

# The valence bond solid in quasicrystals

Anatol N. Kirillov and Vladimir E. Korepin

*Steklov Mathematical Institute,  
Fontanka 27, St.Petersburg, 191011, Russia*

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## **Abstract**

A generalized model of Heisenberg quantum antiferromagnet on an arbitrary graph is constructed so that the VBS is the unique ground state. The norm of the base state and equal time multi point correlation functions are computed in terms of generalized hyper geometric functions. For the one-dimensional periodic Heisenberg model we present a method of computing multi point correlation functions based on the study of a commuting family of transfer matrices. The connection of multi point correlators with Young tableaux and Gegenbauer polynomials is found.

## **Introduction.**

Theory of antiferromagnetism is very important. We consider the models with Valence Bond Solid ground state [2]. The study of generalized Heisenberg antiferromagnets (see [3]–[9]) is of great interest. A distinguishing feature of these models is that their Hamiltonians can be represented in the form of a linear combination of projections, which makes it possible to explicitly construct the base states for the models in question. The models we study are a generalization and modification of those considered in [2], [5], [6], [7], [8]. They have a valence bound state (abbreviated VBS) as the ground state (which is distinct from the Néel ground state). In the first part of the

work – §§1–7 – we devote our attention mainly to questions of uniqueness of the VBS. We prove that the VBS is the unique ground state of the model in any dimension for a periodic lattice with any coordination number. Thus, the Néel state is never a ground state for the model in question. We have succeeded in reformulating the model on an unclosed chain (by introducing special boundary conditions) in such a way that the VBS is, as before, the unique ground state. We can construct a generalized model of a quantum antiferromagnet (and boundary conditions) on an arbitrary graph, so that the VBS is the unique ground state.

The second part of the paper – §§8–13 – is devoted to the computation of the square of the norm of the VBS of the wave function and of multi point correlators for the  $O(m)$ –Heisenberg model on an arbitrary graph. The computations make essential use of the properties of harmonic polynomials  $\Lambda^{A^n}(\mathbf{n})$  (Theorem 7) introduced in §9, which are a natural generalization of Gegenbauer polynomials to the case of several variables. Theorem B in Appendix B plays a central role; it makes it possible to derive recurrence relations for the generating function of the correlators (Theorem 9). We show that the Heisenberg  $O(m)$ –model on one–dimensional or periodic chain are connected with a commuting family of transfer matrices (§9), which provides an alternative method of computing the correlation functions (§§10, 11).

An outline of our paper is as follows. In the first two sections we present a general construction of models having a unique ground state. The models describe the intersection of quantum spins–distinct spins are situated at the vertices of an arbitrary graph  $\Gamma$ . Further, for the models constructed we explicitly give the Hamiltonian and present the construction of the ground state. Generally speaking, the ground state is not unique. In order to formulate a condition for the VBS, we introduce the required notation. We denote by  $N_0$  the number of vertices and by  $N_1$  the number of edges of the graph  $\Gamma$ . Let  $S_l$  be the value of the spin situated at the vertex of  $\Gamma$  with index  $l$ . We consider the vector  $2\mathbf{S}$  whose components with index  $l$  is equal to  $2S_l$ . We denote by  $\langle kl \rangle$  the edge of  $\Gamma$  joining the vertices  $k$  and  $l$ . With each edge  $\langle kl \rangle$  of  $\Gamma$  we associate a number  $M_{kl}$  which we henceforth call an alternating number. We now consider the vector  $\mathbf{M}$  of dimension equal to the number of edges  $N_1$  of  $\Gamma$ , whose components with index  $\langle kl \rangle$  is equal to  $M_{kl}$ . The (edge–vertex) incidence matrix  $\hat{I}$  [10] is an important geometric characteristic of  $\Gamma$ . This is a rectangular matrix of dimension  $N_0 \times N_1$  whose element with index  $(n, \langle kl \rangle)$  is equal to  $\delta_{n,k} + \delta_{n,l}$ . In the second section we

prove the following theorem.

*Solvability in nonnegative integers  $M_{kl}$  (for fixed spins  $s_l$ ) of the system of linear equations*

$$2\mathbf{S} = \hat{I} \cdot \mathbf{M} \tag{1}$$

*is a necessary and sufficient condition for uniqueness of the ground state of generalized Heisenberg magnet corresponding to spins  $s_l$  and alternating numbers  $M_{kl}$ .*

In §§3, 5–8 we study conditions for solvability of the system (1) for a fixed collection of spins  $s_l$ . This problem naturally decomposes into two problems: *i)* solvability of the system in integers (the  $M_{kl}$  must be integers); *ii)* solvability of the system in positive integers  $M_{kl}$ . We give a complete solution of the first problem. The answer depends in an essential way on whether the graph  $\Gamma$  is bipartite [10]. We recall that a bipartite graph is characterized by the fact that the set of its vertices can be decomposed into two subsets  $A$  and  $B$  so that any edge in  $\Gamma$  joins only vertices of different subsets. In §7 we prove the following theorem.

*For a connected, bipartite graph  $\Gamma$  the system of equations (1) is solvable in integers  $M_{kl}$  if and only if the following relation is satisfied:*

$$\sum_{a \in A} s_a = \sum_{b \in B} s_b. \tag{2}$$

*For a non bipartite graph  $\Gamma$  in §8 we prove that the system (1) is solvable in integers  $M_{kl}$  if and only if the following condition is satisfied:*

$$\sum_{l \in \Gamma} s_l \in \mathbf{Z}. \tag{3}$$

To prove the theorems formulated above it is useful to first consider cases where the graph  $\Gamma$  is a tree (§§3, 5) or a cycle (§6). For such graphs we solve the system (1) explicitly. The result obtained are used in the proof of the theorems in the general case.

As concerns solvability of the system (1) in nonnegative integers, we present only necessary conditions (§6). The question of sufficiency of these conditions remains open.

We consider the model of an antiferromagnet in a quasicrystal separately (§4). The fact of the matter is that the thermodynamic limit for models in a quasicrystal is analogous to the thermodynamic limit in crystal. For a one-dimensional quasicrystal we explicitly compute the multi point correlation

functions. In the second part of our work we study multi point correlation functions for the Heisenberg model on graph  $\Gamma$ . We begin by considering the Heisenberg model on one-dimensional lattice. In this simplest example all spins at the nodes of the lattice are equal to 1. The Hamiltonian of the model is [5]

$$H = \frac{1}{2} \sum_l \left\{ \widehat{\mathbf{S}}(l) \cdot \widehat{\mathbf{S}}(l+1) + \frac{1}{3} (\widehat{\mathbf{S}}(l) \cdot \widehat{\mathbf{S}}(l+1))^2 + \frac{2}{3} \right\}. \quad (4)$$

Here the  $\widehat{\mathbf{S}}(l)$  are the quantum spins (a representation of the algebra  $0(3)$ ;  $l$  is the index of a node of the lattice). The ground state of the model – the valence bond state – is described in §1. The equal time two-point correlator is computed in [2] (in the thermodynamic limit):

$$\langle \widehat{S}^{a_1}(l_1) \cdot \widehat{S}^{a_2}(l_2) \rangle = \frac{4}{3} \delta_{a_2}^{a_1} (-3)^{-|l_1-l_2|}. \quad (5)$$

Here the  $\widehat{S}^a$  are the components of the vector  $\widehat{\mathbf{S}}$  ( $a = 1, 2, 3$ ). We prove that for the model under consideration the equal time multi point correlator reduces to the product of the two-point correlators (5). More precisely, we have the following:

- 1) If the number of spins is odd, then

$$\left\langle \prod_{j=1}^N \widehat{S}^{a_j}(l_j) \right\rangle = 0, \quad N \equiv 1 \pmod{2}. \quad (6)$$

We suppose here and below that all the coordinates  $l_j$  are distinct.

- 2) If the number of spins  $N$  is even,  $l_N > l_{N-2} > \dots > l_2 > l_1$ , then we prove that

$$\left\langle \prod_{j=1}^N \widehat{S}^{a_j}(l_j) \right\rangle = \prod_{r=1}^{N/2} \langle \widehat{S}^{a_{2r}}(l_{2r}) \cdot \widehat{S}^{a_{2r-1}}(l_{2r-1}) \rangle. \quad (7)$$

A generalization of the model (4) to the case of higher spin  $S$  is given in [5], [8]. In this case the Hamiltonian of the model is given by formulas (19) and (21) of the text. The equal time two-point correlator (in the thermodynamic limit) for the Heisenberg model of higher spin  $S$  is computed in [5]:

$$\langle \widehat{S}^{a_1}(l_1) \widehat{S}^{a_2}(l_2) \rangle = \frac{(s+1)^2}{3} \delta_{a_2}^{a_1} \left( \frac{-s}{s+2} \right)^{|l_1-l_2|}. \quad (8)$$

In the present paper we compute the equal time multi point correlator (in the thermodynamic limit) for a model of spin  $S$ . The precise formulation is presented in §11 and differs basically from (7). As an example, we present a formula for the four–point correlator

$$\begin{aligned}
\langle \widehat{S}^{a_1}(l_1) \widehat{S}^{a_2}(l_2) \widehat{S}^{a_3}(l_3) \widehat{S}^{a_4}(l_4) \rangle &= \\
&= \frac{(s+1)^4}{9} \delta_{a_2}^{a_1} \delta_{a_4}^{a_3} \left( \frac{-s}{s+2} \right)^{l_2-l_1+l_4-l_3} \\
&+ \frac{(s+1)^4}{15} \left( \frac{-s}{s+2} \right)^{l_2-l_1+l_4-l_3} \left( \frac{s(s-1)}{(s+2)(s+3)} \right)^{l_3-l_2} \\
&\times \left\{ \delta_{a_4}^{a_1} \delta_{a_3}^{a_2} + \delta_{a_3}^{a_1} \delta_{a_4}^{a_2} - \frac{2}{3} \delta_{a_2}^{a_1} \delta_{a_4}^{a_3} \right\}.
\end{aligned} \tag{9}$$

Here  $l_4 > l_3 > l_2 > l_1$ .

We note that in the case of a one–dimensional, periodic chain the models considered (4) and (19)–(21) have a unique ground state. For a finite open chine the ground state is no longer unique. In the first part of the present paper we analyze conditions under which the models considered have a unique ground state. For example, for the model (19)–(21) on an open chain the two boundary spins  $s$  must be replaced by spins equal to  $s/2$ . These modified models have a unique ground state. such modification does not affect the thermodynamic limit, but it simplifies the computation of the correlation functions for finite chains.

In the first part of the paper we prove that there exists a natural, inhomogeneous generalization of the models considered above (distinct spins are situated at the vertices of an arbitrary graph) with presentation of uniqueness of the ground state. One–dimensional quasicrystals are examples of such models. We also compute the multi point, equal time correlators (in the thermodynamic limit) for an inhomogeneous model on a chain (see §11). In [2] it was shown that the quantum models considered above are equivalent to a one–dimensional, classical, modified Heisenberg model, which is useful in computing the correlators. We give a description of the modified model in §8. Using the integral operators  $\widehat{K}_M$  (see §9), we construct a transfer matrix for the classical model and find the spectrum and eigenfunction of the operators  $\widehat{K}_M$  in §9. This enables us to compute the norm of the wave function (see §10) and also to find the multi point correlators (§11). In the following

§12 we give a generalization to the case of the group  $O(m)$  of the modified Heisenberg model considered earlier. In §13 we consider the quantum and corresponding classical Heisenberg model for a multidimensional lattice. The inhomogeneous Heisenberg model (arbitrary spins are situated at the vertices) for a complete graph (all the vertices of the graph are joined by edges) is a natural algebraic object. We consider the generating function for the correlators and derive for it recurrence relations which make it possible in principle to find it for the complete graph (and hence for an arbitrary graph). We show that the generating function of the correlators can be expressed in terms of generalizes hyper geometric functions.

We consider in more detail a one-dimensional quasicrystal [12]. There exists a quasi periodic covering by a direct infinite sequence of two intervals (short and long). One of the examples of such a covering is connected with the golden section  $\tau = (1 + \sqrt{5})/2$ . The position of the end of the  $l$ th segment of the covering is found by the formula

$$x_l = l + \frac{1}{\tau} \left[ \frac{l}{\tau} + \alpha \right]. \quad (10)$$

Here  $\alpha$  is real parameter. We denote by  $p_L$  ( $p_s$ ) the probability of the occurrence of long (short) interval [12]. We construct a Heisenberg antiferromagnet for the quasi periodic lattice (10). To each long segment we assign an alternating number  $M_L$  and to each short segment we assign a number  $M_s$ . The spins (36) can then assume only the three values

$$s(l) = \{M_L, M_S, (M_L + M_S)/2\} \quad (11)$$

depending on the position of the vertex  $l$ . The Hamiltonian of the model is given by formulas (37), (38), while the ground VBS state is given by formula (39). In §10 we explicitly compute the correlation functions – for a finite chain and in the thermodynamic limit. We present the answer for the asymptotic of the two-point correlator of a quasicrystal:

$$\langle \hat{S}^{a_1}(l_1) \hat{S}^{a_2}(l_2) \rangle = \frac{1}{3} \delta_{a_2}^{a_1} (s(l_1) + 1) (-1)^{l_2 - l_1} \exp \{-m(l_2 - l_1)\}. \quad (12)$$

Here  $(l_2 - l_1) \rightarrow \infty$ ,

$$m = p_s \log \left( 1 + \frac{2}{M_S} \right) + p_L \log \left( 1 + \frac{2}{M_L} \right). \quad (13)$$

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**§1. A generalized model of an antiferromagnet. Uniqueness of the ground state.**

We consider a periodic lattice in  $D$ -dimensional Euclidean space. The interacting quantum spins  $\hat{\mathbf{S}}_j$  are situated at the nodes of the lattice ( $l$  is the index of a node). At each node the spins are a representation of the algebra  $SU(2)$ :

$$[\hat{S}_l^a, \hat{S}_k^b] = i\delta_{lk}\epsilon^{abc}\hat{S}_l^c. \quad (14)$$

Here the spin index  $a$  assumes the values  $a = 1, 2, 3$ . The lower index  $l$  numbers the nodes of the lattice. We express the spins in terms of two independent, canonical Bose fields on the lattice –  $a_l$  and  $b_l$ . Their commutation relations are standard:

$$[a_l, a_k^+] = \delta_k^l, [b_l, b_k^+] = \delta_k^l, [a_l, b_k] = 0. \quad (15)$$

The components of the spin can be expressed as follows:

$$\begin{aligned} \hat{S}_l^+ &= a_l^+ b_l, \quad \hat{S}_l^- = b_l^+ a_l, \\ \tilde{S}_l^3 &= \frac{1}{2}(a_l^+ a_l - b_l^+ b_l). \end{aligned} \quad (16)$$

The value of the spin at the node with index  $l$  is an eigenvalue of the operator

$$\hat{\mathbf{S}}_l = \frac{1}{2}(a_l^+ b_l + b_l^+ a_l). \quad (17)$$

(In a quasicrystal different spins are present at different nodes.)

Thus, the spin operators (16) act in Fock space whose vectors have the form

$$P(a_l^+, b_l^+) | 0 \rangle. \quad (18)$$

Here  $P$  is a polynomial in the variables  $a_l^+$  and  $b_l^+$  ( $l$  runs through all nodes of the lattice). We seek eigenfunction of the operator  $\hat{\mathbf{S}}_l$ :

$$\hat{\mathbf{S}}_l P(a^+, b^+) | 0 \rangle = s_l P(a^+, b^+) | 0 \rangle.$$

This relation means that the polynomial  $P$  is a homogeneous function of the variables  $a_l^+$  and  $b_l^+$  (for given  $l$ ) of degree  $2s_l$ , i.e., the polynomial  $P$  can be represented in the form

$$P = \sum_{k=0}^{2s_l} (a_l^+)^k (b_l^+)^{2s_l-k} \tilde{P}_k.$$

Here the polynomial  $\tilde{P}_k$  does not depend on the variables at the  $l$ th node.

The Hamiltonian of the model describes the interaction only of spins situated at the nearest nodes of the lattice (we denote this by  $\langle kl \rangle$ ):

$$H = \sum_{\langle kl \rangle} H(k, l). \quad (19)$$

The Hamiltonian density  $H(k, l)$  is usually described in terms of powers of the scalar product of spins  $(\mathbf{S}_k \cdot \mathbf{S}_l)^n$ . Instead of this we use a special basis of polynomials in  $(\mathbf{S}_k \cdot \mathbf{S}_l)$ . These are the projections  $\Pi_J(k, l)$  onto the state with fixed spin  $J$ . They can be found from the following system of linear equations:

$$(\mathbf{S}_k \cdot \mathbf{S}_l)^n = \sum_{J=|s_k-s_l|}^{s_k+s_l} \Pi_J(k, l) \left[ \frac{1}{2}J(J+1) - \frac{1}{2}s_l(s_l+1) - \frac{1}{2}s_k(s_k+1) \right]^n, \quad (20)$$

$n = 0, 1, 2, \dots, 2s_{\min}$ .

Here  $s_k$  and  $s_l$  are the magnitudes of the spins at the nodes  $k$  and  $l$ , while  $s_{\min}$  is the least of these two values. The system of equations (20) can easily be solved for projections  $\Pi_J(k, l)$ ; we find the following expression:

$$\Pi_J(k, l) = \prod_{j \neq J, |s_k-s_l| \leq j \leq s_k+s_l} \frac{\hat{S}^2 - j(j+1)}{J(J+1) - j(j+1)}. \quad (21)$$

Here  $\hat{S}^2 = (\hat{S}_k + \hat{S}_l)^2$ . The projection (21) is a polynomial in  $(S_k, S_l)$  of degree  $2s_{\min}$ . In order to produce the Hamiltonian of [1], [3], we suppose that the magnitude of spin at each node is the same and equal to  $s$  (we give up this assumption in next section). The Hamiltonian density is

$$H(k, l) = \sum_{J=2s+1-M}^{2s} A_J \Pi_J(k, l). \quad (22)$$



Here  $M$  is a positive integer,  $1 \leq M \leq 2s$ . (We shall see below that this is an important parameter of the theory.) It is important that the following relation be satisfied:

$$2s = zM. \quad (23)$$

Here  $z$  is the coordination number of the node (the number of nearest neighbors). The coefficients  $A_J$  are positive real numbers—parameters of the model. The model has thus been determined. We call it the AKLT model [2]. We present examples. For a one-dimensional, periodic lattice  $z = 2$ , and for the least value  $s = 1$ ,  $M = 1$

$$H = A \sum_{\langle kl \rangle} \left\{ (\mathbf{S}_l \cdot \mathbf{S}_k)^2 + 3(\mathbf{S}_l \cdot \mathbf{S}_k) + 2 \right\}. \quad (24)$$

In the two-dimensional case on a hexagonal lattice  $z = 3$ , and for the least value  $s = 3/2$ ,  $M = 1$

$$H = A \sum_{\langle kl \rangle} \left\{ (\mathbf{S}_l \cdot \mathbf{S}_k) + \frac{116}{243} (\mathbf{S}_l \cdot \mathbf{S}_k)^2 + \frac{16}{243} (\mathbf{S}_l \cdot \mathbf{S}_k)^3 \right\}. \quad (25)$$

We now return to the Hamiltonian (22) and construct the ground state.

**Theorem 1** *The Hamiltonian (22) has a unique ground state of the following form:*

$$|\psi\rangle = \text{const} \prod_{\langle kl \rangle} (a_k^+ b_l^+ - a_l^+ b_k^+)^M. \quad (26)$$

*Proof.* It is obvious that  $H \geq 0$  and also that  $\Pi_J(k, l) \geq 0$ . Thus, if there exists a solution  $|\psi\rangle$  of the equation

$$H|\psi\rangle = 0 \quad (27)$$

then it is ground state of the Hamiltonian (22). Due to positivity equation (27) is equivalent to the set of equations

$$\Pi_J(k, l)|\psi\rangle = 0 \quad (28)$$

for any pair of nearest nodes  $\langle kl \rangle$  and for any  $J$  in the interval  $2s + 1 - M \leq J \leq 2s$ . This implies that in adding spins  $S_k$  and  $S_l$  there are no projections onto states with complete spin  $J$ , where  $2s + 1 - M \leq J \leq 2s$ . We now use the following theorem, whose proof we present in Appendix A.

**Theorem 2** (on addition of spins). We suppose that after addition of spins  $\widehat{S}_l$  and  $\widehat{S}_k$  there arises a state  $|\psi\rangle$  with zero projections onto spins  $J$  of the interval

$$s_l + s_k + 1 - M \leq J \leq s_l + s_k. \quad (29)$$

Then the polynomial  $|\psi\rangle$  is divisible by  $(a_k^+ b_l^+ - a_l^+ b_k^+)^M$ .

We continue the proof of Theorem 1. We recall that we are seeking a ground state  $|\psi\rangle$  in the form (18). It follows from Theorem 2 that the polynomial  $P(a^+, b^+)$  is divisible by  $(a_k^+ b_l^+ - a_l^+ b_k^+)^M$  for each edge  $\langle kl \rangle$ . Hence,

$$|\psi\rangle = P(a^+, b^+) |0\rangle = \left\{ \prod_{\langle kl \rangle} (a_k^+ b_l^+ - a_l^+ b_k^+)^M \right\} \widetilde{P}(a^+, b^+) |0\rangle. \quad (30)$$

Here  $\widetilde{P}(a^+, b^+)$  is another polynomial. We now compute the magnitude of the spin of the state  $|\psi\rangle$  at the node with index  $m$ . This is an eigenvalue of the operator

$$\widehat{S}_m = \frac{1}{2}(a_m^+ a_m + b_m^+ b_m), \quad (31)$$

in other words,  $\widehat{S}_m |\psi\rangle = s_m |\psi\rangle$ .

From this it follows that the polynomial  $P$  is a homogeneous function of the variables  $a_m^+$  and  $b_m^+$ . Applying the operator (31) and (30), we obtain

$$2s_m = Mz + \Delta_m. \quad (32)$$

Here  $\Delta_m \geq 0$  is the degree of homogeneity of the new polynomial  $\widetilde{P}$  in the variables  $a_m^+$  and  $b_m^+$ . Comparing (32) and (23), we find that  $\Delta_m = 0$ , i.e., the polynomial  $\widetilde{P}$  does not depend on the variables  $a_m^+$  and  $b_m^+$ . Thus,  $\widetilde{P} = \text{const}$  as asserted. This means that equation (27) has a unique solution

$$|\psi\rangle = \prod_{\langle kl \rangle} (a_k^+ b_l^+ - a_l^+ b_k^+)^M. \quad (33)$$

The theorem on the existence and uniqueness of a ground state of the Hamiltonian (22) has thus been proved. ■

The wave function (33) realized the valence bound state. In [1], [2], [5], [6] it is called a VBS (valence bond state).

We shall now try to construct a model for an unclosed chain so that the uniqueness theorem is preserved. We thus consider a one-dimensional chain of  $N$  nodes  $l = 1, 2, \dots, N$ . We observe immediately that for the interior nodes ( $l = 2, \dots, N - 1$ ) the coordination number  $z = 2$ , while for the boundary nodes ( $l = 1, N$ )  $z = 1$ . From the relation  $2s_l = z_l M$  (for  $M = 1$ ) it follows that for interior nodes  $s_l = 1$  while for the boundary nodes  $s_N = 1/2$ .

We take the Hamiltonian in the form

$$H = \sum_{l=1}^{N-1} \Pi_{s_l+s_{l+1}}(l, l+1). \quad (34)$$

We note that for interior edges  $\Pi$  is the projection onto spin 2:  $\Pi_{s_l+s_{l+1}} = \Pi_2$ , while for boundary edges it is the projection onto spin 3/2 ( $\Pi_{s_l+s_{l+1}} = \Pi_{3/2}$  for  $l = 1$  and  $l = N - 1$ ). This is the difference between our Hamiltonian and that proposed in [2], [5], [6], [7]. In those papers it was assumed that  $s_1 = s_N = 1$ , and the ground state is fourfold degenerate. For the hamiltonian (34) it is easy to prove that the ground state is unique. In the next section we construct a model with a unique ground state on an arbitrary graph.

## §2. The model on an arbitrary graph.

We shall construct a generalized model of an antiferromagnet on an arbitrary graph  $\Gamma$ . We do this so that the theorem on existence and uniqueness of the wave function of ground state (VBS) is preserved. We consider an arbitrary graph. It consists of  $N_0$  vertices. Some of the vertices are connected by edges. We denote the number of edges by  $N_1$ . The edge-vertex incidence matrix  $\hat{I}$  plays an important role below. This is matrix with  $N_0$  rows and  $N_1$  columns. Each of its matrix elements is equal to 0 or 1. We define  $\hat{I}$  more precisely. Each row of  $\hat{I}$  is connected with specific vertex of the graph  $\Gamma$ ; each column is connected with an edge. If a vertex belongs to an edge, then the corresponding matrix element is equal to 1; otherwise it is equal to 0. We have thus defined the incidence matrix. We number all vertices of the graph, for example, by the letter  $l$ . We number edges by pair of letters (for example,  $\langle kl \rangle$ ) denoting the vertices belonging to the given edge. We begin the construction of a generalized model of an antiferromagnet. To each vertex  $l$  we assign a spin  $s_l$  (the integer  $2s_l > 0$ ), and to each edge  $\langle kl \rangle$  we

assign another positive integer  $M_{kl} > 0$ . At each node these numbers must be connected by relation

$$2s_l = \sum_{\langle kl \rangle} M_{kl}. \quad (35)$$

Here the summation goes over all edges abutting the node  $l$ . This relation can be written with the help of the incidence matrix as

$$2\mathbf{S} = \hat{I} \cdot \mathbf{M}. \quad (36)$$

Here  $\mathbf{S}$  is an  $N_0$ -component vector (its components are equal to  $s_l$ ), while  $\mathbf{M}$  is an  $N_1$ -component vector (its components are equal to  $M_{kl}$ ). We discuss the solution of equation (36) below, but now we continue the construction of the model. We place a quantum spin  $s_l$  at each vertex of the graph. We construct the Hamiltonian describing the interaction of nearest neighbors  $\langle kl \rangle$ :

$$H = \sum_{\langle kl \rangle} H(k, l). \quad (37)$$

Here the summation goes over the edges. The Hamiltonian density is

$$H(k, l) = \sum_{J=s_k+s_l+1-M_{kl}}^{s_k+s_l} A_J(k, l) \Pi_J(k, l). \quad (38)$$

Here  $\Pi_J$  is the projection (21) while the coefficients  $A_J(k, l)$  are real and positive-parameters of the model (depending on  $J$  and on the edge). The model is thus determined. In analogy to §1 we prove the existence and uniqueness of an eigenfunction describing the ground state. This VBS state is such that

$$H|\psi\rangle = 0,$$

where

$$|\psi\rangle = \prod_{\langle kl \rangle} (a_k^+ b_l^+ - a_l^+ b_k^+)^{M_{kl}} |0\rangle. \quad (39)$$

We anticipate that Hamiltonian (38) has a gap in the spectrum; the correlators (see §11) decay exponentially for quasi periodic coverings,  $M_{kl} \geq 1$ .

Several subsequent sections are devoted to the solution of equations (35) and (36). The explicit form of the wave function (39) shows that nullification

of any of the numbers  $M_{kl}$  is equivalent to the absence of an edge. It is therefore a question of solving system (35) in positive integers  $M_{kl} \geq 1$ . Of course, the integers  $M_{kl}$  can be prescribed arbitrarily, and the values of the spins can be computed. It is, however, interesting to solve the inverse problem and clarify what restrictions the system (36) imposes on the permitted values of the spins. For example, it is clear that the spin can be equal to  $1/2$  only when a vertex has one nearest neighbor. We shall consider only connected graphs. For disconnected graphs the problem reduces to several independent problems.

### §3. Linear graphs.

We shall first solve system (36) for linear graphs. We consider a non closed, one-dimensional chain with  $N$  nodes. Let  $l$  be the index of a node,  $l = 1, 2, \dots, N$ . We complete the relations (35) at the ends as follows:

$$M_{01} = 0 \text{ and } M_{N,N+1} = 0. \quad (40)$$

A solution of equation (35) has the form

$$M_{l,l+1} = \sum_{k=1}^l (-1)^{k-l} \cdot 2s_k. \quad (41)$$

For  $l = N$  it follows from this that

$$\sum_{k=1}^N (-1)^k \cdot 2s_k = 0. \quad (42)$$

From the positivity of  $M$  it follows that

$$M_{l,l+1} = \sum_{k=1}^l (-1)^{k-l} \cdot 2s_k \geq 1. \quad (43)$$

We now consider a periodic chain with an even number of nodes  $N$ . The equation

$$2s_l = M_{l,l+1} + M_{l-1,l} \quad (44)$$

has a non unique solution. A solution of the homogeneous equation

$$0 = M_{l,l+1} + M_{l-1,l} \quad (45)$$

has the form

$$M_{l,l+1} = (-1)^l \alpha. \quad (46)$$

The nodes with indices 1 and  $N + 1$  are identified. This can be used to break the chain, i.e., to set, for example,

$$M_{1,N} = 0. \quad (47)$$

This reduces the problem to an unclosed chain, which has already been solved. Thus, a solution exists only in the case

$$\sum_{l=1}^N (-1)^l \cdot 2s_l = 0. \quad (48)$$

It has the form

$$M_{l,l+1} + (-1)^{l+1} M_{1,N} = \sum_{k=1}^l (-1)^{k-l} \cdot 2s_k. \quad (49)$$

Here  $M_{1,N}$  is an arbitrary positive integer. For odd  $l$  the left side of (49) is positive, which imposes restrictions on the right side of (49). Thus, recalling positivity, it is possible to change  $\{M_{l,l+1}\}$  for fixed values of the spin  $s$ . This means that for the same choice of spins it is possible to produce several distinct Hamiltonians (38) whose definition contains the collection  $\{M_{l,l+1}\}$ . Each of these Hamiltonians will have a unique eigenfunction of the ground state.

We now consider a cycle with an odd number of nodes. Equation (44) has the unique solution

$$M_{l,l+1} = \sum_{k=1}^l (-1)^{l-k} \cdot 2s_k - \sum_{k=1}^N (-1)^{l-k} \cdot s_k \geq 1. \quad (50)$$

From this it is clear that the sum  $\sum_{k=1}^N (-1)^k \cdot s_k = (-1)^N M_{1,N}$  must be an integer. Since we have the equality

$$\sum_{k=1}^N s_k + \sum_{k=1}^N (-1)^k \cdot s_k = 2 \sum_{k \equiv 0 \pmod{2}} s_k, \quad (51)$$

it follows that the sum  $\sum_{k=1}^N s_k$  is an integer. Hence,

$$\sum_{k=1}^N 2s_k = \text{an even number.} \quad (52)$$

#### §4. Quasicrystals.

There are now a large number of works devoted to quasicrystals (see, for example, [11]–[14]). Analysis of dynamical system in quasicrystals is also of broad interest. For example, it has been possible to solve the Ising model and the eight–vertex Baxter model in two–dimensional quasicrystal [14].

In connection with the problem of finding the densest packing specific associated with quasicrystals are of major interest [15].

Here we shall construct a generalized model of a quantum antiferromagnet in a quasicrystal of two or higher number of dimensions. The simplest example of a quasi periodic covering is the Penrose tiling of the plane by translations of ten rhombuses. Corresponding figures can be found in [13], [14]. In the situation of general position a quasicrystal (its vertices and edges) is bipartite graph [16].

An important characteristic of a quasicrystal is the number of nearest neighbors of vertices (the coordination number  $z_l$ ). It can run through only a finite number of values. For example, for the Penrose tiling of the plane by translations of 10 rhombuses we have [15]  $3 \leq z_l \leq 7$ . For filling out three–dimensional space by translations of 20 parallelepipeds of special form (rhomboids) we have [15]  $4 \leq z_l \leq 12$ . This filling has the symmetry of a right icosahedron and realizes the crystal structure of the rapidly cooled alloy  $\text{Al}_6\text{Mn}$  [17], [18]. A generalized quantum antiferromagnet can be constructed in a quasicrystal in the same way as on an arbitrary graph. The general outline was described above. We shall present only the simplest example.

At the vertices of the quasicrystal we place spins which, generally speaking, are different in magnitude. The spin situated at a given vertex is equal to half the correlation number:

$$s_l = \frac{1}{2}z_l. \quad (53)$$

For Penrose rhombuses  $3/2 \leq s_l \leq 7/2$ , while for the icosahedral filling of space by rhomboids  $2 \leq s_l \leq 6$ . The interaction Hamiltonian for these spins can be taken, for example, in the form

$$H = \sum_{\langle kl \rangle} \Pi_{s_k+s_l}(k, l). \quad (54)$$

The Hamiltonian density is the projection onto the highest possible value of the spin arising in adding spins at two neighboring nodes  $k$  and  $l$ . We note that the model constructed here of a quantum antiferromagnet in a quasicrystal has a unique ground state (the valence bond state), which differs from the Néel state.

A quasicrystal is a quasi periodic filling of all space. An important characteristic of it is the following. Any finite part of quasicrystal has an infinite set of copies, and these copies repeat throughout space with a particular probability. It is just this that makes it possible to justify the presence of a thermodynamic limit in quasicrystal. In analogy to [5] it is possible to show that there is a gap in the spectrum of the Hamiltonian (54), and the correlators of the spins decay exponentially. In analogy to [4] it can be shown that the model of quantum antiferromagnet in quasicrystal is equivalent to a model of classical statistical physics in the same quasicrystal with Hamiltonian density  $-\log((1 - \mathbf{n}_k \mathbf{n}_l)/2)$ . Here  $\mathbf{n}$  is a unit vector on the sphere.

### §5. Tree graphs.

For a disconnected graph the system of equations (35) decomposes into several independent systems. Therefore we henceforth consider only connected graphs. The procedure of "cutting-off branches" is useful in the investigation of the system (35) for an arbitrary graph; we proceed to a description of it. Here an important role is played by bipartite graphs. by definition, the set of vertices  $\{l\}$  of bipartite graph can be broken into two non intersecting subsets  $\{l\} = \{A\} \cup \{B\}$  so that an edge of the initial graph joins only vertices of different subsets. To each vertex  $l$  of the graph  $\Gamma$  it is possible to ascribe a parity

$$\epsilon_l = \begin{cases} 1, & \text{if } l \in A, \\ -1, & \text{if } l \in B. \end{cases} \quad (55)$$



We now consider a graph which decomposes into two disconnected subgraphs  $\Gamma_1$  and  $\Gamma_2$  when one edge is removed. here  $\Gamma_1$  is a bipartite graph. we denote vertices belonging to the edge  $b$  by  $a$  and  $c$  ( $a \in \Gamma_1, c \in \Gamma_2$ ). It is easy to compute the integer  $M$  corresponding to the edge  $b$ :

$$M_b = \sum_{l \in \Gamma_1} \epsilon_l \cdot 2s_l. \quad (56)$$

The contributions of the edges of the bipartite graph cancel on the right side of (56). The signs are chosen so that  $\epsilon_a = 1$ . We now redefine the magnitude of the spin at the nodes  $a$  and  $c$ :

$$2s_a \mapsto 2s_a - M_b; \quad 2s_c \mapsto 2s_c - M_b. \quad (57)$$

Thus, solution of equation (36) for the entire graph has reduced to the solution of two independent equations of the type (36) for each subgraphs  $\Gamma_1$  and  $\Gamma_2$  which are not connected with one another. This procedure of reducing the system (35) to two simpler independent systems is naturally called the procedure of "cutting-off branches". We apply this procedure to the investigation of system (35) for the tree graphs. Thus, we consider the graphs without cycles (tree graphs). We denote them by the letter  $\mathcal{D}$ . It is well known that any tree is a bipartite graph [10]. We shall study equation (36) in this case. We consider any edge  $\langle kl \rangle$ . Removing it leads to a decomposition of the tree into two disconnected trees  $\mathcal{D}_1 \cup \mathcal{D}_2$  where  $k \in \mathcal{D}_1, l \in \mathcal{D}_2$ .

We choose an alternating sum of spins along the tree  $\mathcal{D}_1$ , so that  $s_k$  enters with sign  $+1$ . It is clear that

$$M_{kl} = \sum_{p \in \mathcal{D}_1} \epsilon_p \cdot 2s_p. \quad (58)$$

This is a solution of equation (36). it is also possible to express  $M_{kl}$  in terms of an analogous sum over the second tree. Comparison of the two expressions leads to the relation

$$2 \sum_{l \in \mathcal{D}} \epsilon_p \cdot 2s_p. \quad (59)$$

Moreover, the following inequality must be satisfied (the condition of positivity):

$$M_{kl} = 2 \sum_{p \in \mathcal{D}_1} \epsilon_p s_p \geq 1. \quad (60)$$

The Cayley tree is a particular example of a tree.

We note that in [5] the case where all spins  $s = 3/2$  is considered for a finite Cayley tree (with  $z = 3$ ). It is easy to see that then the system of equations (36) has positive integer solutions. The spin must be equal to  $3/2$  only for interior nodes, while on the boundary  $s = 1/2$ . This guarantees the uniqueness theorem.

We note that it is just the requirement of positivity which does not permit taking all spins equal to one another for a tree of general position.

### §6. Bipartite connected graphs. A criterion of positivity.

We remark that if there is a cycle of even length in an arbitrary graph, then a solution of equation (36) is not unique. Indeed, along a cycle it is always possible to add to the quantities  $M$  a quantity  $(-1)^l \alpha$  analogous to (46). It does not change  $s_l$ . The quantity  $\alpha$  can be chosen so that one of the quantities  $M$  vanished. This corresponds to removing an edge and breaking the cycle. Thus, all even cycles can be broken without changing the spins but by changing  $M$ . For a bipartite graph any cycle is even [10]; therefore, the problem on a bipartite graph reduces to the problem on a tree graph. Hence, equation (36) is solvable in integers if and only if

$$\sum \epsilon_l s_l = 0. \tag{61}$$

The positivity condition  $M_{kl} \geq 1$  imposes more complicated restrictions on the spin. An obvious consequence of (35) is the condition

$$2s_l \geq z_l.$$

Here  $z_l$  is the coordination number of the  $l$ th node.

For bipartite graphs we shall derive a more refined necessary condition. For this we make several definitions. We denote our graph by  $\Gamma$ . Suppose that by cutting  $n$  edges it can be broken into two independent subgraphs  $\Gamma_1$  and  $\Gamma_2$ . Suppose that the following conditions is thereby satisfied. All vertices belonging simultaneously to  $\Gamma_1$  and the cut edges belong only to one sub lattice, for example,  $A$ . We chose an alternating sum of spins along the subgraph  $\Gamma_1$ :

$$\sum_{l \in \Gamma_1} \epsilon_l \cdot 2s_l = \sum_{l \in \Gamma_1} \sum_{k \in \Gamma_2} M_{lk}. \tag{62}$$

Here  $\epsilon_l = 1$  for the sub lattice  $A$  and  $\epsilon_l = -1$  for the sub lattice  $B$ . The sum of  $M_{kl}$  along the cut edges stands on the right side of (62).

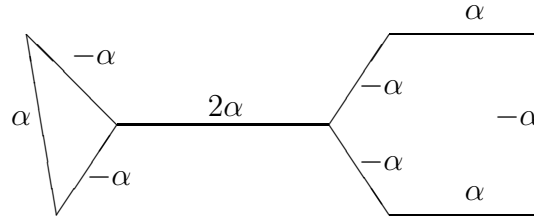
From (62) it is evident that

$$\sum_{l \in \Gamma_1} \epsilon_l \cdot 2s_l \geq n. \quad (63)$$

Here  $n$  is the number of cut edges. This condition is necessary. We have been unable to prove sufficiency of this condition.

### §7. Non bipartite, connected graphs.

It was shown in §6 that the presence of an even cycle in a graph leads to non uniqueness of the system (36). It turns out that two odd cycles joined by a chain lead to analogous degeneracy.



The scheme shown admissible changes of the number  $M$  (corresponding to edges) which do not change the spins. (Indeed, the sum of the changes at each node is equal to 0.) It suffices that the graph shown in the scheme be subgraph of  $\Gamma$ ; this already implies degeneracy of the system (36) on  $\Gamma$ . It is easy to see that the graph in the scheme was obtained from an even cycle (of length 10) by gluing together two sides. This is a degenerate even cycle. Thus, we consider the system (36) on a non bipartite graph. We begin to simplify it. We first break all even cycles. We then remove all degenerate even cycles so that the connectivity of the graph is preserved. For this we remove only non degenerate edges in the operation of breaking cycles in the degenerate case. Under this method of breaking degenerate cycles the connectivity of the graph is preserved [10]. In summary we break all degenerate even cycles. After applying the procedure of "cutting-off branches" (see §6) we arrive at a connected graph consisting of a single odd cycle. For one odd

cycle the problem has already been solved (see (50)). The condition for solvability in integers is the parity of the sum  $\sum 2s_k$  (see (52)). The condition of positivity of  $M$  requires further study.

### §8. The modified classical Heisenberg model.

In [6] it was shown that the quantum Heisenberg model considered in §§1,2 is equivalent to a modified classical Heisenberg model. This model can be described as follows. The three-components unit vector  $\mathbf{n}(l)$  (classical) is situated at the  $l$ th node of the lattice. The statistical sum of the classical model is

$$\Phi = \int_{S^2} \prod_{l=-L}^L d\Omega_l \prod_{l=-L}^{L-1} \left( \frac{1 - \mathbf{n}(l)\mathbf{n}(l+1)}{2} \right)^{M(l)} \quad (64)$$

and is equal to the square of the norm of the VBS wave function

$$|\psi\rangle = \prod_{l=-L}^{L-1} (a_l^+ b_{l+1}^+ - a_{l+1}^+ b_l^+)^{M(l)} |0\rangle. \quad (65)$$

The multi point correlation functions are computed by the formula

$$\begin{aligned} \langle \psi | \prod_{j=1}^N \widehat{S}^{a_j}(r_j) | \psi \rangle &= \prod_{j=1}^N (s(l_j) + 1) \int_{S^2} \prod_{l=-L}^L d\Omega_l \quad (66) \\ &\times \prod_{l=-L}^L \left( \frac{1 - \mathbf{n}(l) \cdot \mathbf{n}(l+1)}{2} \right)^{M(l)} \prod_{j=1}^N \mathbf{n}^{a_j(r_j)}. \end{aligned}$$

Here  $r_N > r_{N-1} > \dots > r_2 > r_1$ ,  $s(l)$  is the magnitude of the spin at the  $l$ th node, and  $n^a(l)$  is the component with index  $a$  ( $a = 1, 2, 3$ ) of the vector  $\mathbf{n}(l)$ .

We introduce the integral operator  $\widehat{K}_M$ . It acts in the space of functions  $f(\mathbf{n})$  on the unit sphere  $S^2$  according to the formula

$$(\widehat{K}_M f)(\mathbf{n}_2) = \int_{S^2} d\Omega \left( \frac{1 - \mathbf{n}_1 \cdot \mathbf{n}_2}{2} \right)^M f(\mathbf{n}_1). \quad (67)$$

In the next section we diagonalize the operator  $\widehat{K}_M$  and show that the operators  $\widehat{K}_M$  form a commuting family. An individual operator  $\widehat{K}_M$  is linear combination of  $M + 1$  projections. Using the properties of the operators  $\widehat{K}_M$ ,

we compute in explicit form the norm of the function (65) and the correlation functions.

We note that the VBS wave function of the inhomogeneous Heisenberg model for an arbitrary graph was found in (39), §2. The statistical sum for the modified classical Heisenberg model is equal to the norm of the VBS wave function (39) and can be computed by the formula

$$\Phi = \langle \psi | \psi \rangle = \int_{S^2} \prod_l d\Omega_l \prod_{\langle kl \rangle} \left( \frac{1 - \mathbf{n}_k \cdot \mathbf{n}_l}{2} \right)^{M_{kl}}. \quad (68)$$

Here  $M_{kl} \geq 0$  is an arbitrary choice of alternating numbers. For the complete graph all the vertices are joined by edges. If for some edge  $\langle kl \rangle$  we have  $M_{kl} = 0$ , then this is equivalent to the absence of the edge  $\langle kl \rangle$  in the graph. Hence, we can consider an arbitrary graph as special case of a complete graph.

### §9. Commuting transfer matrices.

We consider the integral operator  $\widehat{K}_M$  with kernel

$$K_M(\mathbf{n}_1, \mathbf{n}_2) = \left( \frac{1 - \mathbf{n}_1 \cdot \mathbf{n}_2}{2} \right). \quad (69)$$

We shall first find its eigenfunction. We denote them by  $\Lambda_M^A(\mathbf{n})$  – these are symmetric, traceless tensors of rank  $N$ . For example,

$$\Lambda_0 = 1, \quad \Lambda_1^a = n^a, \quad \Lambda_2^{a_1 a_2} = n^{a_1} n^{a_2} - \frac{1}{3} \delta_{a_2}^{a_1}, \quad (70)$$

$$\Lambda_3^{a_1 a_2 a_3} = n^{a_1} n^{a_2} n^{a_3} - \frac{1}{5} (n^{a_1} \delta_{a_3}^{a_2} + n^{a_2} \delta_{a_3}^{a_1} + n^{a_3} \delta_{a_2}^{a_1}).$$

Before writing out the general formula for  $\Lambda^A(\mathbf{n})$ , it is useful to introduce some notation. For a collection of indices  $A_N = \{a_j\}$  we set

$$n^{A_N} = \prod_{j=1}^N n^{a_j}. \quad (71)$$

For even  $N$ ,  $N = 2k$ , an important role is played below by a partition of the set  $A_N$  into pairs  $A_{2k} = \cup_{\alpha} \{a_{\alpha} b_{\alpha}\}$ . We define the delta function of the set  $A_N$  by the formula

$$\delta(A_{2k}) = \frac{1}{k!} \sum_{A_{2k} = \cup_{\alpha} \{a_{\alpha} b_{\alpha}\}} \prod_{\alpha} \delta_{b_{\alpha}}^{a_{\alpha}}. \quad (72)$$

We note that the number of terms in (72) is equal to

$$(2k - 1)!! = (2k)! / (2^k \cdot k!).$$

**Theorem 3** *The eigenfunction  $\Lambda^{A_N}$  of the integral operator  $\widehat{K}_M$  are given by the formula*

$$\Lambda^{A_N}(\mathbf{n}) = \sum_{k=0}^{[N/2]} \gamma_k(N) \sum_{A_{2k} \subset A_N} n^{A_N/A_{2k}} \delta(A_{2k}). \quad (73)$$

Here  $(A_N/A_{2k}) \cup A_{2k} = A_N$ . The coefficient  $\gamma_k(N)$  is

$$\gamma_k(N) = \prod_{j=1}^k \left( \frac{-1}{2N - 2j + 1} \right); \quad \gamma = 1. \quad (74)$$

The tensor  $\Lambda^{A_N}$  is a polynomial in  $n^a$  whose leading component is equal to  $n^{A_N}$ .

**Theorem 4** *The polynomial  $\Lambda^{A_N}(\mathbf{n})$  is an eigenfunction of the operator  $\widehat{K}_M$ ,*

$$\int_{S^2} d\Omega_l \left( \frac{1 - \mathbf{n}_1 \cdot \mathbf{n}_2}{2} \right)^M \Lambda^{A_N}(\mathbf{n}_1) = z(N, M) \Lambda^{A_N}(\mathbf{n}_2), \quad (75)$$

with eigenvalue

$$z(N, M) = \frac{1}{M+1} \prod_{j=0}^{N-1} \frac{j - M}{j + 2 + M}, \quad z(0, M) = \frac{1}{M+1}. \quad (76)$$

It is interesting to note that

$$z(N, M) = 0, \quad \text{if } N \geq M + 1.$$

Hence, the integral operator  $\widehat{K}_M$  is a linear combination of  $M + 1$  projections. The details of the computation of  $z(N, M)$  are presented in Appendix B. We note that the  $\Lambda^{A_N}(\mathbf{n})$  are eigenfunction of the Laplace operator on the sphere  $S^2$  (for details see §12).

The of the operator  $\widehat{K}_M$  do not depend on  $M$  (see (73)). This implies that the operators  $\widehat{K}_M$  commute for different values of  $M$ :

$$[\widehat{K}_{M_1}, \widehat{K}_{M_2}] = 0.$$

The norm of the VBS wave function or the statistical sum for the modified Heisenberg model (64) and the correlation functions (see §10) can be expressed in terms of the transfer matrices

$$T(r_2, r_1) = \prod_{l=r_1}^{r_2-1} \widehat{K}_{M(l)}. \quad (77)$$

All factors in (77) commute with one another. The set of eigenfunctions of the transfer matrix  $T$  is given by (73) and does not depend on the collection of numbers  $\{M(l)\}$ . Hence, the transfer matrices (77) form a commuting family for different collections of the numbers  $\{m(l)\}$ .

The number of independent components of the tensor  $\Lambda_N^{A_N}$  (which is symmetric and traceless) is equal to  $2N + 1$ . The tensor  $\Lambda_N^{A_N}$  generates an irreducible representation of the algebra  $o(3)$  of spin  $N$ . The product of irreducible representations decomposes into a direct sum of irreducible representations. We shall need the explicit form of this decomposition in the case where one of the spins is equal to 1:

$$\begin{aligned} \Lambda_N^{A_N} \cdot n^{a_{N+1}} &= \Lambda_{N+1}^{A_N \cup a_{N+1}} + \frac{1}{2N+1} \sum_{j=1}^N \Lambda_{N-1}^{a_1 \dots \widehat{a}_j \dots a_N} \delta_{a_{N+1}}^{a_j} \\ &- \frac{2}{(2N+1)(2N-1)} \sum_{1 \leq i < j \leq N} \Lambda_{N-1}^{a_1 \dots \widehat{a}_i \dots \widehat{a}_j \dots a_{N+1}} \delta_{a_j}^{a_i}. \end{aligned} \quad (78)$$

We recall that  $A_N = \{a_1, \dots, a_N\}$ . We note that in adding spin  $N$  and spin 1 only the spins  $N + 1$  and  $N - 1$  occur. We further define the projections  $P$  onto spin  $N + 1$  and spin  $N - 1$  by the formulas

$$P_{N+1}(\Lambda_N^{A_N} \cdot n^{a_{N+1}}) = \Lambda_{N+1}^{A_N \cup a_{N+1}}, \quad (79)$$

$$\begin{aligned}
P_{N-1}(\Lambda_n^{A_N} \cdot n^{a_{N+1}}) &= \frac{1}{2N+1} \sum_{j=1}^N \Lambda_{N-1}^{a_1 \dots \widehat{a}_j \dots a_N} \delta_{a_{N+1}}^{a_j} \\
&- \frac{2}{(2N+1)(2N-1)} \sum_{1 \leq i < j \leq N} \Lambda_{N-1}^{a_1 \dots \widehat{a}_i \dots \widehat{a}_j \dots a_{N+1}} \delta_{a_j}^{a_i}.
\end{aligned} \tag{80}$$

We remark that  $P_{N-1} + P_{N+1} = 1$  which follows from (79). The value of the projections with other indices on the polynomial  $\Lambda_N^{A_N} \mathbf{n}^{a_{N+1}}$  we define to be zero:

$$P_J(\Lambda_N^{A_N} n^{a_{N+1}}) = 0, \quad \text{if } j \neq N \pm 1.$$

It is natural to consider the number  $N$  to be the spin of the tensor  $\Lambda^{A_N}$ . In computing multi point correlators it turns out to be useful to consider the following combination of projections:

$$P_0(n^{a_N} P_{k_{N-1}}(n^{a_{N-1}} P_{k_{N-2}}(n^{a_{N-2}} \dots P_{k_4}(n^{a_4} P_{k_3}(n^{a_3} P_{k_2} n^{a_2 a_1}))) \dots)). \tag{81}$$

The expression (81) is well defined and is a linear combination of products of the delta functions  $\delta_{a-k}^{a_j}$ . It is different from 0 only if  $k_2 = 0, 2$  and for  $j = 2, \dots, N-2$  the equalities  $|k_{j+1} - k_j| = 1$  are satisfied. We present some examples:

$$\begin{aligned}
P_0(n^{a_1 a_2}) &= \frac{1}{3} \delta_{a_2}^{a_1}, \quad P_2(n^{a_1 a_2}) = \Lambda_2^{a_1 a_2}, \quad P_0(n^{a_3} P_k(n^{a_1 a_2})) = 0, \\
P_1(\Lambda_2^{a_1 a_2} n^{a_3}) &= \frac{1}{5} (n^{a_1} \delta_{a_3}^{a_2} + n^{a_2} \delta_{a_3}^{a_1}) - \frac{2}{15} n^{a_3} \delta_{a_2}^{a_1}, \\
P_0(n^{a_4} P_1(n^{a_3} P_2(n^{a_1 a_2}))) &= \frac{1}{15} (\delta_{a_4}^{a_1} \delta_{a_3}^{a_2} + \delta_{a_3}^{a_1} \delta_{a_4}^{a_2}) - \frac{2}{45} \delta_{a_2}^{a_1} \delta_{a_4}^{a_3}.
\end{aligned} \tag{82}$$

A combination of the projections (81) is a collection of nonnegative integers  $\{k_0, k_1, k_2, \dots, k_{N-1}, k_N\}$  such that  $k_0 = 0$ ,  $|k_{i+1} - k_i| = 1$ ,  $i = 0, 1, \dots, N-1$ ,  $k_N = 0$ . It is clear that it is also possible to consider such collections with another boundary condition  $k_N = l$ . It is obvious that  $l \leq N$ ,  $l \equiv N \pmod{2}$ . In §11 we show that for fixed  $k_N := l$  such sequences can be parameterized by standard Young tableaux of the form  $((N+l)/2, (N-l)/2)$ . Hence, the number of different combinations of projections of the form (52) is equal to 0 for  $N$  odd and equal to the Catalan number  $C_k = (2k)!/k!(k+1)!$  if  $N = 2k$ .

We finish this section with the remark that in principle, using the rules of passing from functions of spin variables to functions of spherical coordinates



on the sphere  $S^2$  described in [19], it is possible to find an expression for  $\Lambda^{A_N}$  in terms of the original quantum spins  $\widehat{S}^a$  (see §1). We illustrate the nature of the answer only with one example. We consider  $\Lambda_2^{a_1 a_2} = n^{a_1 a_2} - \frac{1}{3}\delta_{a_2}^{a_1}$ . In terms of the quantum spins  $\Lambda_2$  can be written as follows:

$$\Lambda_2^{a_1 a_2} = \frac{\widehat{S}^{a_1} \widehat{S}^{a_2} + \widehat{S}^{a_2} \widehat{S}^{a_1} - \frac{2}{3}s(s+1)\delta_{a_2}^{a_1}}{(s+1)(2s+3)}.$$

### §10. The norm and the two-point correlator.

We first consider an open chain and compute the norm of the VBS wave function (65). We shall proceed from formula (64). We remark that it is possible to rewrite the expression for the square of the norm of the wave function in terms of the transfer matrix (77):

$$\begin{aligned} \langle \psi | \psi \rangle &= \int \prod_{l=-L}^L d\Omega_l \prod_{l=-L}^{L-1} \left( \frac{1 - \mathbf{n}(l) \cdot \mathbf{n}(l+1)}{2} \right)^{M(l)} \\ &= \psi_0 = \Lambda_0 \prod_{l=-L}^{L-1} \widehat{K}_{M(l)} \Lambda_0. \end{aligned} \tag{83}$$

Using (75) and (76), we find

$$\langle \psi | \psi \rangle_0 = \prod_{l=-L}^{L-1} \frac{1}{M(l)+1}. \tag{84}$$

The norm of the wave function for a periodic chain can be computed in a similar way. Indeed,

$$\langle \psi | \psi \rangle_{\text{reg}} = \text{tr} \prod_{l=-L}^{L-1} \widehat{K}_{M(l)} = \text{tr} T(L, -L). \tag{85}$$

All the operators  $\widehat{K}$  commute, and their spectrum is known. The transfer matrix  $T(L, -L)$  has  $m+1$  nonzero eigenvalues. Here  $m = \min\{M(l)\}$ . According to (76), these eigenvalues are equal to

$$Z_N = \prod_{l=-L}^{L-1} Z(N, M(l)), \quad N = 0, 1, \dots, m. \tag{86}$$

Hence, the square of the norm of the wave function for a periodic chain is

$$\langle \psi | \psi \rangle_{\text{reg}} = \sum_{N=0}^m (2N + 1) \cdot Z_N. \quad (87)$$

The factor  $2N + 1$  in (87) describes the multiplicity of the degeneracy of the eigenvalue  $Z_N$ .

An especially simple formula for  $\langle \psi | \psi \rangle_{\text{reg}}$  is obtained for  $m = 1 = M(L)$ :

$$\langle \psi | \psi \rangle_{\text{reg}} = \prod_{l=-L}^{L-1} \frac{1}{M(l) + 1} - \frac{1}{2} \prod_{l=-L}^{L-1} \frac{-M(L)}{(M(l) + 1)(M(l) + 2)}.$$

We now consider the thermodynamic limit for  $\langle \psi | \psi \rangle_{\text{reg}}$  as  $L \rightarrow \infty$ . It is easy to see that

$$\langle \psi | \psi \rangle_{\text{reg}} = \sum_{N=0}^m (2N + 1) Z_N \rightarrow Z_0 = \prod_{l=-L}^{L-1} \frac{1}{M(l) + 1} = \langle \psi | \psi \rangle_0. \quad (88)$$

Hence, the thermodynamic limit does not depend on the boundary conditions. Computation of the limit in (88) is based on the inequality

$$Z(N_2, M) < Z(N_1, M) \quad \text{if } N_1 < N_2. \quad (89)$$

We now proceed to the computation of the two-point correlator for an open chain. We use the representation (66):

$$\psi_0^{-1} \int \prod_l d\Omega_l \prod_{l=-L}^{L-1} \left( \frac{1 - \mathbf{n}(l) \cdot \mathbf{n}(l+1)}{2} \right)^{M(l)} \mathbf{n}^{a_1}(r_1) \cdot \mathbf{n}^{a_2}(r_2). \quad (90)$$

Here  $r_2 > r_1$ . Formula (90) for the correlator can be written in terms of the transfer matrix (77) in the following manner:

$$\psi_0^{-1} \cdot \Lambda_0 \cdot T(L, r_2) n^{a_2}(r_2) T(r_2, r_1) n^{a_1}(r_1) T(r_1, -L) \Lambda_0. \quad (91)$$

In (91) the transfer matrices are defined in analogy to (77). For example,

$$T(L, r_2) = \prod_{l=r_2}^{L-1} \widehat{K}_{M(l)}, \quad r_2 < L. \quad (92)$$

The function  $\Lambda_0$  is an eigenfunction for the transfer matrix  $T(r_1, -L)$  with eigenvalue

$$\prod_{l=-L}^{r_1-1} \frac{1}{M(l) + 1}. \quad (93)$$

It is easy to see that (93) is contained as a factor in  $\langle \psi | \psi \rangle_0$  and hence cancels in (91). The function  $n^{a_1}(r_1) = \Lambda_1^{a_1}$  is an eigenfunction for (92) with eigenvalue

$$\prod_{l=r_1}^{r_2-1} \frac{-M(l)}{(M(l) + 1)(M(l) + 2)}. \quad (94)$$

Further, it is clear that  $T(L, r_2)$  does not contribute to the correlator. Finally, for (90) we find

$$\langle n^{a_2}(r_2)n^{a_1}(r_1) \rangle = \frac{1}{3} \delta_{a_2}^{a_1} \prod_{l=r_1}^{r_2-1} \left( \frac{-M(l)}{M(l) + 2} \right). \quad (95)$$

Formula (95) gives an expression for two-point correlator for finite open chain. Using (66), we find the correlator of two spins

$$\langle S^{a_1}(r_1)S^{a_2}(r_2) \rangle = \frac{(s(r_1) + 1)(s(r_2) + 1)}{3} \delta_{a_2}^{a_1} \prod_{l=r_1}^{r_2-1} \left( \frac{-M(l)}{M(l) + 2} \right). \quad (96)$$

Formula (96) gives an expression for the two-point correlator for the quantum model (38). In the thermodynamic limit formula (86) is preserved (the thermodynamic limit does not depend on the boundary conditions). It also gives an answer for the one-dimensional quasicrystal (10) in which  $M(l)$  assumes only the two values  $M_s$  and  $M_l$  in a quasi periodic manner. The asymptotic (12) follow directly from (96). If all the numbers  $M(l)$  are equal, then  $M = s$  and (96) reproduces the result (8).

We now rewrite formula (97) for the square of the norm of the VBS wave function in terms of generalized hyper geometric functions  ${}_pF_q$ . We recall their definitions (see [20]):

$${}_pF_q \left( \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \middle| x \right) := \sum_{k \geq 0} \frac{(\alpha_1)_k \cdots (\alpha_p)_k x^k}{(\beta_1)_k \cdots (\beta_q)_k k!}. \quad (97)$$

The symbol  $(\alpha)_k$  is defined as follows:

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \alpha(\alpha + 1) \cdots (\alpha + k - 1), \quad k \geq 0. \quad (98)$$

In terms of hypergeometric series, formula (87) takes the form

$$\begin{aligned} \langle \psi | \psi \rangle_{\text{reg}} &= \\ &= \left\{ \prod_{l=-L}^L \frac{1}{M(l) + 1} \right\} {}_{2L+3}F_{2L+2} \left( \begin{matrix} 3/2, 1, -M(-L), \dots, -M(-L) \\ 1/2, M(-L) + 2, \dots, M(L) + 2 \end{matrix} \middle| 1 \right). \end{aligned} \quad (99)$$

Using Dougall's formula [20] for the completely balanced series  ${}_5F_4$  we obtain an especially simple expression for the square of the norm of the wave function for a periodic chain with three nodes ( $L = 1$ ):

$$\begin{aligned} \langle \psi | \psi \rangle_{\text{reg}} &= \\ &= \frac{\Gamma(M(-1) + 1)\Gamma(M(0) + 1)\Gamma(M(1) + 1)\Gamma(M(-1) + M(0) + M(1) + 2)}{\Gamma(M(-1) + M(0) + 2)\Gamma(M(-1) + M(1) + 2)\Gamma(M(0) + M(1) + 2)}. \end{aligned} \quad (100)$$

### §11. The multi point correlator.

In §8 it was shown that the correlator for the quantum Heisenberg model and the correlator for the modified classical Heisenberg model are connected by the relation

$$\left\langle \prod_{j=1}^N \widehat{S}^{a_j}(r_j) \right\rangle_{kb} = \sum_{j=1}^N (s(r_j) + 1) \left\langle \prod_{j=1}^N n^{a_j}(r_j) \right\rangle. \quad (101)$$

Here  $r_N > r_{N-1} > \dots > r_2 > r_1$ . It is clear that the correlator is equal to 0 for odd  $N$ . The right side of formula (101) can be computed by means of commuting family of transfer matrices in analogy to the computation of the two-point correlator on the basis of (91). The transfer matrix between two nearest nodes  $n^{a_j}(r_j)$  and  $n^{a_{j+1}}(r_{j+1})$  can be by means of the polynomials  $\Lambda_{k_j}^{A_{k_j}}$  (we recall that we call  $k_j$  the spin of the tensor  $\Lambda^{A_{k_j}}$ ). The difference of two nearest spins  $k_j$  is equal to  $k_{j+1} - k_j = \pm 1$  (see (79)). Thus, the spins of the tensors  $\Lambda^A$  form a sequence  $k_0, k_1, \dots, k_N$  of nonnegative integers such that

$$k_0 = 0, |k_{j+1} - k_j| = 1, j = 0, \dots, N - 1, k_N = 0. \quad (102)$$

We use the sequences (102) to write out a formula for the multipoint correlator in explicit form. Since the function  $\Lambda_{k_{j-1}}^A$  is multiplied by  $n^{a_j}(r_j)$  at

the vertex  $r_j$ , we must use the projections (79). We denote by  $X(k, M)$  the normalized eigenvalue of (76):

$$X(k, M) = \frac{Z(k, M)}{Z(0, M)} = \frac{(-M)_k}{(M+2)_k}. \quad (103)$$

The symbol  $(\alpha)_k$  is given by formula (98).

We first write out the answer for correlation functions in the one dimensional case (all the  $M(l)$  are the same,  $M(l) = M = s$ ; see (22)).

**Theorem 5** *We have the equality*

$$\left\langle \prod_{j=1}^N n^{a_j}(r_j) \right\rangle = \sum_{\{k\}} \prod_{j=1}^{N-1} \{X(k_j, M)\}^{r_{j+1}-r_j} P_0(n^{a_N} \dots P_{k_j}(n^{a_1 a_2})) \dots. \quad (104)$$

The summation in (104) goes over all possible sequences (102). We remark that the last factor on the right side of (104) is a  $c$ -number equal to a linear combination of products of the delta functions  $\delta_{a_k}^{a_j}$  (and does not depend on the component  $n^a$  of the vector  $\mathbf{n}$  and the indices of the nodes  $r_j$ ). It is easy to see that

$$X(1, M) = \frac{-M}{M+2}, \quad X(2, M) = \frac{M(M-1)}{(M+2)(M+3)}.$$

Using equalities (82) for the projections, we obtain formulas (8) and (9) ( $s = M$ ). We note that (104) for even  $N$  gives an answer for multipoint correlation functions also for an open chain (in the thermodynamic limit everything remains unchanged).

We now present an answer for correlation functions in the inhomogeneous case (all the  $M(l)$  are distinct; see §2).

**Theorem 6** *We have the equality*

$$\left\langle \prod_{j=1}^N n^{a_j}(r_j) \right\rangle = \sum_{\{k\}} \prod_{j=1}^{N-1} \prod_{l=r_j}^{r_{j+1}-1} X(k_j, M(l)) P_0(n^{a_N} \dots P_{k_3}(n^{a_3} P_{k_2}(n^{a_1 a_2})) \dots). \quad (105)$$

The summation in (105) goes over all possible sequences (102).

The formula for the correlators (12) for quasicrystals follows from (105).

We make some remarks regarding the sequences (102) and the projections (81). First of all, it is natural to consider sequences (102) of length  $N + 1$  with the boundary condition  $k_N = l$  for fixed  $l$ . Let  $a_{l,N}$  be the number of them. It is clear that  $l \equiv N \pmod{2}$ . Using induction to  $N$ , it is possible to show that

$$a_{l,N} = \binom{(N-l)/2}{N} - \binom{(N-l)/2-1}{N}. \quad (106)$$

The full number of sequences (102) of length  $N + 1$  ( $k_N$  is not fixed) is equal to

$$\sum_{l \leq N} a_{l,N} = \binom{N}{[N/2]}. \quad (107)$$

From (106) it is evident that  $a_{l,N}$  is equal to the number of standard Young tableaux of the form  $((N+l)/2, (N-l)/2)$  and is also equal to the multiplicity of degeneracy of the irreducible representation of the Lie algebra  $o(3)$  of signature  $(N+l)/2$  in the  $N$ th tensor power of the fundamental representation. In our special case  $l = 0$ , and hence

$$a_{0,N} = \begin{cases} 0 & \text{if } N \equiv 1 \pmod{2} \\ \frac{(2k)!}{k!(k+1)!} & \text{if } N = 2k, k \text{ an integer.} \end{cases} \quad (108)$$

We shall construct a bijection between the set of sequences  $(k_0, \dots, k_N)$  such that  $k_0 = 0$ ,  $k_N = l$ ,  $k_j \in \mathbf{Z}_+$ ,  $|k_i - k_{i+1}| = 1$ ,  $i = 0, 1, \dots, N - 1$ , and the set of standard Young tableaux  $T$  of the form  $\lambda = ((N+l)/2, (N-l)/2)$ . To this end we consider the sequence  $k_0, k_1, \dots, k_N$ . We write the number  $i \in [1, N]$  in the first row of the diagram  $\lambda$  if  $k_{i-1} - k_i = -1$  and in the second row otherwise. The bijection has been constructed. We present an example. Suppose  $\{k_i\} = (0, 1, 2, 3, 2, 1, 0, 1, 0, 1)$ . Then

$$T = \begin{array}{cccccc} 1 & 2 & 3 & 7 & 9 & \\ 4 & 5 & 6 & 8 & & \end{array}.$$

We have thus obtained a representation for the  $N$ -point correlator (105) in the form of a sum over standard Young tableaux of the form  $(N/2, N/2)$ . It would be desirable to have a more explicit formula for projection (81), proceeding directly from the standard table  $T$  of the form  $((N+l)/2, (N-l)/2)$ . This problem is in the stage of solution. We only make several remarks. We denote by  $P_T$  the projection (81) corresponding to the

table  $T$ . It is clear that  $P_T$  is a linear combination of the polynomials  $\Lambda_l^A$ . For example, if

$$T = \begin{array}{cccccc} 1 & 3 & \dots & N-1 & N-l+1 & \dots & N \\ 2 & 4 & \dots & N-l & & & \end{array}$$

then

$$P_T = \left( \frac{1}{2\lambda+2} \right)^{(N-l)/2} \Lambda_l^{\{N-l+1, \dots, N\}} \prod_{j=1}^{(N-1)/2} \delta_{a_{2j}}^{a_{2j-1}}, \quad (109)$$

if

$$T = \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array},$$

then

$$P_T = \frac{1}{(2\lambda+2)(2\lambda+4)} \left\{ \delta_{a_4}^{a_1} \delta_{a_3}^{a_2} + \delta_{a_3}^{a_1} \delta_{a_4}^{a_2} - \frac{1}{\lambda+1} \delta_{a_2}^{a_1} \delta_{a_4}^{a_3} \right\}, \quad (110)$$

if

$$T = \begin{array}{ccc} 1 & 2 & 4 \\ 3 & & \end{array},$$

then

$$P_T = \frac{1}{2\lambda+4} \left\{ \Lambda^{a_1 a_4} \delta_{a_3}^{a_2} + \Lambda^{a_2 a_4} \delta_{a_3}^{a_1} - \frac{1}{\lambda+1} \delta_{a_2}^{a_1} \delta_{a_4}^{a_3} \right\}.$$

For the Lie algebra  $o(3)$  in the preceding formulas it is necessary to set  $\lambda = 1/2$ .

## §12. The $o(m)$ -generalization of the modified classical Heisenberg model.

In the preceding sections we considered the  $o(3)$ -modified classical Heisenberg model. There is a natural generalization to the case of the Lie algebra  $o(m)$  and irreducible representations corresponding to one-sided Young diagrams. The statistical sum for one-dimensional, classical Heisenberg model has the form

$$\Phi = \int_{S^{m-1}} \prod_{l=-L}^L d\Omega_l \prod_{l=-L}^{L-1} \left( \frac{1 - \mathbf{n}(l) \cdot \mathbf{n}(l+1)}{2} \right)^{M(l)}. \quad (111)$$

In (111) the vector  $\mathbf{n} \in S^{m-1}$ , and the invariant measure  $d\Omega$  is presented in Appendix B. We emphasize that now  $\mathbf{n}$  is an  $m$ -component unit vector.

The integral operator  $\widehat{K}_M$  is defined in analogy to (69):

$$(\widehat{K}_M f)(\mathbf{n}_2) = \int_{S^{m-1}} d\Omega_l \left( \frac{1 - \mathbf{n}_1 \cdot \mathbf{n}_2}{2} \right)^M f(\mathbf{n}_1). \quad (112)$$

The eigenfunctions of the operator  $\widehat{K}_M$  are also given by formula (73) in which the coefficient  $\gamma_k(N)$  are

$$\gamma_k(N) = \prod_{j=1}^k \frac{-1}{m + 2(N - j - 1)}, \quad \gamma_0(N) = 1. \quad (113)$$

The tensor  $\Lambda^{A_N}$  (see (73), (113)) is characterized uniquely by the conditions of symmetry and tracelessness, and its principal part  $n^{A_N}$ . We note that the dimension of the tensor  $\Lambda^{A_N}$  (for fixed  $N$ ) is

$$\begin{aligned} \binom{N+m-1}{m-1} - \binom{N+m-3}{m-1} &= \frac{2N+m-2}{m-3} \binom{N+m-3}{m-3} \\ &= \binom{N+m-2}{m-2} + \binom{N+m-3}{m-2}, \end{aligned} \quad (114)$$

and coincides with the dimension of the irreducible representation of the Lie algebra  $o(m)$  with highest weight  $(N, 0, \dots, 0)$ . We shall formulate the basic properties of the polynomials  $\Lambda^{A_N}$  as a theorem (in which  $\lambda = m/2 - 1$ ).

We consider the polynomials

$$\tilde{\Lambda}^{A_N}(\mathbf{n}) = \sum_{l \geq 0} \frac{r^{2l}}{2^l (-\lambda - N + 1)_l} \sum_{A_{2l} \subset A_N} n^{A_N/A_{2l}} \delta(A_{2l}). \quad (115)$$

Here  $\mathbf{n} \in \mathbf{R}^m$ ,  $r^2 = \sum_{a=1}^m (n^a)^2$ . Hence,  $\Lambda^{A_N} = \tilde{\Lambda}^{A_N}|_{S^{m-1}}$ .

**Theorem 7 1.** *The tensor  $\tilde{\Lambda}^{A_N}$  is traceless; in other words, if  $A_N = \{a_1, a_2, \dots\}$ , then*

$$\text{tr}_{\{a_1, a_2\}} \tilde{\Lambda}^{A_N} := \sum_{a_1=1}^m \sum_{a_2=1}^m \tilde{\Lambda}^{A_N}(\mathbf{n}) \delta_{a_2}^{a_1} = 0. \quad (116)$$



2. Let  $X$  be a symmetric polynomial in the variables  $n^1, \dots, n^m$ , and suppose  $\Pi_{\{a_1, a_2\}} x = 0$ . We suppose that  $X^{\max} = \sum_{A_N} \alpha_{A_N} \cdot n^{A_N}$ . Then

$$X = \sum_{A_N} \alpha_{A_N} \cdot \tilde{\Lambda}^{A_N}(\mathbf{n}).$$

3.  $\Delta \tilde{\Lambda}^{A_N} \equiv 0$ , where  $\Delta$  is the Laplace operator for  $\mathbf{R}^m$ . Hence,  $\tilde{\Lambda}^{A_N}$  is a harmonic polynomial of degree  $N$ , and the polynomials  $\tilde{\Lambda}^{A_N}$ ,  $A_N \subset [1, \dots, m]^N$  with the condition  $\#\{a_i \in A_N : a_i = 1\} \leq 1$ , form a basis in the space of harmonic polynomials of degree  $N$  in  $\mathbf{R}^m$ .

4. Let  $\Delta_S$  be the Laplace operator on the sphere  $S^{m-1}$ . Then  $A_S \Lambda^{A_N} = -N(N + m - 2)$ .

5. We consider the operators

$$E_{jk} = n^k \frac{\partial}{\partial n^j} - n^j \frac{\partial}{\partial n^k}, \quad 1 \leq j \neq k \leq m. \quad (117)$$

The generators  $E_{jk}$  generate the Lie algebra  $SO(m)$ :

$$[E_{jk}, E_{pq}] = \delta_{kp} E_{jq}, \quad \text{if } j < k, p < q. \quad (118)$$

The action of the generators  $E_{jk}$  on the polynomials  $\Lambda^{A_N}$  (for fixed  $N$ ) is given by the formula

$$E_{jk} \Lambda^{A_N} = \sum_{a \in A_N} \{ \Lambda^{(A_N \cup \{k\}) \setminus \{a\}} \delta_j^a - \Lambda^{(A_N \cup \{j\}) \setminus \{a\}} \delta_k^a \}. \quad (119)$$

6. For each polynomial  $f(\mathbf{n})$  of degree  $N$ ,  $\mathbf{n} \in \mathbf{R}^m$ , we define its harmonic projection (see [19]; here  $\lambda = m/2 - 1$ ):

$$(Hf)(\mathbf{n}) = \sum_{l=0}^{[N/2]} \frac{r^{2l} (\Delta^l f)(\mathbf{n})}{2^{2l} l! (-\lambda - N + 1)_l}. \quad (120)$$

Then the polynomial  $\tilde{\Lambda}^{A_N}$  is the harmonic projection of the monomial  $n^{A_N}$ :

$$\tilde{\Lambda}^{A_N}(\mathbf{n}) = (H n^{A_N})(\mathbf{n}).$$

Moreover, it is easy to see that

$$\Delta^k n^{A_N} = 2^k \cdot k! \sum_{A_{2k} \subset A_N} n^{A_N \setminus A_{2k}} \delta(A_{2k}). \quad (121)$$

7. Let  $A_N = \underbrace{\{a, \dots, a\}}_N$ . Then

$$\tilde{\Lambda}^{A_N}(\mathbf{n}) = \frac{N! \Gamma(\lambda) r^N}{2^N \Gamma(\lambda + N)} \cdot C_N^\lambda \left( \frac{n^a}{r} \right). \quad (122)$$

Here the  $C_N^\lambda$  are the Gegenbauer polynomials whose definition is presented in Appendix B.

It is interesting to note that formula (73) for  $\Lambda^{A_N}$  can be inverted, and the monomials  $n^{A_N}$  (see (71)) can be expressed in terms of the harmonic polynomials  $\tilde{\Lambda}^{A_N}$ :

$$n^{A_N} = \sum_{l=0}^{[N/2]} r^{2l} \beta_l(N) \sum_{A_{2l} \subset A_N} \tilde{\Lambda}^{A_N \setminus A_{2l}} \delta(A_{2l}). \quad (123)$$

Here  $\beta_l(N) = \gamma_l(M - l + 1)$ , where

$$\gamma_l(M) = \frac{1}{2^l (-\lambda - M + 1)_l}. \quad (124)$$

The identity (123) plays an important role in further considerations, since it makes it possible to solve the problem of decomposing the tensor product of two irreducible (single-sided) representations of the lie algebra  $o(m)$  into irreducible components. In particular,

$$\begin{aligned} \Lambda_N^{A_N} \cdot n^{a_{N+1}} &= \Lambda_{N+1}^{A_N \cup a_{N+1}} + \frac{1}{2(\lambda + N)} \sum_{j=1}^N \Lambda_{N-1}^{a_1 \dots \hat{a}_j \dots a_N} \delta_{a_{N+1}}^{a_j} \\ &- \frac{1}{2(\lambda + N)(\lambda + N - 1)} \sum_{1 \leq i < j \leq N} \Lambda_{N-1}^{a_1 \dots \hat{a}_i \dots \hat{a}_j \dots a_{N+1}} \delta_{a_i}^{a_j}. \end{aligned} \quad (125)$$

We thus consider projections analogous to (79) and (80):

$$P_{N+1}(\Lambda_N^{A_N} \cdot n^{a_{N+1}}) = \Lambda_{N+1}^{A_N \cup a_{N+1}}, \quad (126)$$

$$P_{N-1}(\Lambda_N^{A_N} \cdot n^{a_{N+1}}) = \Lambda_N^{A_N} \cdot n^{a_{N+1}} - \Lambda_{N+1}^{A_N \cup a_{N+1}}, \quad (127)$$

$$P_k(\Lambda_N^{A_N} \cdot n^{a_{N+1}}) = 0, \quad \text{if } k \neq N \pm 1.$$

We note that the operators  $\widehat{K}_M$ , as in the case  $m = 3$ , form a commuting family:

$$[\widehat{K}_{M_1}, \widehat{K}_{M_2}] = 0. \quad (128)$$

It follows from Theorem 111 that the polynomials  $\Lambda^{A_N}$  are eigenfunctions of the operators  $\widehat{K}_M$ :

$$\widehat{K}_M \Lambda_N^{A_N} = Z_m(N, M) \Lambda_N^{A_N}. \quad (129)$$

The eigenvalue  $Z_m(N, M)$  can be computed from the formula

$$Z_m(N, M) = \frac{\Gamma(2\lambda + 1)}{\Gamma(\lambda + \frac{1}{2})} \frac{\Gamma(M + \lambda + \frac{1}{2})(-M)_N}{\Gamma(M + 2\lambda + 1)(M + 2\lambda + 1)_N}. \quad (130)$$

We have used here the notation (98). The proof of formulas (129) and (130) is presented in Appendix B. Another important characteristic is

$$X_m(k, M) = \frac{Z_m(k, M)}{Z_m(0, M)} \quad (131)$$

which can be computed by means of (130) and is equal to

$$X_m(k, M) = \frac{(-M)_k}{(M + 2\lambda + 1)_k}. \quad (132)$$

We can now conclude the computation of the multipoint correlation functions for the one-dimensional, modified, classical Heisenberg  $O(m)$ -model. The answer is given by formulas (104) and (105) in which the  $X(k_j, M)$  are now replaced by (132), the projections (79) and (80) are replaced by the projections (126) and (127), and the summation both in (104) and (105) goes over sequences  $(k_0, k_1, \dots, k_N)$  such that  $k_0 = k_n = 0$ ,  $k_i \in \mathbf{Z}_+$ ,  $|k_i - k_{i+1}| = 1$ ,  $i = 0, 1, \dots, N - 1$ .

In conclusion we present formulas for square of the norm of the VBS wave function of the classical Heisenberg  $O(m)$ -model for an open and periodic one-dimensional chain (we recall that  $\lambda = m/2 - 1$ ):

$$\langle \psi | \psi \rangle_0 = \left\{ \frac{\Gamma(2\lambda + 1)}{\Gamma(\lambda + \frac{1}{2})} \right\}^{2L} \prod_{l=-L}^{L-1} \frac{\Gamma(M(l) + \lambda + \frac{1}{2})}{\Gamma(M(l) + 2\lambda + 1)}. \quad (133)$$

We note that if all  $M(l) \in \mathbf{Z}_+$ , then

$$\langle \psi | \psi \rangle_0 = \prod_{l=-L}^{L-1} \frac{(\lambda + \frac{1}{2})_{M(l)}}{(2\lambda + 1)_{M(l)}}. \quad (134)$$

Formulas (133), (134) for  $\lambda = 1/2$  go over into (84):

$$\begin{aligned} \langle \psi | \psi \rangle_{\text{reg}} &= \prod_{l=-L}^L \frac{\Gamma(2\lambda + 1)\Gamma(M(l) + \lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2})\Gamma(M(l) + 2\lambda + 1)} \\ &\times {}_{2L+3}F_{2L+2} \left( \begin{matrix} 2\lambda, \lambda + 1, -M(-L), \dots, -M(L) \\ \lambda, M(-L) + 2\lambda + 1, \dots, M(L) + 2\lambda + 1 \end{matrix} \middle| 1 \right). \end{aligned} \quad (135)$$

### §13. An algebraic approach to the computation of correlators for the multidimensional Heisenberg model.

We consider the statistical sum for the classical Heisenberg  $O(m)$ -model on a multidimensional lattice

$$\Phi(M) = \int_{S^{m-1}} \prod_{\text{sites}} d\Omega \prod_{\langle ij \rangle} \left( \frac{1 - \mathbf{n}_i \mathbf{n}_j}{2} \right)^M. \quad (136)$$

It is clear that  $\Phi(M)$  is equal to the square of the norm of the VBS wave function (39) for quantum Heisenberg model describing the interaction of spins  $s = Mz/2$  (here  $z$  is the coordination number of the lattice). Computation of correlation functions of the quantities  $(1 - \mathbf{n}_k \cdot \mathbf{n}_l)/2$ , where  $k, l$  are nearest neighbors, is of major interest. First of all the mean value of  $(1 - \mathbf{n}_1 \cdot \mathbf{n}_2)/2$  is the Néel order parameter

$$\left\langle \frac{1 - \mathbf{n}_1 \mathbf{n}_2}{2} \right\rangle = \frac{\int \prod_j d\Omega \prod_{\langle ij \rangle} ((1 - \mathbf{n}_i \cdot \mathbf{n}_j)/2)^M ((1 - \mathbf{n}_1 \cdot \mathbf{n}_2)/2)}{\int \prod_j d\Omega \prod_{\langle ij \rangle} ((1 - \mathbf{n}_i \cdot \mathbf{n}_j)/2)^M}. \quad (137)$$

Equality to one of the mean of  $(1 - \mathbf{n}_1 \cdot \mathbf{n}_2)/2$  is equivalent to the existence of a Néel order in the system. For one-dimensional lattice of  $O(m)$ -spins  $s$  the Néel order parameter is computed in (134):

$$\left\langle \frac{1 - \mathbf{n}_1 \cdot \mathbf{n}_2}{2} \right\rangle = \frac{s + \lambda + \frac{1}{2}}{s + 2\lambda + 1} < 1. \quad (138)$$

Here  $\lambda = m/2 - 1$ . Equality (138) shows the absence of Néel order for the model considered. It is interesting to note that for one-dimensional Heisenberg chain there is no correlation between Néel order parameters:

$$\langle (1 - \mathbf{n}(1) \cdot \mathbf{n}(2))(1 - \mathbf{n}(j) \cdot \mathbf{n}(j+1)) \rangle = \langle 1 - \mathbf{n}(1) \cdot \mathbf{n}(2) \rangle \langle 1 - \mathbf{n}(j) \cdot \mathbf{n}(j+1) \rangle.$$

To compute the correlation functions of several quantities  $(1 - bf n_k \cdot \mathbf{n}_l)/2$  we consider the generating function for the correlators

$$\Phi_\Gamma(\{M_{ij}\}) = \int_{S^{m-1}} \prod_{\text{sites}} d\Omega \prod_{\langle ij \rangle} \left( \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right)^{M_{ij}}. \quad (139)$$

In the general case all the alternating numbers  $M_{ij}$  are assumed to be distinct. It is most natural to consider the Heisenberg model for the complete graph (all the vertices are joined by edges) with  $p + 1$  vertices

$$\Phi_{p+1}(\{M_{ij}\}) = \int_{S^{m-1}} \prod_{j=1}^{p+1} d\Omega \prod_{1 \leq i < j \leq p+1} \left( \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right)^{M_{ij}}. \quad (140)$$

We emphasize again that we consider the alternating numbers  $M_{ij}$  as independent parameters. In this section we shall prove that  $\Phi_\Gamma$  is a rational function of (integer) parameters  $M_{ij}$ . We note that an arbitrary graph can be obtained from complete graph by annihilating some of the parameters  $M_{ij}$ . Indeed, the condition  $M_{ij} = 0$  is equivalent to the absence of the edge  $\langle ij \rangle$  in the graph.

We proceed to the investigation of the statistical sum (140) for the Heisenberg  $O(m)$ -model on a complete graph with  $p + 1$  vertices. We wish to find recurrence relations between the functions  $\Phi$ . For this we first compute the integral with respect to a fixed invariant measure  $d\Omega_j$  of the integral (140) in terms of generalized hypergeometric functions. We must thus compute the integral

$$\int_{S^{m-1}} d\Omega_0 \prod_{j=1}^p \left( \frac{1 - \mathbf{n}_0 \cdot \mathbf{n}_j}{2} \right)^{M_{0j}}. \quad (141)$$

Before presenting an explicit expression for the integral (141), we give some definitions and examples. We consider collections of complex numbers  $z_{ij}$  and nonnegative integers  $k_{ij}$  ( $1 \leq i < j \leq p$ ). We denote these collections by  $z^{(p)}$  and  $k^{(p)}$ . For the collection  $k^{(p)}$  we define

$$|k^{(p)}| := \sum_{1 \leq i < j \leq p} k_{ij}, \quad (k^{(p)})! := \prod_{1 \leq i < j \leq p} (k_{ij})!, \quad (142)$$

$$k_i^{(p)} = \sum_{1 \leq l < i} k_{li} + \sum_{i \leq l \leq p} k_{il}.$$

We now define the generalized hypergeometric series

$$F^{(p)} \left( \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta \end{matrix} \mid z^{(p)} \right)$$

depending on  $p(p-1)/2$  complex variables  $z^{(p)}$  and on  $p+1$  real parameters  $\alpha_1, \dots, \alpha_p, \beta$ . The function  $F^{(p)}$  is given by means of  $\frac{1}{2}p(p-1)$ -fold sum over the collections  $k^{(p)}$ :

$$F^{(p)} \left( \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta \end{matrix} \mid z^{(p)} \right) := \sum_{k^{(p)}} \frac{\prod_{i=1}^p (\alpha_i)_{k_i} k_i^p}{(\beta)_{|k^{(p)}}} \prod_{1 \leq i < j \leq p} \frac{z_{ij}^{k_{ij}}}{(k_{ij})!}. \quad (143)$$

Here we use the notation (98):

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}.$$

We note that if the  $\alpha_j$  are nonnegative integers, then the series (143) breaks off and the function

$$F^{(p)} \left( \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta \end{matrix} \mid z^{(p)} \right)$$

can be represented as a finite sum. We present examples of the functions  $F^{(p)}$  for  $p = 2, 3$  ( $F^{(1)} \equiv 1$ ):

$$F^{(2)} \left( \begin{matrix} \alpha_1, \alpha_2 \\ \beta \end{matrix} \mid z \right) = \sum_{k \geq 0} \frac{(\alpha_1)_k (\alpha_2)_k}{(\beta)_k} \cdot \frac{z^k}{k!}. \quad (144)$$

$$\begin{aligned} F^{(3)} \left( \begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta \end{matrix} \mid z_1, z_2, z_3 \right) &= \sum \frac{(\alpha_1)_{k_2+k_3} (\alpha_2)_{k_1+k_3} (\alpha_3)_{k_1+k_2}}{(\beta)_{k_1+k_2+k_3}} \\ &\times \frac{z_1^{k_1}}{k_1!} \cdot \frac{z_2^{k_2}}{k_2!} \cdot \frac{z_3^{k_3}}{k_3!}. \end{aligned} \quad (145)$$

We now formulate the result of evaluating the integral (141).

**Theorem 8** Let  $\mathbf{n}_0, \mathbf{n}_1, \dots, \mathbf{n}_p \in S^{m-1}$ . Then

$$\int_{S^{m-1}} d\Omega \prod_{j=1}^p \left( \frac{1 - \mathbf{n}_0 \cdot \mathbf{n}_j}{2} \right)^{M_{0j}} = \frac{\Gamma(2\lambda + 1)}{\Gamma(\lambda + \frac{1}{2})} \cdot \frac{\Gamma(M_{01} + \dots + M_{0p} + \lambda + \frac{1}{2})}{\Gamma(M_{01} + \dots + M_{0p} + 2\lambda + 1)}$$

$$\times F^{(p)} \left( \begin{matrix} -M_{01}, -M_{02}, \dots, -M_{0p} \\ -M_{01} - M_{02} - \dots - M_{0p} - \lambda + \frac{1}{2} \end{matrix} \middle| \left\{ \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right\}_{1 \leq i < j \leq p} \right). \quad (146)$$

We recall that  $\lambda = m/2 - 1$ . We again emphasize that if all the  $M_{0j} \in \mathbf{Z}_+$ , then the series on the right side of (146) contains only a finite number of terms.

From Theorem 8 we obtain a recurrence relation for the statistical sum (140) of the classical Heisenberg  $o(m)$ -model on a complete graph. We denote the vertices of the complete graph with  $p + 1$  vertices by  $l = 0, 1, \dots, p$ . We decompose the set of alternating numbers  $\{M_{lm}\} = \{M_{0j}\} \cup \{M_{ij}\}$  where  $1 \leq i, j \leq p$ . We consider the complete graph with  $p$  vertices as a subgraph of the complete graph with  $p + 1$  vertices for which the vertex  $l = 0$  and the edges  $\langle 0, 1 \rangle, \dots, \langle 0, p \rangle$  have been removed.

**Theorem 9** The statistical sums  $\Phi_p$  and  $\Phi_{p+1}$  are connected by the relation

$$\Phi_{p+1}(\{M_{0j}\} \cup \{M_{ij}\}) = \frac{\Gamma(2\lambda + 1)}{\Gamma(\lambda + \frac{1}{2})} \cdot \frac{\Gamma(M_{01} + \dots + M_{0p} + \lambda + \frac{1}{2})}{\Gamma(M_{01} + \dots + M_{0p} + 2\lambda + 1)}$$

$$\times \sum_{k^{(p)}} \frac{\prod_{j=1}^p (-M_{0j})_{k_j^{(p)}}}{(k^{(p)})! (-\sum_{j=1}^p M_{0j} - \lambda + \frac{1}{2})_{|k^{(p)}|}} \cdot \Phi_p(\{M_{ij} + k_{ij}\}). \quad (147)$$

The proof follows immediately from Theorem 8 and formula (143). As an example we present the formulas for  $\Phi_p$  for  $p \leq 3$ :

$$\Phi_0 = \Phi_1 = 1, \quad \Phi_2(M) = \frac{\Gamma(2\lambda + 1)}{\Gamma(\lambda + \frac{1}{2})} \cdot \frac{\Gamma(M + \lambda + \frac{1}{2})}{\Gamma(M + 2\lambda + 1)}, \quad (148)$$

$$\Phi_3(M_1, M_2, M_3) = \frac{[\Gamma(2\lambda + 1)]^2 \Gamma(M_1 + M_2 + M_3 + 2\lambda + 1)}{[\Gamma(\lambda + \frac{1}{2})]^3}$$

$$\times \prod_{j=1}^3 \frac{\Gamma(M_j + \lambda + \frac{1}{2})}{\Gamma(M_1 + M_2 + M_3 - M_j + 2\lambda + 1)}.$$

It should be noted that Theorem 8 can be used not only for complete graphs. We consider, for example, a graph  $\Gamma$  containing a vertex  $l$  of multiplicity 2. It may be assumed that  $l = 1$  and the vertex 1 is joined only with vertices 2 and 3. We denote by  $\widehat{\Gamma}$  the graph obtained from  $\Gamma$  by removing vertex 1 and the edges  $\langle 12 \rangle$  and  $\langle 13 \rangle$ . The statistical sums corresponding to the graphs  $\Gamma$  and  $\widehat{\Gamma}$  are connected by relation

$$\begin{aligned} \Phi_{\Gamma}(M_{12}, M_{13}, M_{23}, \dots) &= \frac{\Gamma(2\lambda + 1)}{\Gamma(\lambda + \frac{1}{2})} \cdot \frac{\Gamma(M_{12} + M_{13} + \lambda + \frac{1}{2})}{\Gamma(M_{12} + M_{13} + 2\lambda + 1)} \quad (149) \\ &\times \sum_{k \geq 0} \frac{(-M_{12})_k (-M_{13})_k}{k! (-M_{12} - M_{13} - \lambda + \frac{1}{2})_k} \Phi_{\widehat{\Gamma}}(M_{23} + k, \dots). \end{aligned}$$

The statistical sum  $\Phi_{\Gamma}$  was computed in §§10, 12 for a periodic, one-dimensional chain (a polygon). If (135) is substituted into (149) we obtain a new identity for generalized hypergeometric series:

$$\begin{aligned} & {}_{p+2}F_{p+1} \left( \begin{matrix} 2\lambda, \lambda + 1, -m_1, \dots, -m_p \\ \lambda, m_1 + 2\lambda + 1, \dots, m_p + 2\lambda + 1 \end{matrix} \middle| 1 \right) \quad (150) \\ &= \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda + 1)} \cdot \frac{\Gamma(m_{p-1} + 2\lambda + 1) \Gamma(m_p + 2\lambda + 1) \Gamma(m_{p-1} + m_p + \lambda + \frac{1}{2})}{\Gamma(m_{p-1} + \lambda + \frac{1}{2}) \Gamma(m_p + \lambda + \frac{1}{2}) \Gamma(m_{p-1} + m_p + 2\lambda + 1)} \\ &\times \sum_{k \geq 0} \frac{(-m_{p-1})_k (-m_p)_k (\lambda + \frac{1}{2})_k}{k! (-m_{p-1} - m_p - \lambda + \frac{1}{2})_k (2\lambda + 1)_k} \\ &\times {}_{p+1}F_p \left( \begin{matrix} 2\lambda, \lambda + 1, -m_1, \dots, -m_{p-2}, -k \\ \lambda, m_1 + 2\lambda + 1, \dots, m_{p-2} + 2\lambda + 1, k + 2\lambda + 1 \end{matrix} \middle| 1 \right). \end{aligned}$$

In particular, for  $p = 3$  we arrive at Doudall's formula [20] for the completely balanced series  ${}_5F_4$ . For  $p = 4$  we arrive at the formula for the transformation of the completely balanced series  ${}_6F_5$  into the Saalschütz series  ${}_4F_3$ :

$$\begin{aligned} & {}_6F_5 \left( \begin{matrix} 2\lambda, \lambda + 1, -m_1, -m_2, -m_3, -m_4 \\ \lambda, m_1 + 2\lambda + 1, \dots, m_4 + 2\lambda + 1 \end{matrix} \middle| 1 \right) \\ &= \frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(m_1 + 2\lambda + 1) \Gamma(m_2 + 2\lambda + 1) \Gamma(m_1 + m_2 + \lambda + \frac{1}{2})}{\Gamma(2\lambda + 1) \Gamma(m_1 + \lambda + \frac{1}{2}) \Gamma(m_2 + \lambda + \frac{1}{2}) \Gamma(m_1 + m_2 + \lambda + 1)} \\ &\times {}_4F_3 \left( \begin{matrix} -m_1, m_2, m_3 + m_4 + 2\lambda + 1, \lambda + \frac{1}{2} \\ -m_1 - m_2 - \lambda + \frac{1}{2}, m_3 + 2\lambda + 1, m_4 + 2\lambda + 1 \end{matrix} \middle| 1 \right). \quad (151) \end{aligned}$$



We note that the recurrence relation (149) makes it possible to compute the statistical sum  $\Phi_\Gamma$  for the following graph  $\Gamma$ :

$$\Gamma = \left\{ \begin{array}{ccc} & m_1 & \\ m_4 & \boxed{\begin{array}{c} \diagdown \\ m_5 \\ \diagup \end{array}} & m_2 \\ & m_3 & \end{array} \right\}.$$

$$\begin{aligned} \Phi_\Gamma &= \frac{[\Gamma(2\lambda + 1)]^3 \Gamma(m_1 + m_2 + \lambda + \frac{1}{2})\Gamma(m_3 + \lambda + \frac{1}{2})\Gamma(m_4 + \lambda + \frac{1}{2})}{[\Gamma(\lambda + \frac{1}{2})]^4 \Gamma(m_1 + m_2 + 2\lambda + 1)\Gamma(m_3 + m_4 + 2\lambda + 1)} \\ &\times \frac{\Gamma(m_5 + \lambda + \frac{1}{2})\Gamma(m_3 + m_4 + m_5 + 2\lambda + 1)}{\Gamma(m_3 + m_5 + 2\lambda + 1)\Gamma(m_4 + m_5 + 2\lambda + 1)} \\ &\times {}_4F_3 \left( \begin{array}{c} -m_1, m_2, m_3 + m_4 + m_5 + 2\lambda + 1, m_5 + \lambda + \frac{1}{2} \\ -m_1 - m_2 - \lambda + \frac{1}{2}, m_3 + m_5 + 2\lambda + 1, m_4 + m_5 + 2\lambda + 1 \end{array} \middle| 1 \right). \end{aligned}$$

The proof of Theorem 8 makes use of a large number of different identities for sums of products of binomial coefficients; they are rather cumbersome and will be presented in a separate paper. Here we note only that Theorem 8 follows from the more general Theorem B, whose formulation is presented in Appendix B.

### Appendix A.

We represent the operator  $a_l^+$  as the operator of multiplication by  $x_l$  and the operator  $b_l^+$  as the operator of multiplication by  $y_l$ . The spin operators at the  $l$ th node can then be represented in the form

$$S_l^+ = x_l \frac{\partial}{\partial y_l}; \quad S_l^- = \frac{\partial}{\partial x_l}; \quad (A.1)$$

$$2S_l^z = x_l \frac{\partial}{\partial x_l} - y_l \frac{\partial}{\partial y_l}.$$

We consider the space of the irreducible representation  $V_{s_l}$  of spin  $s$ . A basis can be taken in the form

$$v_\mu = x_l^{s_l + \mu} y_l^{s_l - \mu}; \quad \mu = -s_l, s_l + 1, \dots, s_l - 1, s_l. \quad (A.2)$$

The spin operators act on the basis in the following manner:

$$\begin{aligned}
S_l^+ v_\mu &= (s_l - \mu) v_{\mu+1}; \\
S_l^- v_\mu &= (s_l + \mu) v_{\mu-1}; \\
S_l^z v_\mu &= \mu v_\mu.
\end{aligned} \tag{A.3}$$

We now consider the tensor product of two irreducible representations  $V_{s_l} \otimes V_{s_k}$ . The spin operators add:

$$S^\pm = S_l^\pm + S_k^\pm, \quad S^z = S_l^z + S_k^z. \tag{A.4}$$

The tensor product decomposes into a direct sum of irreducible representations:

$$V_{s_l} \otimes V_{s_k} = \sum_{J=|s_l+s_k|}^{s_l+s_k} V_J. \tag{A.5}$$

We shall construct the leading vector  $v_J$  of the irreducible representation  $V_J$ . We seek it in the form

$$v_J = \sum_{\mu_l+\mu_k=J} a_{\mu_l,\mu_k} x_l^{s_l+\mu_l} y_l^{s_l-\mu_l} x_k^{s_k+\mu_k} y_k^{s_k-\mu_k}. \tag{A.6}$$

The leading vector is a polynomial satisfying the two equations

$$S^z v_J = J v_J; \quad S^+ v_J = 0. \tag{A.7}$$

These equations reduce to recurrence relations for the coefficients  $a_{\mu_l,\mu_k}$  which have the unique solution

$$v_J = x_l^{2s_l-\mu} x_k^{2s_k-\mu} (y_l x_k - y_k x_l)^\mu. \tag{A.8}$$

Here  $\mu = s_l + s_k - J$ .

The remaining vectors of the representation  $V_J$  can be obtained from the leading vector by means of the action of the lowering operator  $S^-$ . However, it should be noted that the operator  $S^-$  commutes with the operator of multiplication by bracket in (A.8):

$$[S^-, y_l x_k - y_k x_l] = 0. \tag{A.9}$$

From this it follows that all the vectors of the irreducible representation  $V_J$  are divisible by  $(y_l x_k - y_k x_l)^\mu$ .

We have thus proved that if it is known that after addition of two spins  $s_l$  and  $s_k$  there is no projection onto states with full spin

$$s_l + s_k + 1 - \mu \leq s_l + s_k,$$

then all the vectors of the representation are divisible by  $(y_l x_k - y_k x_l)^\mu$ . We recall that  $a_k^+ = x_k$ ,  $b_l^+ = y_l$ .

## Appendix B.

To compute the eigenvalues of the integral operator  $\widehat{K}_M$  for the Heisenberg  $O(m)$ -model on a one-dimensional chain we use properties of the Gegenbauer polynomials  $C_N^\lambda(x)$ . We recall [19] that

$$C_N^\lambda(x) = \sum_{l \geq 0} \frac{(-1)^l (\lambda)_{N-l}}{l!(N-2l)!} (2x)^{N-2l}. \quad (B.1)$$

We assume below that  $\lambda = m/2 - 1$ . The relation between the Gegenbauer polynomials  $C_N^\lambda(x)$  and harmonic polynomials  $\Lambda^{A_N}(\mathbf{n})$  defined by formulas (73) and (113) consists in the following equality:

$$\sum_{A_N} \Lambda_{A_N}^{A_N}(\mathbf{n}_1) n_2^{A_N} = \frac{N!}{2^N (\lambda)_N} C_N^\lambda(\mathbf{n}_1 \cdot \mathbf{n}_2). \quad (B.2)$$

Formula (B.2) can be verified directly. We note that

$$C_N^\lambda(1) = \frac{(2\lambda)_N}{N!}. \quad (B.3)$$

Hence,

$$\sum_{A_N} \Lambda_{A_N}^{A_N}(\mathbf{n}) n^{A_N} = \frac{N!}{2^N (\lambda)_N} \binom{N + 2\lambda - 1}{N}. \quad (B.4)$$

We now proceed to evaluate the integrals over the  $(m-1)$ -dimensional sphere of functions containing the Gegenbauer polynomials  $C_N^\lambda(\mathbf{n}_i \cdot \mathbf{n}_j)$ . We first recall (see [19]) the definition of spherical coordinates in  $\mathbf{R}^m$  and the invariant

measure on the sphere  $S^{m-1}$ . Cartesian and spherical coordinates are connected by the transformation

$$\begin{aligned}
x_1 &= r \sin \theta_{m-1} \cdots \sin \theta_2 \sin \theta_1, \\
x_2 &= r \sin \theta_{m-1} \cdots \sin \theta_2 \cos \theta_1, \\
&\dots\dots\dots \\
x_{m-1} &= r \sin \theta_{m-1} \cos \theta_{m-2}, \\
x_m &= r \cos \theta_{m-1}.
\end{aligned} \tag{B.5}$$

Here  $0 \leq r \leq \infty$ ,  $0 \leq \theta_1 < 2\pi$ ,  $0 \leq \theta_k < \pi$ ,  $k \neq 1$ .

The invariant measure on the sphere  $S^{m-1}$  is defined by the formula

$$d\Omega = \frac{\Gamma(m/2)}{2\pi^{m/2}} \sin^{m-2} \theta_{m-1} \cdots \sin \theta_2 d\theta_1 \cdots d\theta_{m-1}. \tag{B.6}$$

Using the transition to spherical coordinates, it is not hard to show that

$$\begin{aligned}
&\int_{S^{m-1}} d\Omega_j \left( \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right)^M (\mathbf{n}_i \cdot \mathbf{n}_j)^N \\
&= \frac{\Gamma(2\lambda + 1)\Gamma(M + \lambda + \frac{1}{2})\Gamma(N + \lambda + \frac{1}{2})}{[\Gamma(\lambda + \frac{1}{2})]^2\Gamma(M + N + 2\lambda + 1)} {}_2F_1 \left( \begin{matrix} -N, M + \lambda + \frac{1}{2} \\ -N - \lambda + \frac{1}{2} \end{matrix} \middle| 1 \right).
\end{aligned} \tag{B.7}$$

We recall that  $\lambda = m/2 - 1$ . Further, using (B.7) and (B.1), it is possible to evaluate the integral

$$\begin{aligned}
&\int_{S^{m-1}} d\Omega \left( \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right)^M C_N^\lambda(\mathbf{n}_i \cdot \mathbf{n}_k) \\
&= \frac{\Gamma(2\lambda + 1)\Gamma(M + \lambda + \frac{1}{2})(-M)_N}{\Gamma(\lambda + \frac{1}{2})\Gamma(M + N + 2\lambda + 1)} C_N^\lambda(\mathbf{n}_i \cdot \mathbf{n}_k).
\end{aligned} \tag{B.8}$$

We now proceed to the proof of Theorem 70 and formula (129). To this end we consider the integral

$$\Theta(\mathbf{n}_i) = \int_{S^{m-1}} d\Omega \left( \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right)^M \Lambda^{A_N}(\mathbf{n}_j). \tag{B.9}$$

From the Theorem 7, 2) it follows that  $\Theta(\mathbf{n}_i) = c\Lambda^{A_N}(\mathbf{n}_i)$ . In order to find the constant  $c$ , we consider the convolution (B.2) of the function  $\Theta(\mathbf{n}_i)$  with  $n_i^{A_N}$ . By (B.8) we find

$$c = \frac{\Gamma(2\lambda + 1)\Gamma(M + \lambda + \frac{1}{2})(-M)_N}{\Gamma(\lambda + \frac{1}{2})\Gamma(M + N + 2\lambda + 1)}, \quad (B.10)$$

which coincides with the eigenvalues  $Z_m(N, M)$  given by formula (130). We formulate the result so obtained:

$$\begin{aligned} \int_{S^{m-1}} d\Omega \left( \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right)^M \Lambda^{A_N}(\mathbf{n}_j) \\ = \frac{\Gamma(2\lambda + 1)\Gamma(M + \lambda + \frac{1}{2})(-M)_N}{\Gamma(\lambda + \frac{1}{2})\Gamma(M + N + 2\lambda + 1)} \Lambda^{A_N}(\mathbf{n}_i). \end{aligned} \quad (B.11)$$

We shall say a few words regarding the proof of Theorem 8. The details will be published in a separate paper. From identity (B.11) and the inversion formula (123) we obtain the equality

$$\begin{aligned} \int_{S^{m-1}} d\Omega \left( \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right)^M n_j^{A_N} \\ = \frac{\Gamma(2\lambda + 1)\Gamma(M + \lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2})\Gamma(M + \lambda + 1)} \sum_{l=0}^{[N/2]} Z_m^{(l)}(N, M) \sum_{A_{2l} \subset A_N} n^{A_N \setminus A_{2l}} \delta(A_{2l}), \end{aligned} \quad (B.12)$$

where, by definition,

$$Z_m^{(l)}(N, M) = \frac{2^l (-M)_{N-2l} (M + \lambda + \frac{1}{2})}{(M + 2\lambda + 1)_N}. \quad (B.13)$$

From identity (B.12) it is easy to derive the equality

$$\begin{aligned} \int_{S^{m-1}} d\Omega \left( \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right)^M (\mathbf{n}_j \cdot \mathbf{n}_{A_N}) \\ = \frac{\Gamma(2\lambda + 1)\Gamma(M + \lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2})\Gamma(M + \lambda + 1)} \sum_{A_{2l} \subset A_N} Z_m^{(l)}(N, M) (\mathbf{n}_i \cdot \mathbf{n}_{A_N \setminus A_{2l}}) \delta(\mathbf{n}_{A_{2l}}), \end{aligned} \quad (B.14)$$

Here we have used the notation

$$(\mathbf{n}_0 \cdot \mathbf{n}_A) = \prod_{a \in A} (\mathbf{n}_0 \cdot \mathbf{n}_a),$$

$$\delta(\mathbf{n}_{A_{2l}}) = \frac{1}{l!} \sum_{A_{2l} \cup_{\alpha} \{a_{\alpha}, b_{\alpha}\}} \prod_{\alpha} (\mathbf{n}_{a_{\alpha}} \cdot \mathbf{n}_{b_{\alpha}}) \quad (B.15)$$

(see definition (72)). We note the following consequence of formula (B.14):

$$\int_{S^{m-1}} d\Omega_0 (\mathbf{n}_0 \cdot \mathbf{n}_{A_N}) = 2^{N/2} \frac{\Gamma(2\lambda + 1) \Gamma(M + \frac{N}{2} + \lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2}) \Gamma(M + N + 2\lambda + 1)} \delta(\mathbf{n}_{A_N}). \quad (B.16)$$

The next step is proving Theorem 8 consists in considering the integral operator  $\widehat{K}_{M_1, M_2}: C^{\infty}(S^{m-1}) \rightarrow C^{\infty}(S^{m-1} \times S^{m-1})$ :

$$(\widehat{K}_{M_1, M_2} f)(\mathbf{n}_1 \cdot \mathbf{n}_2) = \int_{S^{m-1}} d\Omega \left( \frac{1 - \mathbf{n}_0 \cdot \mathbf{n}_1}{2} \right)^{M_1} \left( \frac{1 - \mathbf{n}_0 \cdot \mathbf{n}_2}{2} \right)^{M_2} f(\mathbf{n}_0). \quad (B.17)$$

It is clear that the case  $p = 2$  of Theorem 8 corresponds to the evaluation of the function  $(\widehat{K}_{M_1, M_2} \mathbf{1})(\mathbf{n}_1 \cdot \mathbf{n}_2)$ . This function can be most simply computed by using the expansion of the series  $((1 - x)/2)^M$  in the Gegenbauer polynomials  $C_N^{\lambda}(x)$  for  $\lambda = m/2 - 1$ :

$$\left( \frac{1 - x}{2} \right)^M = \frac{\Gamma(2\lambda + 1) \Gamma(M + \lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2}) \Gamma(M + 2\lambda + 1)} \sum_{N \geq 0} \frac{(N + \lambda)(-M)_N}{\lambda(M + 2\lambda + 1)_N} C_N^{\lambda}(x) \quad (B.18)$$

(see, for example, [19]). As a result, we obtain

$$\int_{S^{m-1}} d\Omega \left( \frac{1 - \mathbf{n}_0 \cdot \mathbf{n}_1}{2} \right)^{M_1} \left( \frac{1 - \mathbf{n}_0 \cdot \mathbf{n}_2}{2} \right)^{M_2}$$

$$= \frac{\Gamma(2\lambda + 1) \Gamma(M_1 + M_2 + \lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2}) \Gamma(M_1 + M_2 + 2\lambda + 1)} {}_2F_1 \left( \begin{matrix} -M_1, -M_2 \\ -M_1 - M_2 - \lambda + \frac{1}{2} \end{matrix} \middle| \frac{1 - \mathbf{n}_1 \cdot \mathbf{n}_2}{2} \right). \quad (B.19)$$

It is clear that equality (B.19) corresponds to the case  $p = 2$  of the Theorem 8. The next step consists in computing the function  $\widehat{K}_{M_1, M_2}(\mathbf{n}_0 \cdot \mathbf{n}_{A_N})$  for any set  $A_N$ . This computation is based on the expansion (B.18) and identity (B.14).

We shall not present the formula for this function but rather immediately formulate a general result having Theorem 8 as a special case.

**Theorem B.** *Let  $\mathbf{n}_0, \mathbf{n}_1, \dots, \mathbf{n}_p \in S^{m-1}$ . Then*

$$\int_{S^{m-1}} d\Omega_0 \left( \frac{1 - \mathbf{n}_0 \cdot \mathbf{n}_1}{2} \right)^{M_1} \cdots \left( \frac{1 - \mathbf{n}_0 \cdot \mathbf{n}_p}{2} \right)^{M_p} (\mathbf{n}_0 \cdot \mathbf{n}_{A_N})$$

$$= \frac{\Gamma(2\lambda + 1) \prod_{j=1}^p \Gamma(M_j + 1) (-1)^N}{\Gamma(\lambda + \frac{1}{2}) \Gamma(|M| + N + 2\lambda + 1)} \quad (B.20)$$

$$\times \left\{ \begin{array}{l} \sum_{\substack{k_1, \dots, k_p \\ N \equiv |k| \pmod{2}}} 2^{1/2(N-|k|)} \frac{\Gamma(|M| + \frac{1}{2}(N - |k| + \lambda + \frac{1}{2}))}{\prod_{j=1}^p \Gamma(M_j - k_j + 1)} \\ \\ \times F^{(p)} \left( \begin{array}{c} -M_1 + k_1, \dots, -M_p + k_p \\ -|M| - \frac{1}{2}(N - |k|)_\lambda + \frac{1}{2} \end{array} \mid \left\{ \frac{1 - \mathbf{n}_i \cdot \mathbf{n}_j}{2} \right\} \right) \\ \\ \times \sum_{\substack{\{A_{k_j}\} \\ A_{k_j} \cap A_{k_i} = \emptyset, \\ i \neq j}} \prod_{j=1}^p (\mathbf{n}_j \cdot \mathbf{n}_{A_{k_j}}) \delta(\mathbf{n}_{A_N} \setminus \cup_{j=1}^p A_{k_j}) \end{array} \right\}.$$

Here we have used the notation (B.15), (143) and  $|M| = \sum_{j=1}^p M_j$ ,  $|k| = \sum_{j=1}^p k_j$ .

From formula (B.20) we obtain an expression for the multipoint correlation functions of the classical Heisenberg  $O(m)$ -model on a complete graph (and hence on an arbitrary graph). The answer is a relation of generalized hypergeometric series which is too cumbersome and requires further study and simplification.

In conclusion we note that identity (B.18) gives an alternative means of

computing the statistical sums  $\langle \psi | \psi \rangle_0$  and  $\langle \psi | \psi \rangle_{\text{reg}}$  (see (133) and (135)) of the classical Heisenberg  $O(m)$ -model on open and closed chains respectively.

## References

- [1] A.N.Kirillov, V.E.Korepin *The resonating valence bond in quasicrystals*, Preprint ITP-SB-88-52 **ITP-SB-88-52** (1988).
- [2] Affleck I., Kennedy T., Lieb E.H. and Tasaki H., *Rigorous results on valence-bond ground states in antiferromagnets*, Phys. Rev. Lett. **59** (1987), 799–802.
- [3] Majumdar C.K. and Ghosh D.K., *On nearest-neighbor interaction in a linear chain*, J. Math. Phys. **10** (1969), 1388–1402.
- [4] Klein D.J., *Exact ground states for a class of antiferromagnetic Heisenberg models with short-range interactions*, J. Phys. A **15** (1982), 661–671.
- [5] Affleck I., Kennedy T., Lieb E.H. and Tasaki H., *Valence bond ground states in isotropic quantum antiferromagnets*, Comm. Math. Phys. **115** (1988), 477–528.
- [6] Arovas D.P., Auerbach A. and Haldane F.D.M., *Extended Heisenberg models of antiferromagnetism: analogies to the fractional quantum Hall effect*, Phys. Rev. Lett **60** (1988), 531–536.
- [7] Iske P.L. and Caspers W.J., *Exact ground state of one-dimensional valence-bond-solid Hamiltonians*, Modern Phys. Lett B **1** (1987), 231–237.
- [8] Kennedy T., Lieb E.H. and Tasaki H., *A two-dimensional isotropic quantum antiferromagnet with unique disordered ground state*, Preprint, Princeton University PACS no. 75.10 Im, 1986.
- [9] Chayes J.T., Chayes L. and Kivelson S.A., *Valence bond ground states in a frustrated two-dimensional spin 1/2 Heisenberg antiferromagnet*, Preprint, Stony Brook Univ., 1988.



- [10] Harary F., Graph theory, Addison–Wesley, Reading, Mass., 1969.
- [11] Harary F., J. Physique **47** (1986), Colloque C3, suppl. au no. 7.
- [12] Steinhardt P.J. and Ostlund S., *The physics of quasicrystals*, World Scientific, Singapore, 1987, pp. 60–86, 662.
- [13] Grunbaum B. and Shephard G.C., *Tiling and patterns*, Freeman. New York, 1986.
- [14] Korepin V.E., *completely integrable models in quasicrystals*, Comm. Math. Phys. **110** (1986), 157–171.
- [15] Henley C.L., *Sphere packing and local environments in Penrose tilings*, Phys. Rev B **34** (1986), 797–816.
- [16] De Bruijn N.G., *Algebraic theory of Penrose’s non periodic tilings of the plane*, I, II, Nederl. Acad. Wetensch. Proc. Ser. A **43** (1981), 39-66.
- [17] Shechtman D., Blech L., Gratias D. and Cahn J.W., *Metallic phase with long-range orientational order and no translational symmetry*, Phys. Rev. Lett. **53** (1984), 1951–1953.
- [18] Kalugin P.A., Kitaev A.Yu. and Levitov L.S., *Al–Mn– a six-dimensional crystal*, Pis'ma Zh. Eksper. Teoret. Fiz., **41** (1986), 119-121.
- [19] Vilenkin V.Ya., *Special functions and the theory of group representations*, "Nauka", Moscow, 1965.
- [20] Bailey W.H., *Generalized hypergeometric series*, New York, London, 1964.