Riemann-Hilbert problem for the calculation of the asymptotics of $\det(I+V)$ as $x \to \infty$ (as it was done in [27] for the case $c = \infty$).

III. Asymptotics of correlation functions

This part consists of two sections. In section 9 we study long distance asymptotics of equal time temperature correlator of the impenetrable Bose gas. In section 10 we study density $\psi^+(x,t)\psi(x,t)$ (current) correlations for the impenetrable Bose gas (it is equal to a matrix determinant), we study also $\frac{1}{c}$ corrections.

9. Temperature equal time correlator

Following section 1 and 5 we shall study asymptotics of equal time temperature correlators (1.1), (1.10)

$$\langle\psi^+(z)\psi(-z)\rangle_T = \frac{\sqrt{T}}{4} B_{++}\det(I-\gamma K) \ . \tag{9.1}$$

Let us use our variables (1.2)

$$\beta = \frac{h}{T} \ ; \qquad x = z\sqrt{T} \ . \tag{9.2}$$

The associated Riemann problem approach leads to the following asymptotic formula $z\to\infty, x\to\infty$.

$$\begin{aligned}\langle\psi^+(z)\psi(-z)\rangle_T = \sqrt{\frac{T}{\pi}}\rho_\infty e^{-xc(\beta)} \exp\left\{-\frac{1}{2}\int_\beta^\infty dt(\partial_t c(t))^2\right\}\times \\ \times\left[1 - 2\ \mathrm{Re}\ \frac{i\alpha^2(\lambda_1)}{\lambda_0\lambda_1(\lambda_0+\lambda_1)^2}e^{2ix\lambda_1}\times\right. \\ \left.\times\left\{(\lambda_0^2+\lambda_1^2)\sin\theta + 2i\lambda_0\lambda_1\cos\theta\right\}\right] \ .\end{aligned} \tag{9.3}$$

Here we took ρ_∞ from [4],[5] :

$$\rho_\infty = \pi e^{1/2}2^{-1/3}A^{-6} \ .$$

A is the Glaisher constant. The function $C(t)$ is given by

$$C(t) = \frac{1}{\pi}\int_{-\infty}^\infty d\mu\ \ell n\left|\frac{e^{\mu^2-t}+1}{e^{\mu^2-t}-1}\right| \ . \tag{9.4}$$

In (9.3) $\lambda_0 = \sqrt{\beta}$

$$\lambda_1 = \sqrt{\beta+2\pi i} \ ; \quad \mathrm{Im}\ \lambda_1 > 0 \ ; \quad \mathrm{Re}\ \lambda_1 > 0 \ . \tag{9.5}$$

The function $\alpha(\lambda)$ is given by

$$\alpha(\lambda) = \sqrt{\frac{\lambda+\lambda_0}{\lambda-\lambda_0}}\exp\left\{\frac{i}{2\pi}\int_{-\infty}^\infty\frac{d\mu}{\mu-\lambda}\ell n\left|\frac{e^{\mu^2-\beta}+1}{e^{\mu^2-\beta}-1}\right|\right\} \ . \tag{9.6}$$

The function θ is given by

$$\theta(x,\beta) = 2\lambda_0 x - \frac{\pi}{2} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \ell n \left| \frac{e^{\mu^2-\beta}+1}{e^{\mu^2-\beta}-1} \right| . \tag{9.7}$$

These formulae give us the long distance asymptotics of the equal time temperature correlator. The zero temperature case was considered in [5].

Another important correlator (at $c = \infty$) of the impenetrable Bose gas is $P(x)$ (0.11), (1.15), (4.10), (4.13), the probability of absence of particles in the interval $[-x, x]$. It is given by (1.15)

$$P(z) = \det(I - \gamma K)/_{\gamma=1/\pi} \tag{9.8}$$

here z and x are related by (1.2). The associated Riemann problem gives us

$$\begin{aligned} \ell n P(x) &= -xC_0(\beta) + \frac{1}{2} \int_{-\infty}^{\beta} (\partial_t C_0(t))^2 dt + \\ &+ e^{-2x^2+2\beta} \cdot \left[\frac{-1}{16\pi x^2} + \frac{C_0(\beta)}{8\pi x^3} + \ldots \right] + e^{-4x^2+4\beta} \left[\frac{1}{2^{10}\pi^2 x^4} + \ldots \right] + \ldots . \end{aligned} \tag{9.9}$$

Here

$$C_0(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \ell n \left(1 + e^{\beta - \lambda^2}\right) . \tag{9.10}$$

At zero temperature, the asymptotics are different

$$\ell n\ P(x) = -\frac{(xq)^2}{2} - \frac{1}{8}\ \ell n\ 2qx . \tag{9.11}$$

10. Current correlators

Firt we study time correlators of currents $j(x,t) = \psi^+(x,t)\psi(x,t)$ for the impenetrable Bose gas at finite temperature as well as at zero temperature. Special attention is paid to the asymptotics, and the $\frac{1}{c}$ corrections are derived. We would like to emphasize the contribution of N.M. Bogoliubov in this section. First let us consider the zero temperature case. The density of the gas is given by

$$D = \frac{q}{\pi} . \tag{10.1}$$

Here q is the Fermi momentum. Let us introduce also two functions

$$\mathcal{P}(t,x) = \frac{1}{2\pi} \int_{-1}^{q} d\lambda e^{it(\lambda^2-h)-i\lambda x} \tag{10.2}$$

$$\begin{aligned} \mathcal{H}(t,x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it(\lambda^2-h)+ix\lambda} d\lambda - \frac{1}{2\pi} \int_{-q}^{q} d\lambda e^{-it(\lambda^2-h)+ix\lambda} \\ h &= q^2 . \end{aligned} \tag{10.3}$$

The N-current correlator is equal [36],[37] to the determinant of an $N \times N$ matrix :

$$\langle j(x_N, t_N) j(x_{N-1}, t_{N-1}) \cdot \ldots \cdot j(x_2, t_2) j(x_1, t_1) \rangle = \det_N M_{ik} \; ; \quad c = \infty \tag{10.4}$$

where the matrix M_{iK} is constructed as follows :

$$\begin{aligned} &M_{ii} = D \; ; \quad M_{ki} = \mathcal{P}(t_i - t_k; x_i - x_k) \quad k < i \\ &M_{ki} = -\mathcal{H}(t_k - t_i, x_k - x_i) \quad k > i \; . \end{aligned} \tag{10.5}$$

A similar formula is valid for finite temperature $T > 0$, where the distribution function of particles in momentum space is given by the Fermi weight (0.6)

$$\rho(\lambda) = \left(\frac{1}{1 + e^{(\lambda^2 - h)/T}} \right) \frac{1}{2\pi} \; . \tag{10.6}$$

The density is equal to $D = \int d\lambda \rho(\lambda)$ [see (0.7)]. The functions $\mathcal{P}_T$ and $\mathcal{H}_T$ are defined now as

$$\mathcal{P}_T(t, s) = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) e^{it(\lambda^2 - h) - i\lambda x} \tag{10.7}$$

$$\mathcal{H}_T(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it(\lambda^2 - h) + ix\lambda} d\lambda - \int_{-\infty}^{\infty} \rho(\lambda) e^{-it(\lambda^2 - h) + ix\lambda} d\lambda \; . \tag{10.8}$$

In this notation formula (10.4) is still valid. Euclidean time correlators are also given by similar formulae. Let us replace $t \to -i\tau$ in (10.4) and demand that

$$\frac{1}{T} > \tau_{k+1} - \tau_k > 0 \; . \tag{10.9}$$

It is possible to change the operator ordering in the l.h.s. of (10.4). Using fermionization (1.14) and denoting by : : normal ordering with respect to fermions one obtains

$$\langle : j(x_N, t_N) \cdot \ldots \cdot j(x_1, t_1) : \rangle = \det_N m_{ik} \; , \quad c = \infty \; . \tag{10.10}$$

The matrix elements of m_{ik} are

$$m_{ik} = \mathcal{P}(t_i - t_k, x_i - x_k) \; . \tag{10.11}$$

It is interesting to look how $\frac{1}{c}$ corrections change these formulae. Let us write down the $\frac{1}{c}$ correction for the equal time zero temperature n-point current correlator

$$\begin{aligned} \langle : j(x_1) \cdot \ldots \cdot j(x_n) : \rangle = \Bigg\{ 1 + \\ + \frac{1}{c} \sum_{m=1}^{n} \left[\frac{2q}{\pi} + \frac{1}{2} \sum_{\ell=1}^{n} \varepsilon(\tilde{x}_\ell - \tilde{x}_m) \left(\frac{\partial}{\partial \tilde{x}_\ell} - \frac{\partial}{\partial \tilde{x}_m} \right) + \frac{1}{\pi} \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \tilde{x}_m} \right] \Bigg\} \times \\ \times \det_n \left(\frac{\sin q \tilde{x}_{jk}}{\pi \tilde{x}_{jk}} + \alpha \tau(\tilde{x}_{jm}, \tilde{x}_{km}) \right) \Big/_{\alpha = 0} \; . \end{aligned} \tag{10.12}$$

Here $\varepsilon(x)$ is the sign function, and

$$\tilde{x} = x\left(1+\frac{2q}{\pi c}\right) \ ; \quad \tilde{x}_{jk} = \tilde{x}_j - \tilde{x}_k$$
$$\tau(\tilde{x},\tilde{y}) = \frac{1}{2\pi i}\int_{-q}^{q} d\lambda_1 d\lambda_2 \frac{e^{i\lambda_1\tilde{x}} - e^{i\lambda_2\tilde{y}}}{\lambda_1-\lambda_2} \ . \tag{10.13}$$

Let us consider also the $1/c$ correction to the two point time dependent field correlator (zero temperature):

$$\begin{aligned}\langle : j(x,t)j(0,0) : \rangle = D^2 + \left(\frac{1+\frac{4q}{\pi c}}{2\pi}\right) \int_\Lambda d^2\lambda e^{it(\lambda_1^2-\lambda_2^2)} \times \\ \times (\cos\lambda_{12}\tilde{x})\left[1+\frac{\lambda_{12}}{\pi c}\fint_{-q}^{q} d\lambda_3\left(\frac{1}{\lambda_{31}}-\frac{1}{\lambda_{32}}\right)\right] + 0\left(\frac{1}{c^2}\right) \\ \tilde{x} = x\left(1+\frac{2q}{\pi c}\right) \ ; \qquad \lambda_{jk} = \lambda_j - \lambda_k \ .\end{aligned} \tag{10.14}$$

The intgration domain Λ is defined by $\Lambda = \{|\Lambda_1| > q, |\lambda_2| < q\}$.

Let us study the asymptotics in different directions in x,t plane. Simplest example is the two-point correlator

$$\langle : j(x_2,t_2)j(x_1,t_1) : \rangle = D^2 - \mathcal{P}(t_{21},x_{21})\mathcal{P}(t_{12},x_{12}) \ . \tag{10.15}$$

An important role is played by the Fermi velocity $v = 2q$. In the space like directions $x_{21} \to \infty, t_{21} \to \infty, \left|\frac{x_{21}}{t_{21}}\right| > v$ integration by parts gives the complete asymptotic expansion

$$\mathcal{P}(t,x) = \frac{e^{iqx}}{2\pi i}\sum_{n=0}^{\infty}\frac{(-2it)^n(2n-1)!!}{(x+vt)^{2n+1}} - \frac{e^{-iqx}}{2\pi i}\sum_{n=0}^{\infty}\frac{(-2it)^n(2n-1)!!}{(x-vt)^{2n+1}} \ . \tag{10.16}$$

This is in agreement with conformal results [38-40]. In time-like directions the asymptotics are different. One should represent $\mathcal{P}$ as follows :

$$\mathcal{P}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it(\lambda^2-h)-i\lambda x}d\lambda - \frac{1}{2\pi}\left(\int_{-\infty}^{-q} + \int_q^{\infty}\right) e^{it(\lambda^2-h)-i\lambda x}d\lambda \ . \tag{10.17}$$

The first term here is a decaying wave packet ($t = t_{21}, x = x_{21}$):

$$\frac{1}{2\pi}\sqrt{\frac{i\pi}{t_{21}}}\exp\left\{-\frac{ix_{21}^2}{4t_{21}} - iht_{21}\right\} \ . \tag{10.18}$$

Integration by parts of two other terms in (10.17) gives an expansion similar to (10.16). So the difference between space and time asymptotics is in the wave packet. A similar difference in space and time asymptotics in the XY model was found in [8]. Combination of (10.4), (10.10) and (10.16), (10.18) gives the complete asymptotic expansion for the N current correlator.

Now let us discuss the asymptotics of temperature correlators. In (10.7) we move the integration contour down in the complex plane and obtain the asymptotic expansion

$$\mathcal{P}(t_{21}, x_{21}) = T \text{ Re } \sum_{K=0}^{\infty} \frac{1}{i\lambda_K} e^{-\pi T t_{21}(1+2K)+ix_{21}\lambda_K} \tag{10.19}$$

$$\lambda_K = \sqrt{h + i\pi T(1 + 2_K)} \; ; \quad \text{Re } \lambda_K \geq 0 \; ; \quad Im\lambda_K \geq 0 \; ; \quad K = 0, 1, \ldots \; . \tag{10.20}$$

This gives us the complete asymptotic expansion of N-current correlator (10.4), (10.10). For small temperature the series (10.19) is a geometric one. We can summarise

$$\mathcal{P}(t_{21}, t_{21}) = \left(\frac{T}{v}\right) \frac{\sin qx_{21}}{sh\frac{\pi T}{v}(x_{21} + vt_{21})} \tag{10.21}$$

where $v = 2q$. This is also in agreement with conformal results [38-40].

To conclude, let us emphasize once more that the problem of calculating correlation functions for the quantum Nonlinear Schrödinger equation is a rich interesting and important problem [41].

We are grateful to L.D. Faddeev, A.S. Fokas, M.J. Ablowitz, T. Miwa, J.H.H. Perk, B.M. McCoy and N.Yu. Reshetigkhin for discussions.

References

1. Mathematical Physics in One Dimension. Lieb E.JH. Mattis D.C. (eds.) New York, London : Academic Press 1966.
2. Yang C.N., Yang C.P. Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction. J. Math. Phys. 10 1115-1122 (1969).
3. Lenard A. One-dimensional impenetrable Bosons in Thermal equilibrium. J. Math. 7 Phys. 1268-1272 (1965).
4. Vaidya H.G., Tracy C.A. Phys. Rev. Lett. 42 (1979) 3.
5. Jimbo M., Miwa T., Mori Y., Sato M.: Density matrix of an impenetrable Bose gas and the fifth Painlève transcendent. Physica 1D, 80-158 (1980).
6. McCoy B.M., Wu T.T. The two-dimensional Ising model. Cambridge, Mass. Harvard University Press 1973.
7. McCoy B.M., Perk J.H.H., Shrock R.E. Time-dependent correlation functions of the transverse Ising chain at the critical magnetic field. Nucl. Phys. B220 FS8 (1983) 35-47.
8. Perk J.H.H., Capel H.W., Quispel G.R.W., Nijhoff F.W. Finite-temperature correlations for the Ising Chain in a transverse field. Physica 123 A (1984) 1-49.
9. Au-Yang H., Perk J.H.H. Critical Correlations in a z-invariant Inhomogenous Ising Model. Physica 144 A (1987), 44-104.
10. Gelfand I.M., Levitan R.M. Dokl. Akad. Nauk SSSR, 88, 593-596 (1953).
11. Faddeev L.D., Takhtajan L.A. "Hamiltonian Approach to the Soliton Theory" "Nauka", Moscow, 1986.
12. Faddeev L.D. Sov. Sci. Rev. Math. Phys. C1, 107-160 (1981).
13. Faddeev L.D. Sklyanin E.K., Dokl. Akad. Nauk SSSR 243, 1430-1433 (1978).
14. Izergin A.G. Korepin V.E. Lattice Versions of Quantum field theory Models in Two Dimensions. Nucl. Phys. B205 FS5 (1982), 401-413.
15. Izergin A.G., Korepin V.E. The most general L opertor for the R-matrix of the XXX model. Letters Mathem. Physics. 8 (1984), 259-265.
16. Korepin V.E. Calculation of Norms of Bethe wave function. Commun. Math. Phys. 86, 391-418 (1982).
17. Drinfel'd V.G. Hopf algebras and the quantum Yang-Baxter equations. Dokl. Akad. Nauk SSSR 283 (1985) (Russian) = Soviet Math. Dokl. 32 (1985) 254-256.
18. Drinfel'd V.G. A new realization of Yangians and quantized affine algebras. Dokl. Akad. Nauk. SSSR 296 (1987) Russian = Soviet Math. Dokl. 36 (1988) 212-216.
19. Tarasov V.O. On the structure of quantum L-operator for the R-matrix of the XXZ model Teor.Mat.Fiz. 61 (1984), 163-173 (Russian) = Theor. Math. Phys 61 (1984).

20. Tarasov V.O. Irreducible monodromy matrices for the R-matrix of the XXZ-model and lattice local quantum Hamiltonians. Teor. Mat. Fiz. 63 (1985), 175-196 (Russian) = Theor.Math.Phys. 363 (1985).
21. Korepin V.E. The analysis of the bilinear relation of the six-vertex model. Dokl. Acad. Nauk SSSR 265 (1982), 1361-1564 (Russian) = Soviet Phys. Dokl. 27 (1982), 612-613.
22. Krein M.G. DAN SSSR p. 97 (1954), 21-24.
23. Krein M.G. DAN SSSR p. 105 (1955), 433-436.
24. Fokas A.S., Ablowitz M.J. Linearization of the Karteweg-de Vries and Painlève II Equations. Phys. Rev. Lett. 47 (1981), 1096-1100.
25. Santini P.M., Ablowitz M.J., Fokas A.S. The direct linearization of a class of nonlinear evaluation equation. J. Math. Phys. 25 (1984) 2614-1619.
26. Ablowitz M.J., Segur H. Solitons and the Inverse Scattering Transform SIAM Philadelphia 1981.
27. Its A.R., Izergin A.G., Korepin V.E. Completely integrable differential equations for temperature correlation functions of an impenetrable Bose gas. Submitted to Phys. Lett. A.; Preprints Intern. Centre. Theor. Phys. Trieste, Italy. IC/89/107, IC/89/120.
28. Korepin V.E. Dual Field Formulation of Quantum Integrable Modes. Commun. Math. Phys. 113, 177-190 (1987).
29. Korepin V.E. Generating Functional for correlation functions of Nonlinear Schrödinger Model. Functional Analis i Prilogh. (Russian) 23, 15-23 (1989).
30. Mehta M.L. Random Matrices (Academic Press, New York, 1967).
31. Creamer, D.B., Thacker H.B., Wilkinson D. Phys. Rev. D21, 1523-1526 (1980).
32. Korepin V.E., Slavnov N.A. Time Correlation function of Impenetrable Bose gas as Fredholm minor. Submitted to Commun. Math. Phys.
33. N.A. Slavnov "Manypoint time field correlation functions of impenetrable Bose gas". Submitted to Commun. Math. Phys.
34. Bogoliubov N.M., Izergin A.G., Korepin V.E. "Quantum Inverse Scattering Method and Correlation Functions". Lecture Notes in Physics 242, 220-316 (1985).
35. Smirnov F.A. Dokl. Akad. Nauk SSSR 262, 78-83 (1982).
36. Bogoliubov N.M., Korepin V.E. Proceedings of II Soviet-Italian Symposium of the Math. Problem, of Stat. Phys. Lvov, 1985.
37. Bogoliubov N.M., Korepin V.E. Zap. Nauch. Semin. LOMI, 156 (1986) 26-36.
38. Bogoliubov N.M., Izergin A.G., Reshetikhin N.Yu. "Finite-size effects and infrared asymptotics of the correlation functions in two dimensions" J.Phys. A. Math. Gen. 20 (1987) 5361-5369.
39. Bogoliubov N.M., Izergin A.G., Korepin V.E. "Critical exponents for Integrable Models"Nuclear Physics B275 FS17 (1986) 687-705.

40. Izergin A.G., Korepin V.E., Reshetikhin N.Yu. "Conformal Dimensions in Bethe Ansatz" Stony Brook ITP-SB-88-47.
41. Hatfield B., Yau S.-T. "An exchange symmetry expansion for the 2-point correlation function of the nonlinear Schrödinger model" Nuclear Physics B305 FS23 (1988), 16-32.
42. Nijhoff, I.F.W., Capel, H.W., Wiersma, G.L. and Quispel, G.R.W. "Linearising integral transformation and partial differential equations" Phys. Lett. 103A (1984) p293.
43. Nijhoff, F.W., "Linear integral transformations and hierarkies of integrable nonlinear evolution equations" Physica D31 (1986) p339.