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QUANTUM THEORY OF SOLITONS

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Contents:

Abstract:

This paper describes the quantum theory of solitons – the localized solutions of the classical field equations. The scattering matrix for the processes with solitons is defined within the functional integral formalism. The Lorentz-invariant perturbation theory for solitons is consistently set up. The physical properties of solitons are calculated for two-dimensional scalar theories in the one-loop approximation.

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Introduction

The quantum field theory in its present state of progress had begun the search of non-perturbative methods. They seem to provide the only hope for this elegant mathematical scheme to remain the basis of the elementary particles theory.

Indeed, the multiplicity of the elementary particles and the complicate hierarchy of their interactions make useless the concept of the fundamental Lagrangian which is constructed in terms of the independent local fields for each particle. To keep the idea of the fundamental Lagrangian valid one ought to be able to compose it with a small number of fields and make it capable to describe a wide spectrum of particles' masses.

The most popular expectations of a strong interactions theory of this kind are connected with the model known as the "standard theory" or quantum chromodynamics. The fundamental fields of this model are represented by the multiplet of the "colored" quarks and their interactions are maintained by the massless Yang–Mills fields. The special features of these interactions such as the strong infrared divergencies are believed to provide the confinement of the quarks which make up a sufficient number of bound states representing mesons and baryons. The confinement mechanism is not yet worked out but most of the theoreticians support this hypothesis [1, 2, 3]. We are not going to discuss these problems in our survey.

The last three years had revealed the development of some other method for the description of the elementary particles mass spectrum, the method that is quite different from the perturbative ones. It is based upon the existence of the spatially localized solutions of the nonlinear classical field equations for a considered model. These solutions differ from the usual decaying wave packets by keeping the physical quantities, e.g. energy density, in a compact spatial region of a constant size that does not extend during the time evolution. In the simplest case the time-dependence of these solutions appears to be the movement of an object as a whole.

These solutions had been known to exist for about a century in some problems of applied mathematical physics. Some time ago they were named "solitons" after the term "solitary wave", and we are going to use this term also. Between 1958 and 1962 some authors, among them Skyrme [4] and Finkelstein [5] (see also [6]), declared that with every such a solution there can be associated an elementary particle in the quantum version of a model. To their papers had not been paid the proper attention at that time. Three years ago a number of groups of authors almost simultaneously had demonstrated that the quantum particles really correspond to these classical solitons.

Faddeev and Takhtajan [7, 8] proceeded from the exact solution of the sin φ_2-model, having found the action-angle type variables that proved the particle-like behaviour of the soliton solutions. Their arguments will be discussed in section 1. Dashen, Hasslacher, Neveu [9, 10] developed semiclassical methods in the quantum field theory, and within their general method exhibited the correspondence of the particles to the solitons. Jackiw and Goldstone [11] displayed the same in their variational approach. Afterwards, the attention of a great number of investigators was attracted to the soliton quantization problem and quite a few papers on this topic appeared [12–28]. Now the principal points of this problem are clear.

The survey that is presented below sums up this development. We are going to discuss in detail the connections of the solitons and elementary particles and present the methods of calculations of the masses and the scattering amplitudes of these particles. We shall not describe all the methods worked out in the recent literature, but we aim to obtain all the known results by the single method that was developed by the authors at the Leningrad department of the V.A. Steklov Mathematical Institute. This method is based upon the use of the functional integral for the quantum theory formulation. It is the functional integral formalism that enables us to describe the quantum theory in terms of the classical one. No wonder that the specific role of the non-trivial solutions of the classical equations of motion can be displayed in this kind of the formalism most clearly.

Now we shall explain why the spatially localized solutions of the classical equations have an influence on the mass spectrum of the quantum problem. The connection of the particles and the fields in the framework of the perturbations theory can be explained through the asymptotic behaviour at $|t| \to \infty$ of the fields that obey the equations of motion. Let us consider, for example, the scalar field $u(x, t)$ with the equations of motion

$$\Box u + v'(u) = 0. \tag{1.1}$$

Within the limits of the formal perturbation theory all these solutions at $|t| \to \infty$ are solutions of the free equation

$$\begin{aligned} &u(x;t) \xrightarrow[t\to\pm\infty]{} u^0_{\substack{\text{out}\\ \text{in}}}(x,t);\\ &(\Box + m^2)u^0_{\substack{\text{in}\\ \text{out}}} = 0; \qquad v''(0) = m^2. \end{aligned} \tag{1.2}$$

Indeed, u can be obtained from u_{in} by means of the non-linear Yang–Feldman integral equation

$$u(x,t) = u^0_{\text{in}}(x,t) - \int (\Box + m^2)^{-1}_{\text{ret}} \cdot (v'(u) - m^2 u)\, \mathrm{d}y \tag{1.3}$$

and one can see that when $|t| \to \infty$ this u is reduced to the solution of the free equation which we denote by u_{out}. The energy

$$\int \mathscr{H}(u)\, \mathrm{d}x = \int \mathrm{d}x[\tfrac{1}{2}u_t^2 + \tfrac{1}{2}(\nabla u)^2 + v(u)] \tag{1.4}$$

and other observables expressed in terms of $u_{\substack{\text{in}\\ \text{out}}}$ coincide with the corresponding expressions for the free fields:

$$\int \mathscr{H}(u)\mathrm{d}x = \int \mathscr{H}_0(u^0_{\substack{\text{in}\\ \text{out}}})\,\mathrm{d}x = \frac{1}{2}\int \left[\left(\frac{\mathrm{d}u^0_{\substack{\text{in}\\ \text{out}}}}{\mathrm{d}t}\right)^2 + (\nabla u^0_{\substack{\text{in}\\ \text{out}}})^2 + m^2 (u^0_{\substack{\text{in}\\ \text{out}}})^2\right]\mathrm{d}x = \int \mathrm{d}k\sqrt{k^2+m^2}\;\rho_{\substack{\text{in}\\ \text{out}}}(k). \tag{1.5}$$

Here $\rho_{\substack{\text{in}\\ \text{out}}}(k)$ is the spectral density of the in (out) fields. This Hamiltonian when quantized exhibits the spectrum of the particles of a single sort. The only quantum correction is the mass shift due to the self-action effects.

The existence of solitons makes the asymptotic representation (1.2) invalid. Indeed, the main property of a soliton is that it does not decay as the wave packet does and the nonlinear term $-m^2 u + v'(u)$ at $u = u_s$ does not disappear in (1.1) at $|t| \to \infty$. The simplest example of a soliton is the stationary solution with the finite energy

$$-\Delta u_s + v'(u_s) = 0. \tag{1.6}$$

According to the Lorentz invariance every such a solution generates a set of solutions

$$u_s(x, t|v, q) = u_s\left(\frac{x - vt - q}{\sqrt{1 - v^2}}\right), \tag{1.7}$$

which is parametrized by the phase space point (v, q) that is the position and the velocity of the soliton's center of mass. The solitons' energies are

$$\int \mathscr{H}(u_s)\,\mathrm{d}x = M/\sqrt{1 - v^2}. \tag{1.8}$$

Another typical example is given by a family of the time-periodic solutions

$$w\left(x, \frac{t + T}{T}, T\right) = w\left(x, \frac{t}{T}, T\right), \tag{1.9}$$

that generate a set of solutions

$$w(x, t|v, T, q, \alpha) = w\left(\frac{x - vt - q}{\sqrt{1 - v^2}}, \frac{t - vx}{\sqrt{1 - v^2}} - \frac{\alpha}{2\pi}; T\right), \tag{1.10}$$

which are parametrized by points in the four-dimensional phase space (v, q, T, α). The new variables T, α are connected to the internal momentum and the initial phase. In general we can imagine a soliton as a finite-dimensional set of non-decaying classical solution $u_s(x, t|\{p_j\}, \{q_j\})$ that depend on v, q and also on the internal coordinates and momenta. The energy of a soliton of that kind is

$$\int \mathscr{H}(u_s)\,\mathrm{d}x = \frac{M(\{p_j\})}{\sqrt{1 - v^2}}, \tag{1.11}$$

where $M(\{p_j\}), v \notin \{p_j\}$ is the soliton's mass. The important property of this solution is the possibility of making the soliton to rest.

When a soliton solution is properly localized in space, a sum of a number of these solutions with the centres separated sufficiently would satisfy a motion equation with high precision. Solitons that move with different velocities are getting farther and farther as $|t| \to \infty$, so we can assert the asymptotics of eq. (1.1) solutions to have the form

$$u(x, t) \xrightarrow[t \to \mp\infty]{} u^0_{\substack{\text{in}\\\text{out}}}(x, t) + \sum_i u^i_s(x, t|\{p_i\}, \{q_i\}) \tag{1.12}$$

as long as the solitons exist. These asymptotics contain the sum of the one-soliton solutions in addition to the free equation solutions $u^0_{\substack{\text{in}\\\text{out}}}$. In general case we have the different sets of solitons in the (1.12) at $t \to -\infty, t \to +\infty$. The solution energy is expressed through its asymptotics as follows:

$$\int \mathscr{H}(u)\,\mathrm{d}x = \int \mathscr{H}_0(u^0_{\substack{\text{in}\\\text{out}}})\,\mathrm{d}x + \sum_i \int \mathscr{H}(u^i_s)\,\mathrm{d}x = \int \sqrt{k^2 + m^2}\rho_{\substack{\text{in}\\\text{out}}}(k)\,\mathrm{d}k + \sum_i \frac{M_i(\{p_j\})}{\sqrt{1 - v_i^2}}. \tag{1.13}$$

Here we assume the wave packet and the solitons to move away asymptotically with different velocities and so to be localized far apart. We see that the phase space of this system is bigger than that of the free scalar field. It is parametrized by the number of solitons of every kind by their internal and Lorentz momenta and their coordinates in addition to the generalized momenta and coordinates contained in u_{in} and u_{out}.

The energy contribution from the solitons looks like the energy expression in the occupation number representation of the quantum field theory. This implies that the particle spectrum of the system provided by the consistent quantization should contain a set of soliton particles along with the particle, corresponding to the original field in perturbative sense. Every structureless soliton generates particles of one sort while a soliton with internal degrees of freedom generates a family of particles with different internal states. It's worth mentioning that the energy contribution of solitons looks like quantized in the classical field theory already.

These general observations had been taken up as the basis of the soliton quantization problem technical account that is given below in this review. The consistent definition of the S-matrix by the functional integral method makes use of the classical solutions asymptotics at large time. Regarding the nontrivial properties of these asymptotics when the solitons exist we can modify the S matrix definition in a natural way. The stationary phase calculation of a functional integral makes it possible to develop a perturbation theory that manifestly exploits the solitons' presence and that is an expansion over the coupling constant likewise. The physical observables of the solitions such as e.g. particle masses, scattering phases, are found to be not analytically dependent on the coupling constant, they contain a contribution inversely proportional to the coupling constant. It is interesting that all the non-analytic contributions are of a purely classical origin and the quantum corrections to them are analytic. Owing to this semiclassical contribution solitons interact strongly when a coupling constant is small and generate a rich spectrum of bound states [29].

Let us mention that in some models the solitons possess the "topological charge", i.e. they can not be deformed continuously into the vacuum. So we become sure that a quantization will leave these solitons stable and not reduce them by the fluctuations.

At last we should outline the survey contents. In section 1 we are going to describe some well-known classical solitons including the $\sin\varphi_2$ model solutions which will be used to examplify the general expressions later on. The role of the topological charge in the quantization of solitons will be explained in the same section.

In section 2 we give a general definition of the S-matrix for a classical system with solitons.

Section 3 is devoted to the diagrammatic technique of the physical observables calculation for solitons. This technique is based on the stationary phase method calculation of the functional integral which describes the soliton's propagation.

In section 4 the semiclassical contributions to the solitons' masses and scattering phases are derived, they are found to be inversely proportional to the coupling constant. The number of the periodic soliton's quantum states happens to be inversely proportional to the coupling constant too, hence reaffirming the strong interaction of solitons with the small coupling constant. All the general formulae are illustrated by the $\sin\varphi_2$ model.

In section 5 we calculate the one-loop corrections to the physical observables on the pattern of an arbitrary scalar theory of a two-dimensional field.

Throughout this paper we choose $h = 1$, $c = 1$. Every paragraph has its own numeration of the formulae. The index of a formula contains two numbers, the first is the number of the paragraph and the second that of the expression itself. Formulae from another section or appendix are referred to by means of the index of three numbers, the first being the number of the section or of that of the appendix.

1. The classical relativistic solitons

This section has a number of objects. Firstly, it supplies an information on the classical localized solutions that will be used in the discussion of the quantum theory in other sections. Secondly, it will illustrate by the concrete example the introduction's formulae about the asymptotic properties of an arbitrary solution of the classical equations of motion. At last, the important property of the most interesting soliton solutions that is called the topological charge will be discussed.

This section is an auxiliary for the following material, so it will present just a brief survey and the reader can find more details in the original literature we refer to.

1.1. The sin φ_2 *model*

All the soliton activities had a great stimulating encouragement in a field model that supplies most of the soliton solutions calculable analytically. This model is the famous "Sine-Gordon" equation in the two-dimensional space–time that was named so by Rubinstein due to an obvious alliteration in [30]. The title "model sin φ_2" seems to be more rigorous for this system. We will use it everywhere avoiding the slang.

In this paragraph we will describe the known results on the classical sin φ_2 solutions and their interpretations. This model will be used as a main illustration of the general quantum solitons theory formulae in the following sections.

Let us consider, in the two-dimensional space–time, a non-linear chiral field which is associated to the Abelian group U(1), i.e., a complex field $\chi(x, t)$ that satisfies the condition

$$|\chi(x,t)| = 1, \qquad \chi(x,t) \xrightarrow[|x|\to\infty]{} 1. \tag{1.1}$$

It is possible to deal instead with a real $u(x, t)$ such that

$$\chi(x,t) = \exp\{iu(x,t)\}; \tag{1.2}$$

$u(x, t)$ must not vanish at the spatial infinity. The asymptotical condition for u is weaker:

$$u(x,t) \xrightarrow[|x|\to\infty]{} 0 \ (\mathrm{mod}\ 2\pi). \tag{1.3}$$

The Lagrange function

$$\mathscr{L} = \frac{1}{2\gamma}\int_{-\infty}^{\infty} \mathrm{d}x[\partial_\mu\chi\partial_\mu\chi^* + m^2(\chi + \chi^* - 2)] = \frac{1}{\gamma}\int_{-\infty}^{\infty} \mathrm{d}x[\tfrac{1}{2}(\partial_\mu u)^2 - m^2(1 - \cos u)] \tag{1.4}$$

defines the model with the mass m and the coupling constant γ. In the second representation the Lagrangian would take a more convenient form after the renormalization

$$u \to \sqrt{\gamma}u, \tag{1.5}$$

but we shall not do it. The classical motion equation

$$u_{tt} - u_{xx} + m^2 \sin u = 0 \tag{1.6}$$

does not contain γ at all in our formulation. The γ reappears instead in the Poisson brackets

$$\{u_t(x), u(y)\} = \gamma\delta(x - y); \tag{1.7}$$

this demonstrates why the perturbations theory in γ will coincide with the semiclassical expansion in the next sections. The equation (1.6) that looks in the light-cone variables as follows

$$u_{\xi\eta} + \sin u = 0, \qquad \xi = \tfrac{1}{2}m(t + x), \qquad \eta = \tfrac{1}{2}m(t - x) \tag{1.8}$$

and defines here the relativistic quantum field theory model had been known for a long time in various branches of the applied mathematical physics and had enjoyed much attention. Some years ago it was treated successfully by the inverse scattering method [39, 35, 31] (see also [32, 34, 37]). The explicit Hamiltonian dynamics formulae that were obtained in [7, 8] incited our formulation of quantum soliton theory.

Let us now apply the inverse scattering method to equation (1.6). It can be represented as a commutation condition for the operators

$$X = \frac{1}{\mathrm{i}}\frac{\partial}{\partial x} - \frac{1}{2}u_t \cdot S_3 + k_1 \cos\frac{u}{2} S_1 + k_0 \cdot \sin\frac{u}{2} S_2; \tag{1.9}$$

$$T = \frac{1}{\mathrm{i}}\frac{\partial}{\partial t} - \frac{1}{2}u_x \cdot S_3 + k_0 \cos\frac{u}{2} S_1 + k_1 \sin\frac{u}{2} S_2.$$

Here $S_i = \frac{1}{2}\sigma_i$, σ_i are the Pauli matrices, $k = (k_0, k_1)$ is an arbitrary vector on the mass shell, $k_0^2 - k_1^2 = m^2$ and $u(x, t)$ is an arbitrary function. Indeed it is easy to check directly that

$$TX = XT \tag{1.10}$$

if and only if $u(x, t)$ satisfies eq. (1.6). One can understand the last relation in such a way: the operations X and T generate the displacements in the space–time. This observation possibly deserves more attention but we will not employ it here.

Owing to (1.10) the equations

$$\begin{cases} X\psi = 0; & (1.11) \\ T\psi = 0, & (1.12) \end{cases}$$

are compatible.

The first of them can be regarded as an eigenvalue problem with the vector k playing the role of a spectral parameter. Meanwhile, the operator X is defined in terms of the initial data u and u_t for the equation (1.6). The complete analysis of eq. (1.11) can be performed as it is usually done in the potential scattering theory, by introducing the Jost matrix solutions and the transition matrix for any real k and investigating the solutions in search of a discrete spectrum. This program is carried out in the mentioned papers. We shall restrict our attention to listing the scattering data resembling that of the Dirac equation.

The two coefficients $a(k_1)$ and $b(k_1)$ make up the transition matrix

$$T = \begin{pmatrix} a, & b \\ -\bar{b}, & \bar{a} \end{pmatrix}, \tag{1.13}$$

which connects the Jost matrix solutions $G(x, k)$ and $F(x, k)$ of eq. (1.11), defined by the asymptotic conditions

$$F|_{x\to\infty} \to \mathscr{E}(x, k); \qquad G(x, k)|_{x\to-\infty} \to \mathscr{E}(x, k), \tag{1.14}$$

where the matrix $\mathscr{E}(x,k)$ is

$$\mathscr{E}(x,k) = \exp\{-\mathrm{i}k_1 S_1[\cos\tfrac{1}{2}u(\infty)]\cdot x\}, \tag{1.15}$$

in such a way that $F(x,k) = G(x,k)T(k)$. Here $a(k)$ and $b(k)$ should obey the conditions

$$|a(k)|^2 + |b(k)|^2 = 1, \qquad \bar{a}(-k) = a(k), \qquad \bar{b}(-k) = +b(k) \tag{1.16}$$

and $a(k)$ has an analytic continuation in the upper half-plane of k with conditions there:

$$a(k) \to 1, \qquad |k| \to \infty; \qquad \bar{a}(+k) = a(-\bar{k}). \tag{1.17}$$

The zeroes of function $a(k)$ are located symmetrically around the imaginary axis. In the generic situation $a(k)$ has a finite number of simple zeroes with none of them on the real axis and no degenerate ones. Let us denote the purely imaginary zeroes by $k = \mathrm{i}\kappa_l$ and the zeroes that are in the right half-plane by $k = \zeta_m$. The zeroes of $a(k)$ correspond to the discrete spectrum of the problem (1.11). More strictly, at $k = \mathrm{i}\kappa_l$ there exists a vector-solution $\psi_l(x)$ of eq. (1.11) such that

$$\psi_l \xrightarrow[x\to-\infty]{} \begin{pmatrix}1\\1\end{pmatrix} \mathrm{e}^{\kappa_l\cdot x}; \qquad \psi_l \xrightarrow[x\to\infty]{} C_l \begin{pmatrix}1\\1\end{pmatrix} \mathrm{e}^{-\kappa_l\cdot x} \tag{1.18}$$

with C_l a real number, and at $k = \zeta_m$ there is a ψ_m that

$$\psi_m \xrightarrow[x\to-\infty]{} \begin{pmatrix}1\\1\end{pmatrix} \mathrm{e}^{-\mathrm{i}\zeta_m x}; \qquad \psi_m \xrightarrow[x\to\infty]{} d_m \begin{pmatrix}1\\1\end{pmatrix} \mathrm{e}^{\mathrm{i}\zeta_m x} \tag{1.19}$$

with d_m the complex transition coefficient. The data

$$\mathrm{S} = (b(k), \kappa_l, C_l, \zeta_m, d_m) \tag{1.20}$$

define the set of so called scattering data for the problem (1.11). The pair of functions $u(x)$, $u_0(x) = u_t(x)$ from some special class and the scattering data set S are in one to one correspondence which is a nonlinear generalization of the Fourier transform

$$u, u_0 \leftrightarrow \mathrm{S}. \tag{1.21}$$

Almost all the conditions on the u, u_t and $b(k)$ concern their smoothness and their Fourier transforms' smoothness properties. The only condition of another sort

$$|b(k)| < 1 \tag{1.22}$$

follows from the "unitarity" condition $|a|^2 + |b|^2 = 1$. We did not include $a(k)$ into the scattering data because it is defined uniquely by $b(k)$ and the zeroes κ_l and ζ_m. The connection (1.21) is not trivial, its evaluation requires the solution of the linear integral equation. Though it makes possible to express the Poisson bracket of the initial data and even the Hamiltonian

$$H = \frac{1}{\gamma}\int_{-\infty}^{\infty} \mathrm{d}x[\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + m^2(1-\cos u)] \tag{1.23}$$

through the scattering data. It was found that the Hamiltonian depends on the canonical momenta only. This means that the scattering data define the variables of the action-angle type. Let us write down the Hamiltonian in terms of the variables that are most convenient for the quantization

$$H = \int \mathrm{d}p\rho(p)\sqrt{p^2 + m^2} + \sum_l \sqrt{p_l^2 + M^2} + \sum_n \sqrt{p_n^2 + (2M\sin\theta_n)^2}, \qquad M = \frac{8m}{\gamma}. \tag{1.24}$$

This expression of the Hamiltonian was the first strict demonstration of the fact that the solitons really correspond to the particles in the quantum field theory [7, 8].

The quantity $0 < \rho(p) < \infty$ means the usual particles density, $-\infty < p_n < \infty$ is the Lorentz momentum of a soliton, $-\infty < p_l < \infty$ is the Lorentz momentum of a periodic soliton, $0 < \theta < \pi/2$ is the internal momentum of a periodic soliton. These canonical momenta can be expressed through $|a(p)|$, κ_l and ζ_n only. In the semiclassical quantization all the canonical variables become operators. The $\rho(p)$ operator has the eigenvalues of the form $\sum_i \delta(p - p_i)$ and displays the contribution of the basic particles, the only particles than can be obtained by the perturbations theory. The eigenvalues of the p_l and p_n operators may be any real number. We shall see later that the θ operator has a finite number of eigenvalues. These variables bring about the solitons' contribution.

The second equation (1.12) describes the time dependence of the scattering data which corresponds to the functions $u(x)$, $u_t(x)$ – change according to eq. (1.6). Noting that the operators X and T look especially simple when $|x| \to \infty$ we find that

$$b(k,t) = \exp\{\mathrm{i}k_0 t\} b(k,0); \qquad C_l(t) = \exp\{\mathrm{i}k_{0_l} t\} C_l; \qquad d_n(t) = \exp\{\mathrm{i}k_{0_n} t\} d_n. \tag{1.25}$$

The canonical momenta are expressed through the variables that do not depend on time:

$$a(k,t) = a(k), \qquad \kappa_l(t) = \kappa_l, \qquad \zeta_n(t) = \zeta_n \tag{1.26}$$

$$k_{0_l} = \sqrt{m^2 - \kappa_l^2}, \qquad k_{0_m} = \sqrt{m^2 + \zeta_m^2}.$$

The square root values are chosen so that $k_{0_l} > 0$ and $\operatorname{Im} k_{0_m} > 0$.

The last formulae enable us to examine the solutions of eq. (1.6) completely and particularly to find the two types of the soliton solutions:

1. A simple soliton without internal degrees of freedom (a structureless soliton)

$$u_s^{\pm}(x,t|v,q_0) = 4\tan^{-1}\exp\left\{\pm m\frac{x - vt - q_0}{\sqrt{1-v^2}}\right\}; \qquad p = \frac{8m}{\gamma}\frac{v}{\sqrt{1-v^2}} \tag{1.27}$$

defined by the parameters p, q_0 and also by an integer-valued parameter $\varepsilon = \pm 1$ with two values that can be interpreted as a charge, as we will show in paragraph 1.3.

2. A periodic soliton of velocity $v = \tanh\varphi$

$$w(x,t|v,T,q,\alpha) = w\left(r, \frac{\tau}{T}, T\right) = 4\tan^{-1}\left\{\tan\theta \frac{\sin[m\cos\theta\cdot\tau - \alpha]}{\cosh[m\sin\theta(r - q\cosh\varphi)]}\right\}; \tag{1.28}$$

$$\tau = t\cosh\varphi - x\sinh\varphi, \qquad r = x\cosh\varphi - t\sinh\varphi$$

$$T = \frac{2\pi}{m\cos\theta}, \qquad M = \frac{16m}{\gamma}\sin\theta. \tag{1.29}$$

One can exhibit the general solution of eq. (1.6) at large values of t as a free wave packet and a linear combination of the solitons mentioned above,

$$u(x,t) \xrightarrow[t\to\pm\infty]{} u^0_{\substack{\text{in}\\\text{out}}}(x,t) + \sum_a u_s^{\pm}(x,t|v_a,q_a)_{\substack{\text{in}\\\text{out}}} + \sum_b w(x,t|v_b,T_b,q_b,\alpha_b)_{\substack{\text{in}\\\text{out}}}. \tag{1.30}$$

The distinctive feature of this model is the conservation of the number of solitons and of the number of the soliton types from $t = -\infty$ to $t = +\infty$ and also of their individual momenta. The only change due to interaction is the additional shift of their coordinates (comparing with their uniform motion without interaction):

$$p_{\text{in}} = p_{\text{out}}, \qquad \theta_{\text{in}} = \theta_{\text{out}}, \qquad q^0_{\text{in}} \neq q^0_{\text{out}}, \qquad \alpha^0_{\text{in}} \neq \alpha^0_{\text{out}}. \tag{1.31}$$

In particular, there are solutions with purely soliton asymptotics. They are called the polysoliton solutions. The inverse scattering method provides the explicit expressions for them. We present two examples now.

1. The solution

$$u_{ss}(x, t|v_1, v_2, q_1, q_2) = 4 \cdot \tan^{-1}\left\{\tanh\frac{\varphi_1 - \varphi_2}{2} \cdot \frac{\sinh((d_1 + d_2)/2)}{\cosh((d_1 - d_2)/2)}\right\}; \tag{1.32}$$

$$d_{1,2} = m\cosh\varphi_{1,2}(x - q_{1,2}) - m\sinh\varphi_{1,2}\cdot t$$

describes the scattering of two solitons of the same charge and behaves asymptotically so:

$$u(x, t|v_1, v_2, q_1, q_2) \xrightarrow[t\to\mp\infty]{} u^+_s(x, t|v^1_{\substack{\text{in}\\ \text{out}}}, q^1_{\substack{\text{in}\\ \text{out}}}) + u^+_s(x, t|v^2_{\substack{\text{in}\\ \text{out}}}, q^2_{\substack{\text{in}\\ \text{out}}}), \tag{1.33}$$

where

$$v^{1,2}_{\text{out}} = v^{1,2}_{\text{in}} = v^{1,2},$$

$$q^2_{\text{out}} - q^1_{\text{in}} = \frac{2}{m\cosh\varphi_1}\ln\coth\left(\frac{\varphi_1 - \varphi_2}{2}\right), \tag{1.34}$$

$$q^1_{\text{out}} - q^2_{\text{in}} = \frac{-2}{m\cosh\varphi_2}\ln\coth\left(\frac{\varphi_1 - \varphi_2}{2}\right), \qquad \varphi_1 > \varphi_2.$$

2. The solution

$$u_{s\bar{s}}(x, t|v_+, v_-, q_+, q_-) = 4\tan^{-1}\left\{\coth\left(\frac{\varphi_+ - \varphi_-}{2}\right)\cdot\frac{\sinh((d_+ - d_-)/2)}{\cosh((d_+ - d_-)/2)}\right\} \tag{1.35}$$

describes the scattering of two solitons of different charges and asymptotically is

$$u_{ss}(x, t|v_+, v_-, q_+, q_-) \xrightarrow[t\to\mp\infty]{} u^+_s(x, t|v^+_{\substack{\text{in}\\ \text{out}}}, qq^+_{\substack{\text{in}\\ \text{out}}}) + u^-_s(x, t|v^-_{\substack{\text{in}\\ \text{out}}}, q^-_{\substack{\text{in}\\ \text{out}}}), \tag{1.36}$$

where

$$v^\pm_{\text{out}} = v^\pm_{\text{in}} = v^\pm,$$

$$q^+_{\text{out}} - q^+_{\text{in}} = \frac{2}{m\cosh\varphi_+}\ln\coth\left(\frac{\varphi_+ - \varphi_-}{2}\right), \tag{1.37}$$

$$q^-_{\text{out}} - q^-_{\text{in}} = \frac{-2}{m\cosh\varphi_-}\ln\coth\left(\frac{\varphi_+ - \varphi_-}{2}\right), \qquad \varphi_+ > \varphi_-.$$

Note that the solution (1.28) is derived from this one by the analytic continuation in the relative rapidity $\varphi_+ - \varphi_- \to \mathrm{i}(\pi - 2\theta)$. It is already clear that w is a classical solution that corresponds to the bound states.

The solution which describes the scattering of a structureless soliton on a periodic one is also revealable but its expression is too cumbersome to be written down here. Just note that the momenta of both solitons remain the same, i.e. $p^{s}_{out} = p^{s}_{in}$, $p^{w}_{out} = p^{w}_{in}$, and the internal motion period stays unchanged, $\theta_{in} = \theta_{out}$ (the internal momentum is conserved). But the coordinates of the simple soliton, q_s, of the periodic soliton, q_w, and the internal phase of the latter α change. Supposing the simple soliton's velocity bigger the coordinate shifts look so:

$$q^{s}_{out} - q^{s}_{in} = \frac{\partial}{\partial p^{s}}\Phi(p^{s}, p^{w}, \theta), \qquad q^{w}_{out} - q^{w}_{in} = \frac{\partial}{\partial p^{w}}\Phi(p^{s}, p^{w}, \theta),$$

$$\alpha_{out} - \alpha_{in} = \frac{\gamma}{16}\frac{\partial}{\partial\theta}\Phi(p^{s}, p^{w}, \theta); \qquad p_s = M\sinh\varphi_s, \qquad p^{w} = 2M\sin\theta\sinh\varphi_w, \tag{1.38}$$

$$\Phi(p^{s}, p^{w}, \theta) = -\frac{8\pi^2}{\gamma} + K(\mathrm{i}e^{-\mathrm{i}\theta}\cdot e^{\varphi_s-\varphi_w}) + K(-\mathrm{i}e^{\mathrm{i}\theta}\cdot e^{\varphi_s-\varphi_w}) \tag{1.39}$$

$$K(x) = \mathrm{i}\frac{8}{\gamma}\int_0^{\pi} d\theta \ln\left(\frac{x\exp\{-\mathrm{i}\theta\}+1}{x+\exp\{-\mathrm{i}\theta\}}\right). \tag{1.40}$$

The scattering of two periodic solitons looks similar. All the momenta are conserved and coordinates obtain the following increments $\varphi_{w_1} > \varphi_{w_2}$:

$$q^{w_1}_{out} - q^{w_1}_{in} = \frac{\partial}{\partial p^{w_1}}F(p_{w_1}; p_{w_2}; \theta_1; \theta_2); \qquad q^{w_2}_{out} - q^{w_2}_{in} = \frac{\partial}{\partial p^{w_2}}F(p_{w_1}; p_{w_2}; \theta_1; \theta_2);$$

$$\alpha^{w_1}_{out} - \alpha^{w_1}_{in} = \frac{\gamma}{16}\frac{\partial}{\partial\theta_1}F(p_{w_1}; p_{w_2}; \theta_1; \theta_2); \qquad \alpha^{w_2}_{out} - \alpha^{w_2}_{in} = \frac{\gamma}{16}\frac{\partial}{\partial\theta_2}F(p_{w_1}; p_{w_2}, \theta_1, \theta_2); \tag{1.41}$$

$$F(p_1, p_2, \theta_1, \theta_2) = K(e^{\varphi_1-\varphi_2}\cdot e^{\mathrm{i}(\theta_1-\theta_2)}) + K(-e^{\varphi_1-\varphi_2}\cdot e^{-\mathrm{i}(\theta_1+\theta_2)}) + \\ + K(-e^{\varphi_1-\varphi_2}\cdot e^{\mathrm{i}(\theta_1+\theta_2)}) + K(e^{\varphi_1-\varphi_2}\cdot e^{\mathrm{i}(\theta_2-\theta_1)}) - 16\pi^2/\gamma. \tag{1.42}$$

Consider now the scattering of any number of solitons. It turns out that all the momenta remain conserved, all the coordinates become shifted due to the interaction, with the total increment of every coordinate being the sum of the two-body shifts

$$\Delta q^{i}_{tot} = \sum_k \Delta q_{ik}, \qquad \Delta\alpha^{i}_{tot} = \sum_k \Delta\alpha_{ik}. \tag{1.43}$$

The reason of the solitons' number, types and momenta conservation is the existence of an infinite number of conservation laws in this model. All the conserved quantities have the local densities that are expressed in terms of u, u_t and their spatial derivatives. Inserting the asymptotics (1.30) of the general solution $u(x, t)$ into these densities we obtain

$$\sum_a p^{2n+1}_{a_{in}} + \int dk\cdot k^{2n+1}\cdot\rho_{in}(k) = \sum_a p^{2n+1}_{a_{out}} + \int dk\cdot k^{2n+1}\cdot\rho_{out}(k);$$

$$\sum_a p^{2n}_{a_{in}}\cdot p^{0}_{a_{in}} + \int dk\cdot k^{2n}\cdot k_0\rho_{in}(k) = \sum_a p^{2n}_{a_{out}}\cdot p^{0}_{a_{out}} + \int dk\cdot k_0\cdot k^{2n}\cdot\rho_{out}(k); \quad p^{0}_{a} = \sqrt{p^2+m_a^2} \tag{1.44}$$

with summing over all the solitons types in the initial and final states. These identities ensure the conservations mentioned above.

1.2. A brief survey of the known classical solutions

Many classical solutions of soliton type have been found up to the present time. We are not going to undertake their general classification which was done in [36–41, 42–51]. Instead we present some remarks and references to original papers.

The fact of solitons existence depends strongly on the dimension d of the space–time, hence the cases of various dimensions ought to be examined separately.

1) $d = 2$.

The existence problem of the structureless solitons for the scalar fields

$$u_{tt} - u_{xx} + v'(u) = 0 \tag{2.1}$$

is simplified by the mechanical analogy. The substitution of $u(x, t)$ in the form

$$u = u_s\left(\frac{x - vt}{\sqrt{1 - v^2}}\right) \tag{2.2}$$

transforms eq. (2.1) into

$$u_s'' = v'(u), \tag{2.3}$$

which is the Newton equation for a particle in the potential $-v(u)$. Thus the soliton solutions appear to exist when $v(u)$ has two nearby minima of equal magnitude.

The periodic solitons do not seem to take the treatment by the general considerations of that simple sort. The numerical experiment [52] proves the existence of the periodic solitons in models other than sin φ_2, but they are not absolutely stable. Such solutions may correspond to the series of resonances in a quantum theory.

2) $d = 3$.

The most interesting example with soliton solutions is the nonlinear chiral field $\boldsymbol{n}(x)$ with the values on the two-dimensional sphere S^2 ($\boldsymbol{n}$-field). In the parametrization of $\boldsymbol{n} = (n_1, n_2, n_3)$, $\boldsymbol{n}^2 = 1$ the Lagrange function is

$$\mathscr{L} = \frac{1}{2}\int \mathrm{d}x(\partial_\mu \boldsymbol{n})^2. \tag{2.4}$$

The classical equations become

$$\Box \boldsymbol{n} + \boldsymbol{n}(\partial_\mu \boldsymbol{n}, \partial_\mu \boldsymbol{n}) = 0. \tag{2.5}$$

The stationary solutions obey the equation

$$\boldsymbol{n}_{xx} + \boldsymbol{n}_{yy} + \boldsymbol{n}[(\boldsymbol{n}_x \cdot \boldsymbol{n}_x) + (\boldsymbol{n}_y \cdot \boldsymbol{n}_y)] = 0. \tag{2.6}$$

It is easy to verify that if $\boldsymbol{n}$ satisfies the system

$$\begin{cases} \boldsymbol{n}_x + \boldsymbol{n} \wedge \boldsymbol{n}_y = 0; \\ \boldsymbol{n}_y - \boldsymbol{n} \wedge \boldsymbol{n}_x = 0, \end{cases} \tag{2.7}$$

then it satisfies eq. (2.6) too. The system (2.7) can be reduced to the Cauchy–Riemann set of equations. It can be checked by regarding S^2 as a complex plane C^1. Substituting formally

$$n_1 = \frac{2u}{1 + u^2 + v^2}, \qquad n_2 = \frac{2v}{1 + u^2 + v^2}, \qquad n_3 = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}, \tag{2.8}$$

we see that the equations (2.7) reduce to the condition for $w = u + iv$ to be the analytic function of the variable $z = x + iy$. Employing this observation we arrive at the infinite set of solutions of eq. (2.6) with the finite energy [53]

$$H = \int d^2x(\boldsymbol{n}_x\boldsymbol{n}_x + \boldsymbol{n}_y\boldsymbol{n}_y). \tag{2.9}$$

Note that this set is degenerate. If $\boldsymbol{n}(x)$ is a solution then $\boldsymbol{n}(\lambda x)$ is a solution too with the same energy. So the $\boldsymbol{n}$-field's solitons do not realize even a local minimum of energy.

3) $d = 4$.

The most popular soliton was found in the system of the Yang–Mills field interacting with the Higgs field. It is the t'Hooft–Polyakov monopole [54–56]. The Lagrange function in the case of O(3) groups looks like

$$\mathscr{L} = \int d^3x \, \mathrm{tr}\left[\frac{-1}{4g^2} F^2_{\mu\nu} + \frac{1}{2}(\nabla_\mu\varphi)^2 - \frac{\lambda}{4}(\varphi^2 - \mu^2)^2\right], \tag{2.10}$$

where φ_a, $a = 1, 2, 3$ is the scalar isovector field and

$$\nabla_\mu\varphi = \partial_\mu\varphi + [A_\mu, \varphi]. \tag{2.11}$$

The stationary solutions are found by

$$\varphi_a = \frac{x_a}{r} u(r); \qquad A^a_i = \varepsilon_{iab}x_b\left(a(r) - \frac{1}{gr^2}\right); \qquad A^a_0 = 0, \tag{2.12}$$

which leads to the next two radial equations

$$\begin{aligned} &u'' + \frac{2}{r}u' + (\mu^2 - 2g^2a^2)u - \lambda u^3 = 0; \\ &a'' + \frac{4}{r}a' - \frac{3}{r^2}a - g^2r^2a^3 - g^2u^2a = 0. \end{aligned} \tag{2.13}$$

They are proved to have the solutions with the boundary conditions

$$u \to \mu\lambda^{-1|2}, \qquad r \to \infty; \qquad a \to 0, \qquad r \to \infty. \tag{2.14}$$

Particularly, the magnetic field $(F^a_{ik}\varphi^a)$ behaves in the infinity as const$/r^2$ and this shows the solution to represent a magnetic monopole. Its mass is of the order of M/g^2, i.e. very big for the usual theory of the electromagnetic and weak interactions. Here M is a mass of a vector particle.

This system also has a time-periodic solution that corresponds to a monopole with electric charge – a dyon. This solution was found by Julia and Zee [57] by the stationary substitution (2.12) changed so:

$$A_0 \neq 0, \qquad A^a_0 = x^a h(r). \tag{2.15}$$

It becomes periodic in the physical gauge $A^a_0 = 0$.

Quite a number of soliton solutions which are stabilized by an extra conservation law was described by Lee et al. [18, 58]. The periodic solutions were investigated in [60, 59].

The nonlinear $\boldsymbol{n}$-field does not produce solitons at $d = 4$, as one can see from the simple scaling considerations [61]. But one may get over this hindrance by changing the Lagrangian, by adding a term with the higher powers of derivatives into it. An example of this was given by Skyrme [4] with the $\boldsymbol{n}$-field on the S^3 sphere. Assume

$$\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4), \qquad \sum \varphi_a^2 = 1 \tag{2.16}$$

and consider the Lagrangian

$$\mathscr{L} = \frac{1}{2\gamma} \int \mathrm{d}^3x[(\partial_\mu \varphi)^2 + \varepsilon^2(\partial_\mu \varphi^b \partial_\nu \varphi^a - \partial_\mu \varphi^a \partial_\nu \varphi^b)^2]. \tag{2.17}$$

The spherically symmetric solitons are found by the substitution

$$\varphi_i = \frac{x_i}{r} f(r), \qquad \varphi_4 = g(r), \qquad f^2 + g^2 = 1. \tag{2.18}$$

One can find the Skyrme model generalization for the gauge fields in [62].

1.3. The topological charge

The soliton solutions had exposed one more interesting aspect of the nonlinear fields theory, the existence and the significance of the so called "topological charge". The nonlinear fields are naturally connected with the maps of the compact manifolds. The space manifold or the vicinity of its infinite point is regarded as a preimage, and the manifold of the field values or its asymptotical values at infinity form the image. The maps of this kind are classified in the topology by integer valued invariants – the homotopy classes. We shall not go deep into this branch of mathematics, one can find a good introduction in Finkelstein's paper [63]. We shall display a number of characteristic examples instead. The dimension of space is of great importance again.

1) $d = 2$.

The field $\chi(x)$ in the model sin φ_2 defines the regular map of the real axis R^1 onto the circle S^1. The regularity means the identity of the values $\chi(-\infty)$ and $\chi(\infty)$, so from the topological point of view R^1 acts as the circle S^1 too. Obviously, an integer number n can be assigned to the field $\chi(x)$, a number that denotes how many times the field circulates while x runs from $-\infty$ to ∞. This number is calculated by

$$n = \frac{1}{2\pi}[u(\infty) - u(-\infty)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \partial_x u \, \mathrm{d}x = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \partial_x \chi \chi^{-1} \, \mathrm{d}x. \tag{3.1}$$

It may be regarded as a charge associated with the current

$$\mathscr{I}_\mu = \frac{1}{2\pi \mathrm{i}} \varepsilon_{\mu\nu} \partial_\nu \chi \chi^{-1} = \frac{1}{2\pi} \varepsilon_{\mu\nu} \partial_\nu u, \tag{3.2}$$

which is conserved irrespective of the motion equations and exhibits the simplest example of a "topological" current. The two characteristic properties: a) the conservation irrespective of the motion equations; b) the integer values of the charge; – may be assumed as a basic of the topological charge definition.

2) $d = 3$.

Regarding the $\boldsymbol{n}$-field example from section 1.2, one can see that the current

$$\mathscr{I}_\mu = \varepsilon_{\mu\nu\sigma}(\partial_\nu \boldsymbol{n} \wedge \partial_\sigma \boldsymbol{n}, \boldsymbol{n}) = \varepsilon_{\mu\nu\sigma}\varepsilon^{abc}\partial_\nu \boldsymbol{n}^a \partial_\sigma \boldsymbol{n}^b \boldsymbol{n}^c \tag{3.3}$$

is conserved. The less obvious fact is that

$$Q = \frac{1}{4\pi}\int \mathscr{I}_0 \,\mathrm{d}x = \frac{1}{4\pi}\int \varepsilon_{ik}(\partial_i \boldsymbol{n} \wedge \partial_k \boldsymbol{n}, \boldsymbol{n})\,\mathrm{d}^2 x \tag{3.4}$$

can have only integer values for the fields $\boldsymbol{n}(x)$ with a fixed asymptotics at $|x| \to \infty$. It can be examined by the parametrization of $\boldsymbol{n}$-field by

$$\boldsymbol{n} = \begin{pmatrix} \sin\rho \quad \cos\varphi \\ \sin\rho \quad \sin\varphi \\ \cos\rho \end{pmatrix}, \tag{3.5}$$

where ρ and φ are the functions of x. In this parametrization Q becomes

$$Q = \frac{1}{4\pi}\int \mathrm{d}\rho \wedge \mathrm{d}\cos\varphi \tag{3.6}$$

and shows how many times n circulates the sphere S^2 while x rans through the plane R^2.

3) $d = 4$.

We can introduce a current similar to the one rendered above for the n-field on the S^3 sphere. In the parametrization

$$\chi = \{\varphi_a\}, \qquad \sum \varphi_a \varphi_a = 1; \qquad a = 1, \dots 4 \tag{3.7}$$

this current looks so:

$$\mathscr{I}_\mu = \frac{1}{2\pi^2}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{abcd}\partial_\nu\varphi^a \cdot \partial_\rho\varphi^b \cdot \partial_\sigma\varphi^c \cdot \varphi^d \tag{3.8}$$

and makes possible an important generalization. Indeed, the S^3 is a manifold of the SU(2) group parameters, and the χ field can be thought of as having its values in the SU(2) so we can rewrite $\mathscr{I}_\mu$ in the parametrization – independent form

$$\mathscr{I}_\mu = \frac{1}{4\pi^2}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\,[L_\nu, L_\rho]L_\sigma, \tag{3.9}$$

where

$$L_\mu = \partial_\mu \chi \chi^{-1}. \tag{3.10}$$

In this form the current can be readily generalized to an arbitrary principal chiral field χ that has its values in a compact group G with the last formulae valid (corrected by another normalization factor) and the tr regarded as the Killing form.

Another series of the topological currents is yielded by the field models with non-trivial behaviour at infinity. In the system of the Yang–Mills A_μ and the Higgs φ fields with the values in the adjoint representation of a gauge group one may construct a current

$$\mathscr{I}_\mu = \varepsilon_{\mu\nu\rho\sigma} \operatorname{tr} F_{\nu\rho} \nabla_\sigma \varphi, \qquad \nabla_\sigma \varphi = \partial_\sigma \varphi + [A_\sigma, \varphi], \tag{3.11}$$

which is conserved. Indeed

$$\partial_\mu \mathscr{I}_\mu = \varepsilon_{\mu\nu\rho\sigma} \operatorname{tr} [(\nabla_\mu F_{\nu\rho} \nabla_\sigma \varphi) + F_{\nu\rho} \nabla_\mu \nabla_\sigma \varphi]. \tag{3.12}$$

The first term in the right side becomes zero according to the Bianci identity. The second term disappears due to the $\nabla_\mu \nabla_\sigma$ antisymmetrization and the fact that

$$(\nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu)\varphi = [F_{\mu\sigma}, \varphi]. \tag{3.13}$$

Unlike the previous examples, the charge density $\mathscr{I}_0$ is the total divergence

$$\mathscr{I}_0 = \partial_i P_i, \qquad P_i = \varepsilon_{ikj} \operatorname{tr} (F_{kj} \varphi) \tag{3.14}$$

and so the corresponding charge Q does not vanish if only the $F_{\mu\nu}$ and φ do not decrease too fast in the infinity. The charge Q is identical to the magnetic charge for the t'Hooft–Polyakov monopole. The integer valuedness of this charge for the SU(2) group follows from the equivalence of the current (3.11) of the φ field with asymptotically vanishing $\nabla_\mu \varphi$ to the current

$$\mathscr{I}_\mu = \varepsilon_{\mu\nu\rho\sigma} \operatorname{tr} [\partial_\nu \varphi, \partial_\rho \varphi] \partial_\sigma \varphi = \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{abc} \partial_\nu \varphi^a \partial_\rho \varphi^b \partial_\sigma \varphi^c. \tag{3.15}$$

Here $\mathscr{I}_0$ is also a total divergence; in this case

$$\mathscr{I}_0 = \partial_i P_i, \qquad P_i = \varepsilon_{ikj} \varepsilon^{abc} \partial_k \varphi^a \partial_j \varphi^b \varphi^c \tag{3.16}$$

and comparing it with (3.3) we see that P_i is analogous to the current $\mathscr{I}_\mu$ defined before for the two-dimensional n-field with values in the S^2. The charge

$$Q = \int \mathscr{I}_0 \, \mathrm{d}^3 x = \int_{S_2} P_i \, \mathrm{d}s_i \tag{3.17}$$

is not zero if only $\varphi \underset{|x| \to \infty}{\neq} 0$ and it is integer if

$$\varphi^a \varphi^a |_{|x| = \infty} = \text{const.} \tag{3.18}$$

We conclude the discussion of the topological charge examples by commenting on their importance in the theory of solitons. It happens so that if a model possesses solitons and permits a topological charge, then usually a soliton has a non-trivial topological charge. What is more, its mass can be estimated from below by means of the topological charge [51, 53, 64, 65, 66, 115].

The first of these properties can be verified directly. The expression (1.27), (3.1) exhibits the structureless solitons of the $\sin \varphi_2$ model to have the charge ± 1. The t'Hooft–Polyakov monopole has the charge 1 that is associated to the current (3.11). At last, the two-dimensional n-field's solitons may have any integer value of the topological charge (3.4).

Let us look more attentively at the second property, beginning from the $\sin \varphi_2$ model. The stationary soliton's mass is expressed by

$$M = \frac{1}{\gamma} \int_{-\infty}^{\infty} \mathrm{d}x [\tfrac{1}{2} u_x^2 + m^2 (1 - \cos u)] \tag{3.19}$$

and allows the estimate for a monotonous function $u(x)$ of the kind

$$M \geqslant \frac{2m}{\gamma} \int \sqrt{\frac{1-\cos u}{2}}\, u_x\, \mathrm{d}x = \frac{2m}{\gamma} \int_{u(-\infty)}^{u(\infty)} \sin\left(\frac{u}{2}\right) \mathrm{d}u = \frac{4m}{\gamma} \cos\frac{u}{2}\bigg|_{\infty}^{-\infty} \tag{3.20}$$

A non-monotonous function requires an obvious generalization. For a soliton with charge 1 the last expression has the value of $8m/\gamma$ what is exactly its mass.

Consider now the two-dimensional n-field. Apparently, we have

$$\frac{1}{4\pi} \int \mathrm{d}^2x (\partial_\mu \boldsymbol{n})^2 \geqslant \frac{1}{4\pi} \int \mathrm{d}^2x (\partial_\mu \boldsymbol{n} \wedge \partial_\nu \boldsymbol{n}, \boldsymbol{n}) \tag{3.21}$$

and so

$$\frac{1}{4\pi} M \geqslant Q. \tag{3.22}$$

This estimate becomes saturated by the exact solutions, because if (2.7) is satisfied then the equality in (3.22) holds.

For the t'Hooft–Polyakov monopole the mass is estimated from below by the current's (3.11) charge; it can be seen immediately by comparing the expressions (3.11) with the energy density

$$H = \operatorname{tr} \int \mathrm{d}^3x \left[\frac{1}{4g^2} F_{ik}^2 + \tfrac{1}{2}(\nabla_i \varphi)^2 + \frac{\lambda}{4}(\varphi^2 - a^2)^2 \right]. \tag{3.23}$$

In the limit of $\lambda = 0$ the estimate is saturated by the exact Sommerfield–Prasad [64] solution which satisfies the equality

$$\tfrac{1}{2}\varepsilon_{ikj} F_{kj} = g \nabla_i \varphi. \tag{3.24}$$

At last, it may be worked out by similar considerations, that the charge of current (3.8) estimates from below the static Hamiltonian of the Skyrme model or its generalizations to an arbitrary chiral field that is defined by the Lagrange function

$$\mathscr{L} = \frac{1}{2\lambda^2} \int \operatorname{tr} L_\mu^2\, \mathrm{d}^2x + \frac{e^2}{2} \int \operatorname{tr}([L_\mu L_\nu]^2)\, \mathrm{d}^3x. \tag{3.25}$$

This property is crucial for the existence and stability of solitons.

2. The *S*-matrix definition within the functional integral formalism

In this section we shall define the S-matrix for solitons and outline a scheme of a modified perturbation theory for its calculation. We shall use the functional integral so that the perturbation theory will be given by the stationary phase method. All the formulations will be presented for a two-dimensional scalar field with the Lagrange function

$$\mathscr{L} = \frac{1}{\gamma} \int_{-\infty}^{\infty} \mathrm{d}x [\tfrac{1}{2}(\partial_\mu u)^2 - v(u)] \tag{0.1}$$

for simplicity but all the results do not depend on this particular example and can be directly generalized to the four-dimensional case.

The coupling constant γ does not enter the classical equations, but it enters the Poisson bracket

$$\{u_t(x), u(y)\} = \gamma\delta(x - y), \tag{0.2}$$

so the quantum model will depend on γ. In our constructions and definitions we shall use the experiences of the functional-integral formulation of the quantum mechanics and of the quantum field theory. The S-matrix is represented there in a path integral form, as a functional integral over the paths that asymptotically coincide with the solutions of the free motion equations, see, e.g., [67, 68].

First we shall recall these formulae and then perform a natural generalization of them for solitons.

2.1. The S-matrix definition within the functional integral formalism

Let us write down a form of the S-matrix in quantum mechanics and in quantum field theory, suitable for the generalization to the soliton case.

The path integral

$$G(t'', q''|t', q') = \int\limits_{t',q'}^{t'',q''} \exp\left\{\mathrm{i}\int\limits_{t'}^{t''} \mathscr{L}(q)\,\mathrm{d}t\right\}\prod_t \mathrm{d}q(t), \qquad \mathscr{L} = \frac{m\dot{q}^2}{2} - v(q), \tag{1.1}$$

$$q(t)|_{t'} = q', \qquad q(t)|_{t''} = q'';$$

expresses the propagator (the transition amplitude) in the configuration representation

$$G(t'', q''|t', q') \equiv \langle q''|\exp\{-\mathrm{i}\mathscr{H}(t'' - t')\}|q'\rangle, \qquad \mathscr{H} = \frac{p^2}{2m} + v(q). \tag{1.2}$$

To obtain the S-matrix in the usual operator formalism we have to evaluate the limit

$$\hat{S} = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \exp(\mathrm{i}\mathscr{H}_0 t'')\exp\{-\mathrm{i}\mathscr{H}(t'' - t')\}\exp(-\mathrm{i}\mathscr{H}_0 t'); \qquad \mathscr{H}_0 = \frac{p^2}{2m}. \tag{1.3}$$

In the functional integral formalism we obtain this result by adopting the following rule: we have to let the time variables in the expression (1.1) go to infinity, $t'' \to \infty$, $t' \to -\infty$, making the q'' and q' variables dependent on t'' and t' according to the classical equations of motion

$$q'' = \frac{p''}{m}t'' + q''_0, \qquad q' = \frac{p'}{m}t' + q'_0. \tag{1.4}$$

The limit of (1.1) will be proportional to the S-matrix in the momentum representation. More strictly, the S-matrix is equal to the limit

$$\langle p''|\hat{S}|p'\rangle = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \frac{G(t'', p''t''/m + q''_0|t', p't'/m + q'_0)}{\sqrt{2\pi}G_0(t'', p''t''/m + q''_0|t_0, q_0)\sqrt{2\pi}G(t_0, q_0|t', p't'/m + q'_0)}. \tag{1.5}$$

The denominator contains the free propagators. This limit does not depend on t_0, q_0, q'_0, q''_0.

We illustrate this by a single-particle problem in quantum-mechanics. The transition amplitude (1.1) can be represented as

$$G(t'', q''|t', q') = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k \exp\{-\mathrm{i}k^2(t''-t')/2m\}}{S(k)} [\tilde{f}_k(q'')\tilde{g}_k(q') + \tilde{g}_k(q'')\tilde{f}_k(q')]$$
$$+ \sum_n \exp\{-\mathrm{i}E_n(t''-t')\}\varphi_n(q'')\varphi_n(q'). \tag{1.6}$$

Functions g, f and φ are the Schroedinger operator eigenfunctions, see Appendix (4.9). Their asymptotic at $|q| \to \infty$ is

$$\tilde{g}_k(q) \to \begin{cases} s(k)\,\mathrm{e}^{-\mathrm{i}kq}, & q \to \infty, \\ \mathrm{e}^{-\mathrm{i}kq} + r(k)\,\mathrm{e}^{+\mathrm{i}kq}, & q \to -\infty \end{cases}$$
$$\tilde{f}_k(q) \to \begin{cases} \mathrm{e}^{+\mathrm{i}kq} - \dfrac{\bar{r}(k)}{\bar{s}(k)} s(k)\,\mathrm{e}^{-\mathrm{i}kq}, & q \to \infty, \\ s(k)\,\mathrm{e}^{+\mathrm{i}kq}, & q \to -\infty; \end{cases} \tag{1.7}$$
$$\hat{k}\varphi_n = -E_n\varphi_n,\ E_n > 0,\ \int \varphi_n^2\,\mathrm{d}q = 1,\ \hat{k}g_k = \frac{k^2}{2m} g_k,\ \hat{k}f_k = \frac{k^2}{2m} f_k,\ \hat{k} = -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + v(q).$$

Let us calculate the transition amplitude (1.5) for a particle in a potential with the initial and final momenta $p'' > 0$, $p' > 0$. We obtain it by evaluating (1.6) in the stationary phase method

$$G(t'', q''|t', q') = \sqrt{\frac{m}{2\pi\mathrm{i}(t''-t')}} \exp\left\{\mathrm{i}m\frac{(q''-q')^2}{2(t''-t')}\right\} s\left(m\frac{q''-q'}{t''-t'}\right) \tag{1.8}$$

and similarly

$$G_0(t'', q''|t', q') = \sqrt{\frac{m}{2\pi\mathrm{i}(t''-t')}} \exp\left\{\mathrm{i}m\frac{(q''-q')^2}{2(t''-t')}\right\}. \tag{1.9}$$

Substituting the expressions (1.8) and (1.9) into the right-hand side of (1.5) and using the formula

$$\lim_{N\to\infty} \sqrt{\frac{\mathrm{i}N}{2\pi}} \exp\left\{-\mathrm{i}N\frac{x^2}{2}\right\} = \delta(x), \tag{1.10}$$

we get the following expression for the transition amplitude

$$\langle p''|\,S\,|p'\rangle = \delta(p''-p')s(p'). \tag{1.11}$$

The formula (1.5) was written for a nonrelativistic particle in a potential, but it can be easily generalised for the many-body problem in quantum mechanics with a translation-invariant interaction. The corresponding formula looks like (A.1.9):

$$\langle\{p_j''\}|\,S\,|\{p_i'\}\rangle = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \frac{G(t'', \{q_j''\}|t', \{q_i'\})}{\prod_j^{N''}\sqrt{2\pi}G_0(t'', q_j''|t_0, q^0)\prod_i^{N'}\sqrt{2\pi}G_0(t_0, q^0|t', q')}; \tag{1.12}$$

$$q_j'' = \frac{p_j''}{m}t'' + q_j^{0''};\qquad q_i' = \frac{p_i'}{m}t' + q_i^{0'}. \tag{1.13}$$

This formula is correct in the space of any dimension. The numerator of (1.12) contains the N-particle propagator and the denominator contains the single-particle propagators. This formula states the S-matrix to be different from the propagator by the external lines amputation.

Let us turn now to quantum field theory. The momentum representation is not natural in this case. The generating functional for the normal form of S-matrix is used most frequenty. This functional makes it possible to define the S-matrix in terms of the functional integral over the paths with the classical asymptotics. The S-matrix generating functional is [67]

$$\mathcal{S}(A^+, A) = \lim_{\substack{t''\to\infty\\ t'\to-\infty}} \int \exp\left\{\frac{1}{4}\int d\beta[(a_\beta^+ a_{-\beta}^+ - a_\beta a_{-\beta})|_{t'}^{t''} + 2a_\beta^+ a_\beta|_{t''} + 2a_\beta^+ a_\beta|_{t'}] + i\int_{t'}^{t''} (\pi\cdot u_t - \mathcal{H})\, d^2x\right\} \prod_{x,t} du(x,t)\, d\pi(x,t); \qquad \mathcal{H} = \frac{\gamma}{2}\pi^2 + \frac{1}{2\gamma}u_x^2 + \frac{1}{\gamma}V(u); \tag{1.14}$$

$$u(x) = \frac{1}{\sqrt{4\pi}}\int d\beta[a_\beta^+ \exp\{-im\sinh\beta x\} + a_\beta \exp\{im\sinh\beta x\}];$$

$$\pi(x) = \frac{im}{\sqrt{4\pi}}\int d\beta \cosh\beta[a_\beta^+ \exp\{-im\sinh\beta x\} - a_\beta \exp\{im\sinh\beta x\}]. \tag{1.15}$$

The quantities $a_\beta^+(t)$, $a_\beta(t)$ at t'' and t' are equal to

$$a_\beta^+|_{t''} = A_\beta^+ \exp\{im\cosh\beta t''\}; \qquad a_\beta|_{t'} = A_\beta \exp\{-im\cosh\beta t'\}, \tag{1.16}$$

when A^+ and A are independent functions here. The S-matrix elements are obtained by the differentiation of the generating functional

$$\langle\{p_j''\}|S|\{p_i'\}\rangle = \left.\frac{\delta^{N'+N''}\mathcal{S}(A^+,A)}{\prod_{j=1}^{N''}\delta A_{p_j}^+ \prod_i^{N'}\delta A_{p_i}}\right|_{A^+=A=0}, \qquad p = m\sinh\beta. \tag{1.17}$$

The relation (1.17) shows that we have to use $A^+(\beta)$ and $A(\beta)$ localized in momentum space. Then u and π will be localized in the configurational space,

$$\int d\beta A^+(\beta)\exp\{im(t''\cosh\beta - x\sinh\beta)\} \to \sqrt{\frac{2\pi i}{m\sqrt{(t'')^2 - x^2}}}\, A_{\beta_{st}}^+ \exp\{im\sqrt{(t'')^2 - x^2}\}; \qquad \tanh\beta_{st} = \frac{x}{t''}, \tag{1.18}$$

and the wave packet goes to infinity faster than it decays. Note that we can perform integration over all the internal π in (1.14) and obtain a manifestly Lorentz-invariant expression (except of the boundary terms).

These examples are sufficient for the S-matrix definitions for processes with structureless solitons. The solitons with an internal degree of freedom correspond to periodic finite motion, and this motion is known to generate bound states in the usual quantum mechanics. So we complete the description of the quantum-mechanical results by the S-matrix definition for processes with bound states. Classical mechanics regards bound states to be the finite periodic motion so we can take a motion of a particle around a circle as a single-particle quantum-mechanical model.

The Schroedinger equation of this model is:

$$\left[-\frac{1}{2}\frac{d^2}{d\alpha^2} + v(\alpha)\right]\psi_E(\alpha) = E\psi_E(\alpha); \qquad \psi_E(\alpha + L) = \psi_E(\alpha). \tag{1.19}$$

We shall start from construction of a single particle propagator which is non-trivial in this case. It is expressed by a functional integral

$$G(t'', \alpha''|t', \alpha') = \int_{t',\alpha'}^{t'',\alpha''} \exp\left\{i\int_{t'}^{t''} \mathscr{L}(\alpha)\,dt\right\}\prod_t d\alpha(t), \tag{1.20}$$

$$\alpha(t'') = \alpha'', \qquad \alpha(t') = \alpha'; \qquad \mathscr{L} = \tfrac{1}{2}\dot{\alpha}^2 - v(\alpha).$$

We have to integrate over paths with an arbitrary number of circulations because any two points separated by an integer number of circulations are identical. This expression can be simplified by the following consideration. Let us take the same equation as (1.19) but on the whole axis and with a periodic potential

$$\begin{gathered}\left[-\frac{1}{2}\frac{d^2}{d\alpha^2} + v(\alpha)\right]\psi_\nu(\alpha) = E_\nu\psi_\nu(\alpha); \qquad v(\alpha + L) = v(\alpha)\\ \psi_\nu(\alpha + L) = e^{i\nu}\psi_\nu(\alpha).\end{gathered} \tag{1.21}$$

The propagator of the last problem

$$\tilde{G}(t'', \alpha''|t', \alpha') = \int_{t',\alpha'}^{t'',\alpha''} \exp\left\{i\int_{t'}^{t''} \mathscr{L}(\alpha)\,dt\right\}\prod_t d\alpha(t) \tag{1.22}$$

is not equival to (1.20); roughly speaking we do not consider identical the two points separated by an integer number of spatial cells, and so we integrate here over the paths with $[(\alpha'' - \alpha')/L]$ "circulations" only. The kernels (1.22) and (1.20) are related by

$$G(t'', \alpha''|t', \alpha') = \sum_{n=-\infty}^{\infty} \tilde{G}(\alpha'' + nL, t''|\alpha', t'). \tag{1.23}$$

The spectrum E_k of the problem (1.19) appears to be easily calculable by examining the function (1.22). It is derived from $\tilde{G}$ by the following formulae:

$$\left\{\begin{array}{l}\dfrac{d}{dT}\left[\dfrac{-i\ln\tilde{G}(nT, nL|0,0) + nET}{n}\right] = 0;\\[2ex] \dfrac{-i\ln\tilde{G}(nT, nL|0,0) + nET}{n} = 2\pi k;\\[2ex] n \to \infty.\end{array}\right. \tag{1.24}$$

Let us explain this expression. We write $\tilde{G}$ in a bilinear form in terms of the Schroedinger equation eigenfunctions

$$\tilde{G}(t'', \alpha''|t', \alpha') = \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi L} \psi_\nu(\alpha'')\bar{\psi}_\nu(\alpha') \exp\{-\mathrm{i}E_\nu(t'' - t')\}, \tag{1.25}$$

$$\int_0^L |\psi_\nu(\alpha)|^2 \,\mathrm{d}\alpha = L.$$

Here ν is the Floquet index entering the condition $\psi_\nu(x + L) = \mathrm{e}^{\mathrm{i}\nu}\psi_\nu(x)$. The spectrum (1.19) is evidently given by the equation

$$\nu(E_k) = 2\pi k. \tag{1.26}$$

The asymptotics of $\tilde{G}$ at large values of t, α is

$$\begin{aligned} &\tilde{G}(t'', \alpha''|t', \alpha') = \frac{\varphi_{\nu_0}(\alpha'')\bar{\varphi}_{\nu_0}(\alpha')}{L\sqrt{2\pi\mathrm{i}E''_{\nu_0}(t'' - t')}} \exp\left\{\mathrm{i}\frac{\nu_0}{L}(\alpha'' - \alpha') - \mathrm{i}E_{\nu_0}(t'' - t')\right\}; \\ &E'_{\nu_0} = \frac{\alpha'' - \alpha'}{L(t'' - t')} = \frac{1}{T}. \end{aligned} \tag{1.27}$$

The function $\varphi_\nu(\alpha) = \psi_\nu(\alpha)\exp\{-\mathrm{i}\nu\alpha/L\}$ is periodic. Substituting (1.27) into (1.24) we find (1.26). From (1.24) we can calculate the spectra of energies E_k and of the corresponding periods T_k. So this is a method to obtain characteristics of a periodic motion by reducing it to an infinite motion in a periodic potential.

Following this idea we can write down a definition of the S-matrix for systems with an internal coordinate which is quite analogous to (1.12), (1.13). Consider a number of particles with the translational and the internal degrees of freedom. Let the function

$$\tilde{G}(t'', \{q''_j\}, \{\alpha''_j\}|t', \{q'_i\}, \{\alpha'_i\}) \tag{1.28}$$

be the corresponding propagator with the internal coordinate having its values on line R^1. Then the S-matrix can be defined as (A.2.3)

$$\langle\{k''_j\}, \{p''_j\}|S|\{k'_i\}, \{p'_j\}\rangle = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \frac{[\prod_{i,j}\Psi_j(0)\Psi_i(0)]\tilde{G}(t'', \{q''_j\}, \{\alpha''_j\}|t', \{q'_i\}, \{\alpha'_i\})}{\prod_j^{N''}\sqrt{2\pi\tilde{G}_0(t'', q''_j, \alpha''_j|t_0, 0, 0)}\prod_i^{N'}\sqrt{2\pi\tilde{G}_0(t_0, 0, 0|t', q'_i, \alpha'_i)}}. \tag{1.29}$$

Here we imply

$$\begin{aligned} q''_j &= p''_j t''/m + q^{0''}_j; \qquad & q'_i &= p'_i t'/m + q^{0'}_i; \\ \alpha''_j &= L_j t''/T_{k_j}; \qquad & \alpha'_i &= L_i t'/T_{k_i}, \end{aligned} \tag{1.30}$$

as we always do. The denominator of (1.29) contains a product of single-particle propagators. The $k_i(k_j)$ in the right-hand side are the quantum numbers of the ith(jth) soliton. The spectra E_k, T_k of each interacting particle are defined by the corresponding $\tilde{G}_0$, and $\Psi(0) = |\psi(0)|$ is the wave function value at $\alpha = 0$.

2.2. The S-matrix of solitons

Now keeping in mind the discussed formalism we are going to deal with the solitons of the field defined by the Lagrange function (0.1). The field propagator is described by the functional integral where $\int \mathcal{H}\,\mathrm{d}x$ is the corresponding Hamiltonian. This function

$$\langle \varphi''|\exp\left\{-\mathrm{i}(t''-t')\int \mathcal{H}\,\mathrm{d}x\right\}|\varphi'\rangle = \int_{t',\varphi'}^{t'',\varphi''} \exp\left\{\mathrm{i}\int^{t''} \mathcal{L}(u)\,\mathrm{d}t\right\}\prod_{x,t}\mathrm{d}u(x,t), \tag{2.1}$$

$$u(t'') = \varphi'', \qquad u(t') = \varphi',$$

defines the soliton S-matrix at $t'' \to \infty$, $t' \to -\infty$ when the functions $\varphi''(x)$, $\varphi'(x)$ are properly constructed. In analogy with (1.13) we have to make the functions $\varphi''(x)$ and $\varphi'(x)$ to be dependent on t'' and t' in such a way that $\varphi''(x, t'')$ and $\varphi'(x, t')$ will be solutions of the classical equations. In the limit they must turn into the sums of purely soliton solutions, and the continuous spectrum degrees of freedom (the usual particles' degrees of freedom) must be frozen.

At first we look at the structureless solitons. Let the function

$$u_s\left(\frac{x-q-vt}{\sqrt{1-v^2}}\right) \tag{2.2}$$

be a single-soliton solution. The configuration

$$\begin{aligned} u_{N''s} &= \sum_{j=1}^{N''} u_s\left(\frac{x-q_j''}{\sqrt{1-(q_j''/t'')^2}}\right); \\ u_{N's} &= \sum_{i=1}^{N'} u_s\left(\frac{x-q_i'}{\sqrt{1-(q_i'/t')^2}}\right) \end{aligned} \tag{2.3}$$

describes N'', N' solitons situated at $\{q_j''\}$ or $\{q_i'\}$ points when $t = t''$ or $t = t'$ and these t'', $|t'|$ are large enough. The function

$$G(t'', \{q_j''\}|t', \{q_i'\}) \equiv \langle u_{N''s}|\exp\left\{-\mathrm{i}(t''-t')\int \mathcal{H}\,\mathrm{d}x\right\}|u_{N's}\rangle \tag{2.4}$$

can be regarded as the soliton propagator in the soliton configuration representation. The S-matrix is defined by a formula analogous to (1.12), (1.13), namely [25]

$$\langle\{v_j''\}|S|\{v_i'\}\rangle = \lim_{\substack{t''\to\infty\\ t'\to-\infty}} \frac{G(t'', \{q_j''\}|t', \{q_i'\})}{\prod_j^{N''}\sqrt{2\pi G(t'', q_j''|t_0, q_0)}\prod_i^{N'}\sqrt{2\pi G(t_0, q_0|t', q_i')}}; \tag{2.5}$$

$$q_j'' = v_j''t'' + q_j^{0\prime\prime};$$

$$q_i' = v_i't' + q_i^{0\prime}, \qquad v = \tanh\varphi.$$

Here in the denominator stands the product of the single-particle propagators. Their asymptotics are, owing to the relativistic invariance:

$$G(t'', q''|t', q') = \sqrt{\frac{M}{2\pi\mathrm{i}\sqrt{(t''-t')^2-(q''-q')^2}}}\exp\{-\mathrm{i}M\sqrt{(t''-t')^2-(q''-q')^2}\}. \tag{2.6}$$

The mass M is not necessarily equal to the classical soliton mass, it is changed by the quantum corrections. In contrast to the quantum mechanics, our S-matrix is the function of velocities but not of momenta; it is the use of velocities what enables us to make the perturbation theory manifestly Lorentz-invariant. The calculation below will show the right-hand side of (2.5) to be a sum of products of the δ-functions which mean the conservation laws, with the factors (the reduced S-matrices) represented by the functional integrals in the vicinity of the purely soliton solutions.

We turn now to the periodic soliton case. The function

$$w(x, t|v, T, q, \alpha) = w\left(\frac{x - vt - q}{\sqrt{1 - v^2}}, \frac{t - vx}{\sqrt{1 - v^2}} - \frac{\alpha}{2\pi}, T\right) \tag{2.7}$$

is a solution of the classical equations. We assume w to be localized at $x = vt + q$. The phase value α should be taken at the spatial density maximum. The function w depends on α periodically with the period 2π. But according to the quantum-mechanical experience we consider the variable α to be varied along the whole axis, thus the points α and $\alpha + 2\pi n$ being not equivalent. We define a single-particle propagator in analogy with (2.4)

$$\begin{gathered}\tilde{G}(t'', q'', \alpha''|t', q', \alpha') = \int\limits_{t',\varphi'}^{t'',\varphi''} \exp\left\{i \int\limits_{t'}^{t''} \mathscr{L}(u)\, \mathrm{d}t\right\} \prod_{x,t} \mathrm{d}u(x, t); \\ \varphi'' = w\left(x, t \left| \frac{q''}{t''}, 2\pi \frac{\sqrt{(t'')^2 - (q'')^2}}{\alpha''}, 0, 0\right.\right), \quad \varphi' = w\left(x, t \left| \frac{q'}{t'}, 2\pi \frac{\sqrt{(t')^2 - (q')^2}}{\alpha'}, 0, 0\right.\right)\end{gathered} \tag{2.8}$$

The periodic soliton mass spectrum should be calculated just as it has been at (1.24), [25]:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}T}\left[\dfrac{-\mathrm{i}\ln \tilde{G}(nT, 0, 2\pi n|0, 0, 0) + MnT}{n}\right] = 0; \\[2ex] \dfrac{-\mathrm{i}\ln \tilde{G}(nT, 0, 2\pi n|0, 0, 0) + MnT}{n} = 2\pi k; \\[2ex] n \to \infty. \end{cases} \tag{2.9}$$

From this we derive M_k and T_k. It should be noted that the number of the internal states $\{k\}$ is finite when the internal momentum varies in the compact domain (as it may happen in the one-dimensional case). Next, the configuration space propagator of an interacting periodic solitons group is:

$$\tilde{G}(t'', \{q_j''\}, \{\alpha_j''\}|t', \{q_i'\}, \{\alpha_i'\}) = \int\limits_{t'\varphi'}^{t''\varphi''} \exp\left\{i \int\limits_{t'}^{t''} \mathscr{L}(u)\, \mathrm{d}t\right\} \prod \mathrm{d}u(x, t); \tag{2.10}$$

$$\begin{aligned} \varphi'' &= \sum_j^{N''} w\left(x, t \left| \frac{q_j''}{t''}, 2\pi \frac{\sqrt{(t'')^2 - (q_j'')^2}}{\alpha_j''}, 0, 0\right.\right); \\ \varphi' &= \sum_i^{N'} w\left(x, t \left| \frac{q_i'}{t'}, 2\pi \frac{\sqrt{(t')^2 - (q_i')^2}}{\alpha_i'}, 0, 0\right.\right) \end{aligned} \tag{2.11}$$

The expressions (2.11) have all the terms separated in space when $|t| \to \infty$. To obtain the S-matrix we perform the usual limit evaluation. In analogy with (1.29) the S-matrix is defined so

$$\langle \{k''_j\},\{v''_j\}|S|\{k'_i\},\{v'_i\}\rangle = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \frac{\tilde{G}(t'',\{q''_j\},\{\alpha''_j\}|t',\{q'_i\},\{\alpha'_i\})\,[\prod_{i,j}\Psi_j(0)\Psi_i(0)]}{\prod_j^{N''}\sqrt{2\pi}\tilde{G}_0(t'',q''_j,\alpha''_j|t_0,0,0)\prod_i^{N'}\sqrt{2\pi}\tilde{G}_0(t_0,0,0|t',q'_i,\alpha'_i)}; \tag{2.12}$$

$$q''_j = v''_j t'' + q_j^{0''}; \qquad q'_i = v'_i t' + q_i^{0'};$$

$$\alpha''_j = 2\pi t''/T_{k_j}; \qquad \alpha'_i = 2\pi t'/T_{k_i}.$$

Calculating the expressions of (2.8), (2.10) by the stationary phase method we have to account for only one stationary phase point, because the points α and $\alpha + 2\pi n$ are not equivalent. We do not need to calculate the values of $\Psi_k(0) = |\psi_k(0)|$, they become ready known from the propagators and the unitarity condition.

At last let us look at the scattering of the usual particles and solitons. We take the structureless solitons for simplicity. Everything necessary for a generalization to the periodic solitons case was discussed above. Consider an object which is an S-matrix element for solitons in velocity representation and a generating functional for the S-matrix of the usual particles [25] (1.14)

$$\langle \{v''_j\}, A^+|S|\{v'_i\}, A\rangle = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \left[\prod_j^{N''}\sqrt{2\pi}G(t'',q''_j|t^0,q^0)\prod_i^{N'}\sqrt{2\pi}G(t^0,q^0|t',q'_i)\right]^{-1}$$

$$\times \int \exp\left\{\tfrac{1}{4}\int d\beta\left[(a^+_\beta a^+_{-\beta} - a_\beta a_{-\beta})\Big|_{t'}^{t''} + 2a^+_\beta a_\beta\Big|_{t'} + 2a^+_\beta a_\beta\Big|_{t'}\right] + i\int_{t'}^{t''}(\pi u_t - \mathscr{H})\,d^2x\right\}\prod_x du\,d\pi. \tag{2.13}$$

The boundary conditions are to be specified in the following way. According to (1.18) we can localize the solitons and the wave packets of the usual particles at different places one far from another at large values of t'', $-t'$ and supply them with the independent boundary conditions (2.3), (1.16). The first boundary term in the exponent of (2.13) is generated only by the wave pockets.

These definitions provide the topological charge conservation. Really, there exists no time-continuous path that joins the field configurations with different topological charges; and the discontinuous paths have an infinite action value so that their contribution to the functional integral is zero [63].

All the definitions given above make it possible to develop a consistent perturbation expansion in powers of γ. To do this we must evaluate the integrals (2.1), (2.8), (2.10), (2.13) by the stationary phase method. The right-hand sides of (2.5), (2.12) turn out to be proportional to the delta-functions with factors that are the functional integrals over the vicinities of the purely soliton solutions. The denominators of (2.5), (2.12), (2.13) and the numerators in the one-loop approximations form the conservation delta-functions. The denominator in the higher approximations accounts for the single-particle states renormalization. This results are manifestly Lorentz-invariant.

The details of this technique and its diagrammatic representation will be described in the next section.

3. The diagram technique

In this section we shall describe the diagram technique for the calculation of the solitons physical observables. The definitions of section 2 will be exploited. The diagram technique arises naturally as a description of the expansion of the functional integral

$$G(t'',\{q_j''\}|t',\{q_i'\}) = \int_{t',\varphi'}^{t'',\varphi''} \exp\left\{\mathrm{i}\int_{t'}^{t''} \mathscr{L}(u)\,\mathrm{d}t\right\} \prod_{x,t} \mathrm{d}u(x,t); \qquad \mathscr{L} = \frac{1}{\gamma}\int \mathrm{d}x(\tfrac{1}{2}u_\mu^2 - v(u)) \tag{0.1}$$

with the boundary conditions

$$\begin{aligned} u(t'') = \varphi'' &= \sum_{j=1}^{N''} u_s\left(\frac{t''(x-q_j'')}{\sqrt{(t'')^2-(q_j'')^2}}\right); \\ u(t') = \varphi' &= \sum_{i=1}^{N'} u_s\left(\frac{t'(x-q_i')}{\sqrt{(t')^2-(q_i')^2}}\right) \end{aligned} \tag{0.2}$$

within the stationary phase method. The stationary phase points of the integral (0.1) are the solutions of the classical equation

$$\Box u^{\mathrm{cl}} + v'(u^{\mathrm{cl}}) = 0 \tag{0.3}$$

with the boundary conditions

$$u^{\mathrm{cl}}|_{t''} = \varphi'', \qquad u^{\mathrm{cl}}|_{t'} = \varphi'. \tag{0.4}$$

For the evaluation of the integral (0.1) we perform the change of variables

$$u(x,t) = u^{\mathrm{cl}}(x,t) + \sqrt{\gamma}\varphi(x,t) \tag{0.5}$$

and expand the action into a series of powers of $\sqrt{\gamma}$. We obtain the expression for G as an integral over φ:

$$\begin{aligned} G(t'',\{q_j''\}|t',\{q_i'\}) &= \exp\left\{\mathrm{i}\int_{t'}^{t''} \mathscr{L}(u^{\mathrm{cl}})\,\mathrm{d}t\right\} \\ &\times \int \exp\left\{-\frac{\mathrm{i}}{2}\int_{t'}^{t''} \varphi H\varphi\,\mathrm{d}^2x - \mathrm{i}\sum_{n=3}^{\infty}\frac{1}{n!}\gamma^{n/2-1}\cdot\int_{t'}^{t''} v^{(n)}(u^{\mathrm{cl}})\varphi^n\,\mathrm{d}^2x\right\} \prod_{x,t}\mathrm{d}\varphi(x,t), \end{aligned} \tag{0.6}$$

with the boundary conditions

$$\varphi|_{t'} = \varphi|_{t''} = 0. \tag{0.7}$$

The quadratic form in φ in the exponent looks as

$$-\tfrac{1}{2}\int_{t'}^{t''} \varphi H\varphi\,\mathrm{d}^2x = \tfrac{1}{2}\int_{t'}^{t''}\mathrm{d}t\int \mathrm{d}x[\partial_\mu\varphi\partial_\mu\varphi - v''(u^{\mathrm{cl}})\varphi^2], \tag{0.8}$$

$$H = \Box + v''(u^{\mathrm{cl}}) \tag{0.9}$$

and does not depend on γ. The higher order terms nonlinear in φ contain the higher powers of γ. In the expansion (0.6) any γ-power term can be expressed as a Gaussian-type integral

$$\int \exp\left\{-\frac{\mathrm{i}}{2}\int \varphi H\varphi\, \mathrm{d}^2x\right\}\prod_k \varphi^{n_k}(x_k)\prod_{x,t}\mathrm{d}\varphi(x,t), \tag{0.10}$$

which is easily calculable and is naturally described in the diagram technique language [69]. Finally G takes the form

$$G(t'',\{q''_j\}|t',\{q'_i\}) = \exp\left\{\sum_{n=-1}^{\infty}\gamma^n\cdot W_n\right\}, \tag{0.11}$$

where W_n is the sum of the connected $(n+1)$-loop vacuum diagrams which are described by the Green function H^{-1} and the n-prong vertices $v^{(n)}(u^{\mathrm{cl}})$. The expressions for the first lowest order W_n are easily written as

$$W_{-1} = \mathrm{i}\int_{t'}^{t''}\mathscr{L}(u^{\mathrm{cl}})\,\mathrm{d}t; \tag{0.12}$$

$$W_0 = -\tfrac{1}{2}\operatorname{Tr}\ln H\cdot H_0^{-1}, \qquad H_0 = \Box + m^2, \qquad m^2 = v''(0). \tag{0.13}$$

This correspondence between the functional integral (0.1) and the diagram technique is generally valid and is not limited to the structureless soliton case. The only thing to be changed in case of the periodic solitons or the continuous spectrum is the boundary condition (0.2), (0.4). The characteristic behaviour of the series

$$\sum_{n=-1}^{\infty}\gamma^n W_n \tag{0.14}$$

as a function of t'' and t' when the solitons are present is that this series grows linearly with t''; t'. We can demonstrate this representing the general classical solution as the sum

$$u^{\mathrm{cl}} = u_{N\mathrm{s}} + \varphi \tag{0.15}$$

of a soliton part $u_{N\mathrm{s}}$ which becomes a sum of single-soliton solutions at large values of the time variables and a function φ which is a packet of plane waves (we shall call φ the continuous spectrum). This packet decays as $1/\sqrt{t}$ at large values of t. The classical equation becomes asymptotically

$$\Box u_{N\mathrm{s}} + v'(u_{N\mathrm{s}}) = 0, \qquad \Box\varphi + v''(u_{N\mathrm{s}})\varphi = 0. \tag{0.16}$$

We substitute (0.15) into (0.12) and represent the action as

$$\begin{aligned}&\mathrm{i}\int_{t'}^{t''}\mathscr{L}(u_{N\mathrm{s}})\,\mathrm{d}t + \frac{\mathrm{i}}{\gamma}\int_{t'}^{t''}\mathrm{d}^2x(\varphi_{\mu}u_{N,\mu} - v'(u_{N\mathrm{s}})\varphi) + \\ &\qquad + \frac{\mathrm{i}}{2\gamma}\int_{t'}^{t''}\mathrm{d}^2x(\varphi_{1\mu}^2 - v''(u_{N\mathrm{s}})\varphi^2) - \frac{\mathrm{i}}{\gamma}\sum_{n=3}^{\infty}\int_{t'}^{t''}v^{(n)}(u_{N\mathrm{s}})\varphi^n\,\mathrm{d}^2x.\end{aligned} \tag{0.17}$$

The last term does not contribute to the linear growth, it tends to a constant at $t'' \to \infty, t' \to -\infty$ because of the decay of the wave packet. The second and the third terms behave similarly which can be proved by the integration by parts and by using the equations (0.16). The leading part of the first term is equal to the sum of the single-soliton contributions which are linear with respect to $t''; t'$ (see, e.g., (1.8)). The same can be demonstrated for any W_n.

We shall use the expansion (0.11) together with the definition (2.2.5) for the S-matrix calculation in paragraphs 3.1 and 3.2. As a matter of fact, the integral (0.1) has more than one stationary phase point; they correspond to the processes of different connectedness. Every term arising from a corresponding stationary phase point will be represented as a product of conservation laws delta-functions and a reduced S-matrix element, which is equal to a functional integral in the vicinity of a fixed classical solution for the process under consideration.

In paragraph 3.3 we shall discuss the zero-mode problem [11], which consists in the following. At first sight it seems that the inverse operator H^{-1} does not exist. Indeed, the quadratic form (0.8) is degenerate; it vanishes on the functions

$$\frac{\mathrm{d}}{\mathrm{d}x} u^{\mathrm{cl}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} u^{\mathrm{cl}}, \tag{0.18}$$

which satisfy the equation

$$H \frac{\mathrm{d}}{\mathrm{d}x} u^{\mathrm{cl}} = 0, \qquad H \frac{\mathrm{d}}{\mathrm{d}t} u^{\mathrm{cl}} = 0. \tag{0.19}$$

Hence the diagram technique seems to be undefined. However we are going to demonstrate that all the zero-modes of type (0.18) belong to the continuous rather than discrete spectrum of the operator H. We shall construct a correct definition of the operator H^{-1} which accounts naturally for the zero-modes.

In paragraph 3.4 we describe the diagram technique for calculation of quantum corrections to the periodic soliton mass and to the two structureless solitons propagators.

In paragraph 3.5 the soliton quantum theory renormalizability is demonstrated.

3.1. *The single soliton propagator*

According to the definition (2.2.4), (2.2.3) the soliton propagator is

$$G(t'', q''|t', q') = \int_{t',\varphi'}^{t'',\varphi''} \exp\left\{\mathrm{i} \int_{t'}^{t''} \mathscr{L}(u)\,\mathrm{d}t\right\} \prod_{x,t} \mathrm{d}u(x,t), \tag{1.1}$$

$$\varphi'' = u_{\mathrm{s}}\left(\frac{t''(x-q'')}{\sqrt{(t'')^2-(q'')^2}}\right), \qquad \varphi' = u_{\mathrm{s}}\left(\frac{t'(x-q')}{\sqrt{(t')^2-(q')^2}}\right); \tag{1.2}$$

we shall calculate G by the stationary phase method. The first two terms are (0.12), (0.13),

$$G(t'', q''|t', q') = \exp\left\{\mathrm{i} \int_{t'}^{t''} \mathscr{L}(u^{\mathrm{cl}})\,\mathrm{d}t\right\} \det{}^{-1/2}(H \cdot H_0^{-1}). \tag{1.3}$$

The stationary phase point $u^{\rm cl}(x, t)$ can not be an exact single-soliton solution. It is obvious because the solitons configurations velocities are not the same at t'' and t' in general $q''|_{t''} \neq q'|_{t'}$. The classical solution is

$$u^{\rm cl} = u_{\rm s}\left(\frac{x - vt - q}{\sqrt{1 - v^2}}\right) + \varphi(x, t);$$
$$v = \frac{q'' - q'}{t'' - t'} + O\left(\frac{1}{t''}\right), \qquad q = \frac{q' \cdot t'' - q'' \cdot t'}{t'' - t'} + O\left(\frac{1}{t''}\right). \tag{1.4}$$

We shall see below that the function φ has no influence on the soliton observables.

It is convenient to perform the Lorentz transformation and use the variables

$$r = (x - vt)/\sqrt{1 - v^2}, \qquad \tau = (t - vx)/\sqrt{1 - v^2}. \tag{1.5}$$

Let us calculate approximately the right-hand side of (1.3), replacing $u^{\rm cl}$ by $u_{\rm s}$:

$$G(t'', q''|t', q') = \exp\left\{ {\rm i} \int_{t'}^{t''} \mathscr{L}(u_{\rm s})\, {\rm d}t \right\} \det{}^{-1/2}(H \cdot H_0^{-1}), \tag{1.6}$$

$$H = \Box + v''(u_{\rm s}). \tag{1.7}$$

The action evaluated on the path $u_{\rm s}$ is

$$\int_{t}^{t''} \mathscr{L}(u_{\rm s})\, {\rm d}t = -M_{\rm s}\sqrt{1 - v^2}(t'' - t') = -M_{\rm s}\sqrt{(t'' - t')^2 - (q'' - q')^2} + {\rm const};$$
$$M_{\rm s} = \frac{1}{\gamma}\int {\rm d}x(\tfrac{1}{2}[u_x^{\rm s}(x)]^2 + v(u_{\rm s}(x))). \tag{1.8}$$

To calculate the second factor in (1.3) we note that

$$H(u_{\rm s}) = \frac{{\rm d}^2}{{\rm d}\tau^2} + \hat{K}(r), \qquad \hat{K}(r) = -\frac{{\rm d}^2}{{\rm d}r^2} + v''(u_{\rm s}(r)). \tag{1.9}$$

The operator H is degenerate and the function $u_{\rm s}'(r)$ is the eigenfunction of H with zero eigenvalue:

$$Hu_{\rm s}'(r) = \hat{K}u_{\rm s}'(r) = 0. \tag{1.10}$$

Indeed, the differentiation of the classical equation

$$-\frac{{\rm d}^2}{{\rm d}r^2} u_{\rm s}(r) + v'(u_{\rm s}(r)) = 0 \tag{1.11}$$

with respect to r leads to

$$-\frac{{\rm d}^2}{{\rm d}r^2} u_{\rm s}'(r) + v''(u_{\rm s}(r)) \cdot u_{\rm s}'(r) = 0. \tag{1.12}$$

We represent the operator H in the form:

$$H = \frac{d^2}{d\tau^2} P + H(I - P), \tag{1.13}$$

with P the projector on the subspace spanned by the functions of the form $F(\tau)u_s'(r)$, $F(\tau)$ arbitrary. The determinant of the operator H is

$$\det(H \cdot H_0^{-1}) = \det\left(\frac{d^2}{d\tau^2}\right) \cdot \det H(I - P)H_0^{-1}. \tag{1.14}$$

The determinant of the operator $d^2/d\tau^2$ which acts in the space of the function of τ can be evaluated most easily as a product of the eigenvalues of the problem:

$$\frac{d^2}{d\tau^2} f_n(\tau) = \lambda_n \cdot f_n(\tau); \qquad f(\tau)|_{t''\sqrt{1-v^2}} = f(\tau)|_{t'\sqrt{1-v^2}} = 0. \tag{1.15}$$

These boundary conditions are derived from (0.7). We obtain the expression

$$\det \frac{d^2}{d\tau^2} = \tau'' - \tau' = \sqrt{1 - v^2}(t'' - t'). \tag{1.16}$$

We will show in section 5 the second factor in (1.14) to be equal to (5.2.12)

$$\det H(u_s(r))(I - P) \cdot H_0^{-1} = \exp\{2i\Delta M \sqrt{(t'' - t')^2 - (q'' - q')^2}\}, \tag{1.17}$$

where ΔM is the soliton mass correction. At last we obtain the expression for (1.6):

$$G(t'', q''|t', q') = \sqrt{\frac{M_s}{2\pi i \sqrt{(t'' - t')^2 - (q'' - q')^2}}} \exp\{-iM_s \sqrt{(t'' - t')^2 - (q'' - q')^2}\}. \tag{1.18}$$

If one takes into account the wave packet φ from (1.4) then the right-hand side of (1.18) will contain the constant factor w,

$$w = \exp\left\{i \int_{t'}^{t''} dt[\mathscr{L}(u^{cl}) - \mathscr{L}(u_s)]\right\} \det{}^{-1/2} H(u^{cl})H^{-1}(u_s). \tag{1.19}$$

So we have proved the soliton propagator to have the form (2.2.6).

Now let us calculate the S-matrix element in a single-particle sector

$$\langle v''| S |v'\rangle. \tag{1.20}$$

Substituting (1.18) into (2.2.5) we obtain

$$\langle v''| S |v'\rangle = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \sqrt{\frac{i\sqrt{(t'' - t_0)^2 - (q'' - q_0)^2}\sqrt{(t_0 - t')^2 - (q_0 - q')^2}}{2\pi M \sqrt{(t'' - t')^2 - (q'' - q')^2}}}$$

$$\times \exp\{-iM_s[\sqrt{(t''-t')^2-(q''-q')^2} - \sqrt{(t''-t_0)^2-(q''-q_0)^2} - \sqrt{(t'-t_0)^2-(q'-q_0)^2}]\}\tilde{w} \tag{1.21}$$

$$q'' = v''t'' + q_0'', \qquad q' = v't' + q_0'.$$

The limit is equal to

$$\langle v''|\,S\,|v'\rangle = \frac{1}{M}\,\delta(\varphi'' - \varphi'). \tag{1.22}$$

Here we have used the rapidity φ, $v = \tanh\varphi$. The factor $\tilde{w}$ became 1 because the u^{cl} from (1.4) is now

$$u^{\mathrm{cl}} = u_{\mathrm{s}}\left(\frac{x - vt - q}{\sqrt{1 - v^2}}\right), \qquad v = \frac{q'' - q'}{t'' - t'}, \qquad q = \frac{t'q'' - t''g'}{t'' - t'}, \qquad \varphi(x, t) = 0. \tag{1.23}$$

The reason for this is the equality of the soliton configurations velocities due to the delta function.

This is a general fact within the stationary phase calculations of (0.1). The classical solution (0.3) which fits the boundary conditions (0.4) is not an exact soliton solution, but differs from it by an additional wave packet $\varphi(x, t)$. But having the propagator (0.1) calculated in this manner as above and substituted into the S-matrix expression (2.2.5) one can neglect $\varphi(x, t)$. Indeed the S-matrix will be a product of some delta functions and a regular factor. The stationary phase point for the regular factor expression will become a pure soliton solution according to the conservation laws and the integration should be performed in the vicinity of this soliton solution.

Returning to the structureless soliton case again, we write the symbolic formula for the mass correction, see (0.11), (1.18)

$$-\mathrm{i}\Delta M\sqrt{1 - v^2} = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \sum_{n=0}^{\infty} \gamma^n W_n \times \left(\frac{1}{t'' - t'}\right). \tag{1.24}$$

Here W_n is a sum of all the $(n + 1)$-loop graphs with n-prong vertices $v^{(n)}(u_{\mathrm{s}}(r))$ and the propagator H^{-1}, (1.7). The method of H^{-1} operator construction will be explained in paragraphs 3.3 and 3.4.

3.2. The S-matrix element for scattering of several solitons

We shall use the definition in section 2 of the soliton S-matrix element (2.2.5):

$$\langle\{v''_j\}|\,S\,|v'_i\}\rangle = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \frac{G(t'', \{q''_j\}|t', \{q'_i\})}{\prod_j \sqrt{2\pi}G(t'', q''|t_0, q^0)\prod_i \sqrt{2\pi}G(t^0, q_0|t', q')}; \tag{2.1}$$

$$q''_j = v''_j t'' + q''_0, \qquad q'_i = v'_i t' + q'_0. \tag{2.2}$$

Let us evaluate the right-hand side by the stationary phase method. In a general case there is a number of the classical solutions which describe a given scattering process. For example, one solution may describe a simultaneous interaction of all solitons, and other solution may describe an interaction of a group of solitons in one point and another group in a different point. These solutions must lead to different terms in the S-matrix with different numbers of delta functions. Everyone of these terms can be represented as a product of the two factors

$$\langle\{v''_j\}|\,S\,|\{v'_i\}\rangle = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \frac{g(t'', \{q''_j\}|t', \{q'_i\})}{\prod_j \sqrt{2\pi}G(t'', q''_j|t^0, q^0)\prod_i \sqrt{2\pi}G(t^0, q^0|t', q'_i)} \tag{2.3}$$

$$\times \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \frac{G(t'', \{q''_j\}|t', \{q'_i\})}{g(t'', \{q''_j\}|t', \{q'_i\})}, \qquad q''_j = v''_j t'' + q''_{0j}; \qquad q'_i = v'_i t' + q'_{0i}. \tag{2.4}$$

The function g is constructed with the purpose to make the first factor a product of delta functions and the second factor a reduced matrix element. These delta functions make the stationary phase point of the functional integral which represents G to be purely soliton solution.

In the case of a simultaneous interaction of all solitons the function g should be taken as

$$g(t'', \{q_j''\}|t', \{q_i'\}) = \int \mathrm{d}q\, \mathrm{d}t \prod_j G(t'', q_j''|t, q) \prod_i G(t, q|t', q_i'). \tag{2.5}$$

The first factor in (2.3) will turn into a product of the two momentum and energy-conserving delta functions.

In the case of the N soliton scattering with the individual momenta conservation the function g should be

$$g(t'', \{q_j''\}|t', \{q_j'\}) = \prod_{j=1}^{N} G(t'', q_j''|t', q_j'). \tag{2.6}$$

The first factor in (2.3) becomes a product of N delta functions. The reduced matrix element is then

$$S(v_1, \dots v_N) = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \frac{G(t'', \{q_j''\}|t', \{q_j'\})}{\prod_{j=1}^{N} G(t'', q_j''|t', q_j')} \tag{2.7}$$

for the structureless solitons case.

For the periodic solitons scattering with the individual Lorentz and internal momenta conservation the reduced S-matrix is, similarly

$$S(v_1, \dots v_N; k_1 \dots k_N) = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \frac{\tilde{G}(t'', \{q_j''\}, \{\alpha_j''\}|t', \{q_i'\}, \{\alpha_i'\})}{\prod_{j=1}^{N} \tilde{G}(t'', q_j'', \alpha_j''|t', q_i', \alpha_i')}. \tag{2.8}$$

In these formulae (2.7), (2.8) we imply the usual limit, (2.2.5), (2.2.12).

3.3. *The zero-mode problem*

Let us examine carefully the operator H (0.9) which is important for the diagram technique construction. Differentiating the classical motion equation, we note that H has some eigenfunctions with zero eigenvalues

$$\frac{\mathrm{d}}{\mathrm{d}x} u^{\mathrm{cl}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} u^{\mathrm{cl}}, \tag{3.1}$$

$$H \frac{\mathrm{d}}{\mathrm{d}x} u^{\mathrm{cl}} = 0, \qquad H \frac{\mathrm{d}}{\mathrm{d}t} u^{\mathrm{cl}} = 0. \tag{3.2}$$

This property is usually understood as an indication that H does not have an inverse operator. Indeed, the zero modes (3.1) situated in the discrete spectrum of H would have made impossible the existence of the operator R such that

$$HR = I. \tag{3.3}$$

Only an operator R would exist which satisfies the condition:

$$HR = I - P, \tag{3.4}$$

where P is the projector on these zero eigenmodes. This would have modified the Feynman rules essentially. Quite a few methods to modify the diagram technique were worked out [13, 14, 17, 70–81], most of them are based on the introduction of collective coordinates [82].

However, we are going to prove that the zero-modes (3.1) are situated in the continuous spectrum of H. As a matter of fact they decrease rapidly with respect to x but do not decrease with respect to t. Hence they are not square integrable and so they make no obstacle for an inverse operator R construction [83, 84].

A correct definition of the H^{-1} operator may be based upon the following observation. At the finite times t'', t' we have to integrate over the fluctuations around the soliton solution which turn to zero at t'' and t' (0.7). The quadratic form

$$\frac{1}{2}\int \varphi H \varphi \, \mathrm{d}^2 x \tag{3.5}$$

is not degenerate because the zero modes do not turn into zero at t'' and t'. So the operator $R = H^{-1}$ is defined uniquely on the finite time interval and the diagram technique is not singular. In the limit $t'' \to \infty$, $t' \to -\infty$ this operator acquires an infinite term. We shall prove this term to make no contribution to the sum of the diagrams. Hence it can be rejected. The resolvent R regularized in this way satisfies the equation

$$HR = I \tag{3.6}$$

and can be used for the diagram technique construction.

Let us realize this program in the structureless soliton case. The operator $R = H^{-1}$ (1.9) with the boundary conditions

$$R(x_2, t_2 | x_1, t_1)|_{t_2 = t''} = R(x_2, t_2 | x_1, t_1)|_{t_2 = t'} = 0 \tag{3.7}$$

looks in this case as (A.5.11)

$$\begin{aligned} R(x_2, t_2 | x_1, t_1) = {} & \frac{u_s'(r_2) u_s'(r_1)}{2\|u_s'\|^2} \left\{ \frac{\tau_2 \tau_1}{\Delta} - \frac{\Sigma}{\Delta}(\tau_2 + \tau_1) + \frac{\Sigma^2}{\Delta} - \Delta + |\tau_2 - \tau_1| \right\} \\ & + \frac{\mathrm{i}}{8\pi} \int \frac{\mathrm{d}\beta \exp\{-\mathrm{i} m \cosh\beta\, |\tau_2 - \tau_1|\}}{S(\beta)} (\tilde{f}_\beta(r_2) \tilde{g}_\beta(r_1) + \tilde{g}_\beta(r_2) \tilde{f}_\beta(r_1)) \\ & + \frac{\mathrm{i}}{2} \sum \frac{\varphi_n(r_2) \varphi_n(r_1)}{\sqrt{m^2 - E_n}} \exp\{-\mathrm{i}\sqrt{m^2 - E_n}\, |\tau_2 - \tau_1|\}, \end{aligned} \tag{3.8}$$

$$r = x \cosh\varphi - t \sinh\varphi, \qquad \tau = t\cosh\varphi - x \sinh\varphi, \qquad \|u_s'\|^2 = \int \mathrm{d}r [u_s'(r)]^2 .$$

Here the functions $\tilde{f}_\beta$, $\tilde{g}_\beta$, φ_n are the eigenfunctions of the Schroedinger operator (A.4.9)

$$\begin{aligned} &\hat{K} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + v''(u_s(r)), \qquad \hat{K}\tilde{f}_\beta = m^2 \cosh^2\beta \tilde{f}_\beta, \qquad \hat{K}\tilde{g}_\beta = m^2\cosh^2\beta\tilde{g}_\beta, \\ &\hat{K}\varphi_n = (m^2 - E_n)\varphi_n, \qquad \hat{K}u_s' = 0, \qquad m^2 = v''(0), \qquad 0 < E_n < m^2, \\ &\Delta = \tfrac{1}{2}(t'' - t'); \qquad \Sigma = \tfrac{1}{2}(t'' + t'). \end{aligned} \tag{3.9}$$

These solutions are described in the Appendix 4. The zero mode $u_s'(r)$ is orthogonal to all other "meson" modes $\tilde{f}_\beta$, $\tilde{g}_\beta$, φ_n at any moment τ.

We affirm that adding the expression $\alpha u_s'(r_1)u_s'(r_2)$ to the Green function (3.8) [83, 84] does not alter the diagrams sum. This will enable us to eliminate the third and the fourth terms from (3.8). Let us show this in the lowest orders of the loops expansion. The principal correction to the structureless soliton propagator is

$$\det{}^{-1/2} H(u_s(r))H_0^{-1}. \tag{3.10}$$

Consider the differential $\mathrm{d}\ln\det HH_0^{-1} = \mathrm{Tr}\, H^{-1}v''(u_s')\,\mathrm{d}u_s$. The right-hand side of this equality contains only one term dependent on α, it is

$$\alpha \int (u_s')^2 v'''(u_s)\,\mathrm{d}u_s\,\mathrm{d}^2x. \tag{3.11}$$

Let us see that it is equal to zero due to the equations of motion. Differentiation of the classical equations (0.3) gives

$$H\,\mathrm{d}u_s = 0, \qquad Hu_s' = 0. \tag{3.12}$$

Differentiating the second of these equations we obtain

$$Hu_s'' = -v'''(u_s)(u_s')^2. \tag{3.13}$$

Multiplying the first of the equations (3.12) by u_s'' and subtracting from it the equation (3.13) multiplied by $\mathrm{d}u_s$ we obtain

$$\int \mathrm{d}^2x\,\mathrm{d}u \cdot v'''(u_s)(u_s')^2 = \int \mathrm{d}^2x \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\mathrm{d}u_s \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}x^\mu} u_s''\right) = 0. \tag{3.14}$$

So the $\det H$ does not depend on α. But we are not going to say that the zero-mode does not contribute to the observables. Its contribution had been written down in the one-loop approximation (1.16), (1.18). With no account of the first term in (3.26) (the zero-mode's contribution) one would obtain a wrong expression of the two-loop mass correction [85, 86, 87].

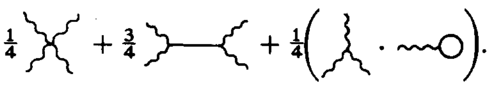
Consider now the two-loop approximation. It is represented by the graphs

(3.15)

The sum of the graphs is a polynomial in α. Let us make it certain that all its coefficients are equal to zero. The coefficient at α^3 is proportional to

$$\int \mathrm{d}r v'''(u_s(r))[u_s'(r)]^3 = 0. \tag{3.16}$$

This is a particular case of (3.14). The coefficient at α^2 can be pictured as



(3.17)

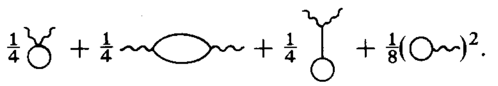
The wavy line is $u_s'(r)$. The last term is equal to zero according to (3.16).

Let us differentiate (3.16) with respect to τ; we obtain

$$\int d\tau \int dr\{(u_s')^4 v^4(u_s) + 3v'''(u_s)(u_s')^2 u_s''\} = 0; \tag{3.18}$$

$$Hu_s'' = -v'''(u_s)(u_s')^2, \qquad u_s'' = -Rv'''(u_s)(u_s')^2. \tag{3.19}$$

Substituting (3.19) into (3.18) we see that the two first terms in (3.17) disappear, hence the coefficient at α^2 is zero. The coefficient at the first power of α is pictured by

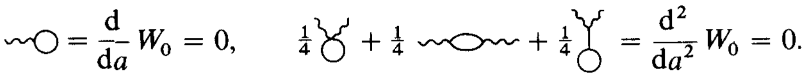


(3.20)

It is zero due to the translational invariance of the one-loop correction (1.6),

$$W_0 = -\tfrac{1}{2}\operatorname{Tr}\ln(\Box + v''(r + a)). \tag{3.21}$$

Really,



(3.22)

In the N-loop approximation we meet the following situation. The approximation is a polynomial in α. These polynomial coefficients can be proved to be the derivatives of W_k, $k < N$ with respect to a, the spatial coordinate of the soliton. All these derivatives are zero due to the translational invariance (see Appendix 3).

So we have demonstrated that the sum of the graphs does not depend on α and the third and the fourth terms in (3.8) can be rejected. It is demonstrated in Appendix 3 that the supplementary term

$$\beta\left[u_s'(r_1)\frac{d}{d\varphi}u_s(r_2) + \frac{d}{d\varphi}u_s(r_1)u_s'(r_2)\right] \tag{3.23}$$

does not alter the graphs sum too. Hence we can reject the second term in (3.8) also. Note that

$$\frac{d}{d\varphi}u_s(r) = -\tau u_s'(r) \tag{3.24}$$

is an example of a growing zero mode. In a general situation of a polysoliton solution the number of growing zero modes is equal to the number of usual zero modes (and is equal to the number of conservation laws). In the case of two zero modes (3.1) we have two growing zero modes, they are the derivatives of the polysoliton classical solution with respect to the total energy and to the total momentum:

$$\frac{d}{dP}u_{Ns}, \qquad \frac{d}{dE}u_{Ns}. \tag{3.25}$$

The first term in (3.8) does not lead to a linearly growing with time term in $\sum_{n=0}^{\infty}\gamma^n W_n$ (1.24), hence it can be rejected too.

Thus we find that the diagram technique for the soliton mass correction (1.24) is to be constructed with the Green function

$$R(x_2, t_2|x_1, t_1) = \frac{u_s'(r_2)u_s'(r_1)}{2\|u_s'\|}|\tau_2 - \tau_1| + \frac{\mathrm{i}}{8\pi}\int \frac{\mathrm{d}\beta \exp\{-im\cosh\beta|\tau_2 - \tau_1|\}}{S(\beta)}$$
$$\times [\tilde{f}_\beta(r_2)\tilde{g}_\beta(r_1) + \tilde{g}_\beta(r_2)\tilde{f}_\beta(r_1)] + \frac{\mathrm{i}}{2}\sum \frac{\varphi_n(r_2)\varphi_n(r_1)}{\sqrt{m^2 - E_n}}\exp\{-\mathrm{i}\sqrt{m^2 - E_n}|\tau_2 - \tau_1|\} \quad (3.26)$$
$$+ \alpha u_s'(r_1)u_s'(r_2) + \beta u_s'(r_1)u_s'(r_2)(\tau_1 + \tau_2),$$

and the sum of the graphs will depend on neither α nor β.

The general situation can be treated similarly. The operator H is degenerate on the zero modes and on the growing zero modes. The method of the resolvent construction is hinted by considering first the case of finite times t'' and t'. At the finite t'', t' the resolvent R exists and is uniquely defined by the boundary conditions

$$H^{-1}(x_2, t_2|x_1, t_1)|_{t_2=t''} = H^{-1}(x_2, t_2|x_1, t_1)|_{t_2=t'} = 0. \quad (3.27)$$

In the limit $t'' \to \infty$, $t' \to -\infty$ a linearly growing with the $(t'' - t')$ term in the resolvent kernel arises; this term is always bilinear in the zero modes. In the case of two zero modes (3.1) this term is

$$au_x(1)u_x(2) + bu_t(1)u_t(2) + c(u_t(1)u_x(2) + u_x(1)u_t(2)). \quad (3.28)$$

Also as it was done above, the addition of terms bilinear in the zero mode can be proved not to alter the sum of graphs, i.e. the growing term in H^{-1} can be thrown out and the regularized resolvent satisfies the equation

$$HR = I. \quad (3.29)$$

This resolvent can be also reduced by using the fact that the graphs sum is not altered by an addition to the resolvent of a product of a zero mode and a growing zero mode (3.1), (3.25).

Returning to the structureless soliton, let us note that the Green function expression contains a term with $|\tau_2 - \tau_1|$. So one may be afraid that the sum $\sum_n \gamma^n W_n$ would grow faster than $(t'' - t')$ at $t'' \to \infty$, $t' \to -\infty$. It can be proved in the perturbations theory that this is not the case. The papers [87, 88] also deal with this method.

3.4. The diagram technique for the observables

The diagram technique construction is reduced to the construction of the resolvent kernel $H^{-1} = R(x_2, t_2|x_1, t_1)$. In this paragraph we describe the expressions for the R kernel in the cases of a periodic soliton and of two interacting structureless solitons.

First we consider a periodic soliton

$$w(r, \tau/T, T), \qquad v = \tanh\varphi, \quad (4.1)$$
$$r = (x - x_0)\cosh\varphi - (t - t_0)\sinh\varphi, \qquad \tau = (t - t_0)\cosh\varphi - (x - x_0)\sinh\varphi.$$

The operator H in this case is

$$H = \Box + v''(w) = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - \frac{\mathrm{d}^2}{\mathrm{d}r^2} + v''\left(w\left(r, \frac{\tau}{T}, T\right)\right). \quad (4.2)$$

The potential here is periodic in τ with the period T and tends rapidly to m^2 at $|r| \to \infty$. The operator H has two zero modes

$$\mathrm{d}w/\mathrm{d}r, \qquad \mathrm{d}w/\mathrm{d}\tau \tag{4.3}$$

and two growing zero modes. Appendix 5 explains that the resolvent R is conveniently constructed using the solutions of the homogeneous equation

$$H\psi = 0. \tag{4.4}$$

We take for this purpose the Floquet solutions

$$\psi_\nu(r, \tau + T) = \mathrm{e}^{-\mathrm{i}\nu}\psi_\nu(r, \tau). \tag{4.5}$$

The Floquet indexes may happen to belong to the "discrete" spectrum, then $\psi_n(r, \tau)$ is square integrable in r. If ν belongs to the continuous spectrum, then $\psi_\nu(r, \tau)$ is

$$(\boldsymbol{u}_\nu(t)\hat{f}_\nu(x)),\ (\hat{g}_\nu^+(x)\boldsymbol{u}^+(t)), \tag{4.6}$$

see Appendix 4. Here $\boldsymbol{u}$ is the vector with the components

$$u_n(t) = \exp\left\{-\mathrm{i}\frac{\nu + 2\pi n}{T}t\right\}, \tag{4.7}$$

and matrices $\hat{g}$ and $\hat{f}$ are the matrix Schroedinger equation solutions:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k_\nu^2 + \hat{v}\right]\hat{f}_\nu = 0, \qquad \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \hat{k}_\nu^2 + \hat{v}\right]\hat{g}_\nu = 0,$$
$$k_{ln} = \sqrt{\left(\frac{\nu + 2\pi n}{T}\right)^2 - m^2\delta_{ln}}, \qquad v_{nl} = \frac{1}{T}\int_0^T \mathrm{d}t \exp\left\{\frac{\mathrm{i}2\pi}{T}(n-l)t\right\}(v''(w) - m^2). \tag{4.8}$$

The solutions are used in Appendix 5 for the resolvent construction with the result:

$$R(\tau_2, r_2|\tau_1, r_1) = \frac{\mathrm{i}}{4\pi T}\sum_\pm \int_{mT}^{mT+2\pi} \mathrm{d}\nu\, \boldsymbol{u}_\nu(\tau_2)\hat{f}_\pm^\nu(r_2)D_{f\pm}^{-1}(\nu)k_\nu^{-1}\hat{g}_\pm^{\nu+}(r_1)\boldsymbol{u}_\nu^+(\tau_1)$$
$$+ \sum_n \frac{\psi_n^-(\tau_2, r_2)\psi_n^+(\tau_1, r_1)}{[\psi_n^+\overleftrightarrow{\partial}_\tau\psi_n^-]} + \frac{w_\varphi(r_2, \tau_2)w'(r_1, \tau_1) - w'(r_2, \tau_2)w_\varphi(r_1, \tau_1)}{2[w'\overleftrightarrow{\partial}_\tau w_\varphi]}$$
$$+ \frac{w_T(r_2, \tau_2)\dot{w}(r_1, \tau_1) - \dot{w}(r_2, \tau_2)w_T(r_1, \tau_1)}{2[\dot{w}\overleftrightarrow{\partial}_\tau w_T]}, \qquad \tau_2 > \tau_1; \qquad \begin{aligned} w_\varphi &= \mathrm{d}w/\mathrm{d}\varphi, \\ w_T &= \mathrm{d}w/\mathrm{d}T. \end{aligned}$$

The sum of the graphs will not be changed by addition of a bilinear form of zero modes (4.3) to the resolvent or a combination of zero modes and growing zero modes products: $\dot{w}w_T$, $\dot{w}w_\varphi$, $w'w_T$, $w'w_\varphi$.

Consider a periodic soliton within the $\sin\varphi_2$ model. The corresponding H operator has no "discrete" spectrum. The matrix Schroedinger equation (4.8) S-matrix is diagonal

$$R_{g,f}^\pm = 0, \qquad [D_+^f(\nu)]_{ln} = [D^g(\nu)]_{ln} = \overline{[D_+^g]_{ln}} = \overline{[D^f]_{ln}} = \delta_{ln}a(\beta_n), \tag{4.10}$$

see (A.4.24). The resolvent is

$$
\begin{aligned}
R(r_2, \tau_2 | r_1, \tau_1) &= \frac{\mathrm{i}}{4\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\beta}{a(\beta)} \psi_\beta^-(\tau_2, r_2) \psi_\beta^+(\tau_1, r_1) \\
&+ \frac{w_\varphi(2) w'(1) - w'(2) w_\varphi(1)}{2[w' \partial_\tau w_\varphi]} + \frac{w_T(2)\dot{w}(1) - \dot{w}(2) w_T(1)}{2[\dot{w} \partial_\tau w_T]}, \qquad \tau_2 > \tau_1 .
\end{aligned} \tag{4.11}
$$

We have introduced a variable β, $v + 2\pi n = mT \cosh\beta$ and used the notation $w(1) = w(r_1, \tau_1)$, $w(2) = w(\tau_2, r_2)$.

Let us look now at the scattering of two structureless solitons. Let the classical scattering occur without reflection, $u_{ss}(x, t | \tanh\varphi_1, \tanh\varphi_2)$ be a corresponding classical solution, the $\tanh\varphi_{1,2}$ the asymptotic soliton velocities. The H operator is

$$
H = \Box + v''(u_{ss}). \tag{4.12}
$$

The zero modes are

$$
u'_{ss} = \mathrm{d}u_{ss}/\mathrm{d}x, \qquad \dot{u}_{ss} = \mathrm{d}u_{ss}/\mathrm{d}t, \tag{4.13}
$$

and the growing zero modes are

$$
u^{ss}_{\varphi_1} = \mathrm{d}u^{ss}/\mathrm{d}\varphi_1, \qquad u^{ss}_{\varphi_2} = \mathrm{d}u^{ss}/\mathrm{d}\varphi_2. \tag{4.14}
$$

With these solutions the resolvent is constructed in the Appendix 5 and it looks as follows (A.5.26)

$$
\begin{aligned}
R(t'', x'' | t', x') &= \frac{\mathrm{i}}{4\pi} \int \frac{\mathrm{d}\beta\,\mathrm{d}\gamma}{a_1(\beta - \varphi_1) a_2(\beta - \varphi_2)} \psi_\gamma^-(x'', t'') (I + C)^{-1}_{\beta\gamma} \psi_\beta^+(x', t') \\
&+ \frac{\mathrm{i}}{2} \sum_n \exp\{-\mathrm{i}\sqrt{m^2 - E_{1n}} |\tau''_1 - \tau'_1|\} \frac{\varphi_{n_1}(r''_1) \varphi_{n_1}(r'_1)}{\sqrt{m^2 - E_{n_1}}} + \frac{\mathrm{i}}{2} \sum_n \frac{\exp\{-\mathrm{i}\sqrt{m^2 - E_{n_2}} |\tau''_2 - \tau'_2|\} \varphi_{n_2}(r''_2) \varphi_{n_2}(r'_2)}{\sqrt{m^2 - E_{n_2}}} \\
&+ \frac{[\cosh\varphi_1 \cdot \dot{u}_{ss}(x'', t'') + \sinh\varphi_1 \cdot u'_{ss}(x'', t'')] u^{ss}_{\phi_2}(x', t')}{2 \| u'_{2s} \|^2 \sinh(\varphi_1 - \varphi_2)} - \frac{u^{ss}_{\phi_2}(x'', t'')}{2 \| u'_{2s} \|^2 \cdot \sinh(\varphi_1 - \varphi_2)} \\
&\times [\cosh\varphi_1 \cdot \dot{u}_{ss}(x', t') + \sinh\varphi_1 \cdot u'_{ss}(x', t')] - \frac{\cosh\varphi_2 \cdot \dot{u}_{ss}(x'', t'') + \sinh\varphi_2 \cdot u'_{ss}(x'', t'')}{2 \| u'_{1s} \|^2 \sinh(\varphi_1 - \varphi_2)} \\
&\times u^{ss}_{\varphi_1}(x', t') + \frac{u^{ss}_{\varphi_1}(x'', t'') [\cosh\varphi_2 \cdot \dot{u}_{ss}(x', t') + \sinh\varphi_2 \cdot u'_{ss}(x', t')]}{2 \| u'_{1s} \|^2 \cdot \sinh(\varphi_1 - \varphi_2)}, \qquad t_2 > t_1 ;
\end{aligned} \tag{4.15}
$$

$$
\| u'_{1s} \|^2 = \int \mathrm{d}x (u'_{1s}(x))^2, \qquad \| u'_{2s} \|^2 = \int \mathrm{d}x (u'_{2s}(x))^2,
$$

see (4.2.3).

This resolvent expression has the same arbitrariness as the previous one.

3.5. The soliton quantum theory renormalizability

In the soliton quantization the ultraviolet divergencies do appear. We are going to show these divergencies to be eliminated by the same counter-terms as the usual particles quantization divergencies [11, 14, 13, 74, 26].

It is clear that at large momenta the soliton diagram technique elements turn into

$$R = (\Box + v''(u_{Ns}))^{-1} \to (\Box + m^2)^{-1}, \qquad v^{(k)}_{(u_{Ns})} \to v^{(k)}_{(0)}, \tag{5.1}$$

because the Fourier transform of u_{Ns} decreases exponentially at large momenta. So the soliton diagram technique asymptotically coincide with the usual diagram technique. These general considerations show why the old counter-terms are sufficient for the soliton divergencies elimination.

For the more rigorous demonstration let us compare the soliton propagator and the S-matrix generating functional for the usual particles [68] (2.1.14)

$$S(A^+, A) = \int \exp\left\{ i \int_{t'}^{t''} \mathscr{L}(u^{cl} + \sqrt{\gamma}\varphi)\, dt \right\} \prod_{x,t} d\varphi(x, t); \tag{5.2}$$

$$u^{cl}(x, t) \xrightarrow[|t| \to \infty]{} u_{as}(A^+, A),$$

which are given by the same integral with different boundary conditions. The only unidentical thing in the perturbations expansions for both integrals is the stationary phase point. For the usual particles let us denote it by u^{cl}, and for the solitons by u_{Ns}. Both diagram techniques are constructed with Green function R and the n-prong vertices:

$$[\Box + v''(u^{st}_{ph})]R = I, \qquad v^{(n)}(u^{st}_{ph}). \tag{5.3}$$

In the diagrams evaluation one meets the same ultraviolet divergencies in both cases: $i\gamma^l \ln \Lambda \int d^2k C(u^{st}_{ph})$. Here $C(u)$ is the local functional calculated on the different classical solutions u^{cl} and u_{Ns}. The both divergencies are eliminated by the same counter-term

$$-\gamma^{l+1} \ln \Lambda \int d^2x C(u). \tag{5.4}$$

This counter-term is, i.e., in the one loop approximation

$$\frac{\gamma}{8\pi^2} D(0) \int d^2x [v''(u) - m^2], \qquad D(0) = \int d^2k (k^2 + m^2)^{-1}. \tag{5.5}$$

These reasonings on the quantum solitons theory renormalizability do not depend on the type of the model and on the dimensions of the space–time and are quite general.

4. The semiclassical approximation

In this section we shall calculate the amplitudes of several processes in the semiclassical approximation (known also as the tree approximation) using the definitions of section 2 and the methods of section 3. As we have said in section 2, to calculate the propagators and the reduced S-matrices one has to evaluate the functional integral

$$G = \int \exp\left\{ i \int_{t'}^{t''} \mathscr{L}(u_{Ns}(x, t) + \sqrt{\gamma}\varphi(x, t))\, dt \right\} \prod_{x,t} d\varphi(x, t). \tag{0.1}$$

Here $u_{Ns}(x, t)$ is the classical equations solution, it defines the soliton process. Note that u_{Ns} is the stationary phase point in the integral (0.1). Calculating (0.1) by the stationary phase method we obtain in the main order of γ:

$$G(t'', \{q''_j\}|t', \{q'_i\}) = \exp\left\{\mathrm{i}\int_{t'}^{t''} \mathscr{L}(u_{Ns})\,\mathrm{d}t\right\}. \tag{0.2}$$

The contents of this section is limited to this approximation; we shall call it the tree approximation in analogy with the quantization of usual particles.

4.1. *The single-particle properties of solitons*

The only property of a structureless soliton is its mass. In our approximation it is equal to the classical value:

$$M^{\mathrm{cl}} = \frac{1}{\gamma}\int\left[\frac{1}{2}\left(\frac{\mathrm{d}u_s(x)}{\mathrm{d}x}\right)^2 + v(u_s(x))\right]\mathrm{d}x, \tag{1.1}$$

which in the sin φ_2 model is equal to

$$M^{\mathrm{cl}} = 8m/\gamma. \tag{1.2}$$

For a periodic soliton the classical soliton is

$$w\left((x-q)\cosh\varphi - t\sinh\varphi, \frac{t\cosh\varphi - x\sinh\varphi}{T} - \frac{\alpha}{2\pi}, T\right), \tag{1.3}$$

where T is the period in the center of mass system and $v = \tanh\varphi$ is the soliton's velocity.

To obtain the mass spectrum we use the formula (2.2.9)

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}T}\,\dfrac{-\mathrm{i}\ln[\tilde{G}(nT, 0, 2\pi n|0, 0, 0)] + nTM}{n} = 0; \\ \dfrac{-\mathrm{i}\ln[\tilde{G}(nT, 0, 2\pi n|0, 0, 0)] + nTM}{n} = 2\pi k; \\ n \to \infty. \end{cases} \tag{1.4}$$

The soliton Green function

$$\tilde{G}(t'', q'', \alpha''|t', q', \alpha') = \int_{t',\varphi'}^{t'',\varphi''} \exp\left\{\mathrm{i}\int_{t'}^{t''} \mathscr{L}(u)\,\mathrm{d}t\right\}\prod \mathrm{d}u \tag{1.5}$$

can be evaluated by the stationary phase method as described in paragraph 2.2,

$$\tilde{G}(t'', q'', \alpha''|t', q', \alpha') = \exp\left\{\frac{\mathrm{i}}{\gamma}\int_{t'}^{t''} \mathrm{d}^2x[\tfrac{1}{2}w_{,\mu}^2 - v(w)]\right\};$$

$$w = w\left(\frac{x - q - vt}{\sqrt{1 - v^2}}, \frac{t - vx}{T\sqrt{1 - v^2}} - \frac{\alpha}{2\pi}; T\right); \tag{1.6}$$

$$v = \frac{q'' - q'}{t'' - t'}; \qquad T = 2\pi\,\frac{\sqrt{(t'' - t')^2 - (q'' - q')^2}}{\alpha'' - \alpha'}, \qquad q = \frac{t'q'' - t''q'}{t'' - t'}.$$

It becomes clear that

$$\lim_{n\to\infty} \frac{-\mathrm{i}\ln G(nT, 0, 2\pi n|0, 0, 0)}{n} = \frac{1}{\gamma}\int_0^T \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x[\tfrac{1}{2}w_{,\mu}^2 - v(w)]\Big|_{v=0}. \tag{1.7}$$

Inserting this into (1.4) we have

$$\frac{1}{\gamma}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^T \mathrm{d}t\dot{w}^2\left(x, \frac{t}{T}, T\right) = 2\pi k; \tag{1.8}$$

$$M = \frac{1}{\gamma}\int_{-\infty}^{\infty} \mathrm{d}x\left[\tfrac{1}{2}\dot{w}^2\left(x, \frac{t}{T}, T\right) + \tfrac{1}{2}w'^2\left(x, \frac{t}{T}, T\right) + v\left(w\left(x, \frac{t}{T}, T\right)\right)\right].$$

Note that (1.8) can be regarded as a direct generalization of the Bohr–Sommerfeld quantization rule. In the sin φ_2 model the classical solution is given in (1.1.28) and the expression (1.7) looks so:

$$\lim_{n\to\infty} \frac{-\mathrm{i}\ln \tilde{G}(nT, 0, 2\pi n|0, 0, 0)}{n} = \frac{32\pi}{\gamma}(\theta - \tan\theta); \tag{1.9}$$

the expressions (1.8) become [9]

$$M_k = 2M_s \sin\theta_k, \qquad \theta_k = k\gamma/16, \qquad M_s = 8m/\gamma. \tag{1.10}$$

4.2. The scattering matrix for solitons

Consider the scattering process of two (in general different) solitons. Let the classical solutions corresponding to them be

$$u_{1s}((x - q_1)\cosh\varphi_1 - t\sinh\varphi_1), \qquad u_{2s}((x - q_2)\cosh\varphi_2 - t\sinh\varphi_2). \tag{2.1}$$

In the semiclassical approximation the only classically permissible processes have non-zero amplitudes. Let the classical solitons pass through one another at some rapidity $\psi_1 - \psi_2$. The classical solution

$$u_{ss}(\tanh\psi_1, \tanh\psi_2, q_1, q_2), \qquad \psi_1 > 0, \psi_2 < 0 \tag{2.2}$$

has the following asymptotics:

$$u_{ss}(\tanh\psi_1, \tanh\psi_2, q_1, q_2) \xrightarrow[t\to-\infty]{} u_{1s}((x - q_1)\cosh\psi_1 - t\sinh\psi_1) + u_{2s}((x - q_2)\cosh\psi_2 - t\sinh\psi_1); \tag{2.3}$$

$$u_{ss}(\tanh\psi_1, \tanh\psi_2, q_1, q_2) \xrightarrow[t\to\infty]{} u_{1s}\left((x-q_1)\cosh\psi_1 - t\sinh\psi_1 - \frac{\Delta(\psi_1-\psi_2)}{M_1}\right) + u_{2s}\left((x-q_2)\cosh\psi_2 - t\sinh\psi_1 + \frac{\Delta(\psi_1-\psi_2)}{M_2}\right).$$

In the classical limit the collision preserves the solitons' individual momenta but produces the complementary coordinates shift, compared to the solitons' uniform motion,

$$\Delta q_1 = \frac{1}{M_1\cosh\psi_1}\Delta(\psi_1-\psi_2); \qquad \Delta q_2 = -\frac{1}{M_2\cosh\psi_2}\Delta(\psi_1-\psi_2). \tag{2.4}$$

The shift is defined uniquely by the Lorentz invariance principle. The solution (2.2) will be used for the semiclassical S-matrix calculation.

For the reduced S-matrix we have (3.2.7)

$$\hat{S}_{-1}(\varphi_1'-\varphi_2') = \lim_{\substack{t''\to\infty\\ t'\to-\infty}} \frac{\exp\{\mathrm{i}\int_{t'}^{t''}\mathscr{L}(u_{ss})\,\mathrm{d}t\}}{G(t'',q_2''|t',q_2')G(t'',q_1''|t',q_1')}, \tag{2.5}$$

$$G(t'',q''|t',q') = \exp\{-\mathrm{i}M^{\mathrm{cl}}\sqrt{(t''-t')^2-(q''-q')^2}\},$$

$$q_1''-q_1' = \tanh\varphi_1'(t''-t'); \qquad q_2''-q_2' = \tanh\varphi_2'(t''-t').$$

The exponent of (2.5) contains the two-soliton solution (2.2) that describes the propagation of the first (second) soliton from the point q_1', q_2' at t' to the point q_1'', q_2'' at t''.

In the other words, the exponent contains the solution with velocities

$$\tanh\psi_{1,2} = \tanh\varphi_{1,2} \mp \frac{1}{t''-t'}\cdot\frac{\Delta(\varphi_1-\varphi_2)}{M_{1,2}\cosh\varphi_{1,2}}. \tag{2.6}$$

Finally we obtain for (2.5):

$$\hat{S}_{-1}(\varphi_1-\varphi_2) = \lim_{\substack{t''\to\infty\\ t'\to-\infty}} \exp\left\{\mathrm{i}\int_{t'}^{t''}[\mathscr{L}(u_{ss}(\tanh\psi_1,\tanh\psi_2)) - \mathscr{L}(u_{1s}(\tanh\varphi_1)) - \mathscr{L}(u_{2s}(\tanh\varphi_2))]\,\mathrm{d}t\right\} = \exp\{-\mathrm{i}K(p_1,p_2)\}; \tag{2.7}$$

$$q_{1,2}'' = v(p_{1,2})t'', \qquad q_{1,2}' = v(p_{1,2})t'.$$

The function K introduced here is the generating function of the canonical transformation that describes the classical scattering. Indeed, one can see that

$$\frac{\partial}{\partial p_{1,2}}K(p_1,p_2) = \Delta q_{1,2}, \qquad p_{1,2} = M_{1,2}\sinh\varphi_{1,2}, \tag{2.8}$$

with $\Delta q_{1,2}$ introduced in (2.4).

The equalities (2.7) and (2.8) have the general meaning. The relation between the semiclassical S-matrix and the canonical transformations' generating function is valid in every Hamiltonian system for the scattering with the conservation of all momenta. We shall exemplify it by a one-dimensional particle in a potential.

The exponent of (2.7) contains the difference between the action calculated on the interacting particles and the action calculated on the free ones that are situated in both cases in the same points $q''_{1,2}$ and $q'_{1,2}$ at t'' and t' moments respectively

$$K(p_1, p_2) = S_0(q''_{1,2}, t''|q'_{1,2}, t') - S(q''_{1,2}, t''|q'_{1,2}, t') \tag{2.9}$$

with $p_{1,2}$ the asymptotical momentum. To prove equality (2.8) we need the derivative of K in the following form

$$\frac{\mathrm{d}}{\mathrm{d}p} K = \frac{\mathrm{d}v}{\mathrm{d}p}\frac{\mathrm{d}}{\mathrm{d}v} K. \tag{2.10}$$

Exploiting the expression for the asymptotical velocity

$$v = (q'' - q')/(t'' - t'), \tag{2.11}$$

substituting the expression (2.9) for K into the right-hand side of (2.10) and using the equality

$$\frac{\mathrm{d}}{\mathrm{d}q''} S = p'' \tag{2.12}$$

we obtain

$$\frac{\mathrm{d}K}{\mathrm{d}p} = -\frac{\mathrm{d}v}{\mathrm{d}p}\cdot(t'' - t')(p_{\mathrm{int}} - p_0). \tag{2.13}$$

Here the p_{int} and p_0 are the momenta of the interacting and of the free particle respectively. Their difference is connected to Δq in such a way

$$p_{\mathrm{int}} - p_0 = \frac{\mathrm{d}p}{\mathrm{d}v}(v_{\mathrm{int}} - v_0) = -\frac{\mathrm{d}p}{\mathrm{d}v}\cdot\frac{\Delta q}{(t'' - t')}. \tag{2.14}$$

And finally we obtain (2.8),

$$\mathrm{d}K(p)/\mathrm{d}p = \Delta q \tag{2.15}$$

Q.E.D. [29, 89].

We have proved the elastic scattering semiclassical matrix phase to be the generating function of the canonical transformation. We shall use this below for calculation of the periodic solitons' S-matrix.

We rewrite finally (2.5) and (2.7) as

$$\hat{S}_{-1}(\varphi_1 - \varphi_2) = c\exp\left\{-\mathrm{i}\int_0^{\varphi_1-\varphi_2} \Delta(\beta)\,\mathrm{d}\beta\right\}. \tag{2.16}$$

For the $\Delta(\beta)$ see (2.4). The constant c is obtained by the limit evaluation in (2.7).

The $\sin\varphi_2$ model solution that describes the soliton–antisoliton scattering had been written down in (1.1.35). The $\Delta(\beta)$ can be extracted from that formula (1.1.37),

$$\Delta(\varphi_+ - \varphi_-) = \frac{16}{\gamma} \ln \coth\left(\frac{\varphi_+ - \varphi_-}{2}\right); \qquad \varphi_+ > \varphi_-. \tag{2.17}$$

The S-matrix calculation by (2.7) and (2.16) gives [29, 13, 14, 89]

$$\hat{S}_-(\varphi_+ - \varphi_-) = \exp\left\{ i\,\frac{8\pi^2}{\gamma} + \frac{8}{\gamma}\int_0^\pi d\theta \ln\left(\frac{\xi e^{-i\theta} + 1}{\xi + e^{-i\theta}}\right)\right\}; \tag{2.18}$$

$$\xi = e^{\varphi_+ - \varphi_-} = \frac{s - 2M^2 + \sqrt{s(s - 4M^2)}}{2M^2}, \qquad s = (p_1^0 + p_2^0)^2 - (p_1 + p_2)^2,$$

$$K(\xi) = \frac{8i}{\gamma}\int_0^\pi d\theta \ln\left(\frac{\xi e^{-i\theta} + 1}{\xi + e^{-i\theta}}\right). \tag{2.19}$$

The scattering of two identical solitons in the $\sin\varphi_2$ model is treated in a similar manner and supplies the following form of the reduced S-matrix expression:

$$\hat{S}_+(\varphi_1 - \varphi_2) = \exp\left\{\frac{8}{\gamma}\int_0^\pi d\theta \ln\left(\frac{\xi e^{-i\theta} + 1}{\xi + e^{-i\theta}}\right)\right\}; \qquad \xi = e^{\varphi_1 - \varphi_2} > 1. \tag{2.20}$$

For the corresponding classical solutions see (1.1.32).

The derived S-matrices obey the crossing-invariance principle

$$S_+(4M^2 - s + i\theta) = \bar{S}_-(s + i\theta). \tag{2.21}$$

This property was first noted by Coleman [89].

Let us attend now to the scattering problem of simple and periodic solitons in the $\sin\varphi_2$ model. It is simplified by exploiting the statement on the generating function of the classical scattering. The first pair of canonically conjugated variables for a periodic soliton is q and p, the coordinate and the Lorentz momentum. The second pair is α and I, the phase of the periodic soliton α and the reduced action

$$I = \frac{1}{2\pi}\int_0^T dt \int_{-\infty}^{\infty} dx\, w_t^2\left(x, \frac{t}{T}, T\right); \tag{2.22}$$

within the $\sin\varphi_2$ model $I = 16\theta/\gamma$. The shifts of the soliton's center and of its internal angular variable have been listed above (1.1.38). We have seen the canonical transformation generating function to be

$$F = K(i\xi\, e^{-i\theta}) + K(-i\xi\, e^{i\theta}) - \frac{8\pi^2}{\gamma}; \qquad \xi = e^{\varphi_s - \varphi_w} > 1. \tag{2.23}$$

The final S-matrix expression is [14]

$$\langle v''_w, v''_s, n'' | \hat{S} | v'_w, v'_s, n' \rangle = \exp\left\{ i\,\frac{8\pi^2}{\gamma} - iK(ie^{-i\theta_{n'}}\xi) - iK(-ie^{i\theta_{n'}}\xi)\right\} \delta(\varphi'_s - \varphi_s)\delta(\varphi'_w - \varphi_w)\delta^{n}_{n'}; \tag{2.24}$$

$$\theta_n = \frac{\gamma}{16} n, \qquad \xi = e^{\varphi_s' - \varphi_w'} > 1, \qquad v = \tanh \varphi;$$

$$\xi = \frac{s - M_s^2 - M_n^2}{2M_s M_n} + \sqrt{\left(\frac{s - M_s^2 - M_n^2}{2M_s M_n}\right)^2 - 1}; \qquad M_n = 2M_s \sin \theta_n,$$

where s is the Mandelstam variable

$$s = (p_1^0 + p_2^0)^2 - (p_1 + p_2)^2 = M_s^2 + M_n^2 + 2MM_n \cosh(\varphi_s - \varphi_w). \tag{2.25}$$

The scattering amplitude of two periodic solitons with velocities $\tanh \varphi_1$ and $\tanh \varphi_2$ and quantum numbers n_1 and n_2 can be obtained in the same method [14], (1.1.42)

$$\begin{aligned} \langle v_2'', n_2'', v_1'', n_1'' | S | v_2', n_2', v_1', n_1' \rangle &= \frac{\delta^{n_1'' n_2'} \delta^{n_2'' n_2'}}{M^2} \delta(\varphi_2'' - \varphi_2') \, \delta(\varphi_1'' - \varphi_1') \\ &\times \exp\{-i[K(\xi \exp\{i(\theta_{n_1'} - \theta_{n_2'})\}) + K(-\xi \exp\{-i(\theta_{n_1'} + \theta_{n_2'})\}) \\ &+ K(-\xi \exp\{i(\theta_{n_1'} + \theta_{n_2'})\}) + K(\xi \exp\{i(\theta_{n_2'} - \theta_{n_1'})\})]\} \exp\left\{i \frac{16\pi^2}{\gamma}\right\}; \end{aligned} \tag{2.26}$$

$$\xi = e^{\varphi_1 - \varphi_2} > 1, \qquad \theta_n = \frac{\gamma}{16} n.$$

The generating function for N solitons scattering is equal to the sum of pair solitons generating functions (1.1.43), (2.18), (2.20), (2.24), (2.26). So the N soliton quasiclassical S-matrix is equal to the product of pair soliton S-matrices [14].

It is worth mentioning that the perturbation theory for the soliton S-matrix is quite different from that of the usual particles. The soliton S-matrix is unitary in every order of the perturbations theory, but its analyticity is restored only by summing of all the orders of the expansion.

4.3. *The scattering of a usual particle on a soliton*

Let us consider the scattering of a usual particle on a soliton, using the definition of section 2. The definition of the S-matrix generating functional for a single soliton looks so:

$$\langle v_s'', A_\beta^+ | S | v_s', A_\beta \rangle = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \frac{G(t'', q'', A_\beta^+ | t', q', A_\beta)}{2\pi G(t'', q' | 0, 0) G(0, 0 | t', q')}; \tag{3.1}$$

and after a transformation similar to (3.1.21) it becomes

$$\begin{aligned} \langle v_s'', A_\beta^+ | S | v_s', A_\beta \rangle &= \frac{1}{M} G^{-1}(t'', \tanh \varphi'' \cdot t'' | t', \tanh \varphi' \cdot t') \delta(\varphi'' - \varphi') \\ &\times \int \exp\left\{\frac{1}{4} \int d\beta [a_\beta^+ a_{-\beta}^+ - a_\beta a_{-\beta}]\Big|_{t'}^{t''} + \frac{1}{2} \int d\beta [a_\beta^+ a_\beta|_{t''} + a_\beta^+ a_\beta|_{t'}] \right. \\ &\left. + i \int_{t'}^{t''} \left[\pi u_t - \left(\frac{\gamma}{2}\pi^2 + \frac{1}{2\gamma} u_x^2 + \frac{1}{\gamma} v(u)\right)\right]\right\} \prod_{x,t} du(x,t) \, d\pi(x,t). \end{aligned} \tag{3.2}$$

We represent the integration variables in the form

$$\begin{cases} u = u_s(x\cosh\varphi' - t\sinh\varphi') + \sqrt{\gamma}\tilde{u}; \\ \pi = \frac{1}{\gamma}\dot{u}_s(x\cosh\varphi' - t\sinh\varphi') + \frac{1}{\sqrt{\gamma}}\tilde{\pi}; \end{cases} \tag{3.3}$$

$$\begin{cases} \tilde{u} = \frac{1}{\sqrt{4\pi}}\int d\beta[a_\beta^+\exp\{-im\sinh\beta\cdot x\} + a_\beta\exp\{im\sinh\beta\cdot x\}]; \\ \tilde{\pi} = \frac{im}{\sqrt{4\pi}}\int d\beta\cosh\beta[a_\beta^+\exp\{-im\sinh\beta x\} - a_\beta\exp\{im\sinh\beta x\}]. \end{cases} \tag{3.4}$$

Then (3.2) can be rewritten as

$$\langle v_s'', A_\beta^+|S|v_s', A_\beta\rangle = \frac{1}{M}\delta(\varphi'' - \varphi')\int\exp\left\{\frac{1}{4}\int d\beta[a_\beta^+ a_{-\beta}^+ - a_\beta a_{-\beta}]\Big|_{t'}^{t''}\right. \tag{3.5}$$

$$\left. + \tfrac{1}{2}\int d\beta[a_\beta^+ a_\beta|_{t''} + a_\beta^+ a_\beta|_{t'}] + i\int_{t'}^{t''}[\tilde{\pi}\tilde{u}_t - \tfrac{1}{2}(\tilde{\pi}^2 + \tilde{u}_x^2 + v''(u_s)\tilde{u}^2) - \sum_{n=3}^{\infty}\gamma^{n/2-1}v^{(n)}(u_s)(\tilde{u})^n]\,d^2x\right\}d\tilde{\pi}\,d\tilde{u}.$$

Here u_s is a solution from (3.3). In the first order of approximation we drop the last sum in the exponent so (3.5) takes the form

$$\langle v_s'', A_\beta^+|S|v_s', A_\beta\rangle = \frac{1}{M}\delta(\varphi'' - \varphi')\int\prod_{\beta,t} da_\beta^+(t)\,da_\beta(t)$$

$$\times\exp\left\{\tfrac{1}{2}\int d\beta[a_\beta^+ a_\beta|_{t''} + a_\beta^+ a_\beta|_{t'}] + i\int_{t'}^{t''} dt\,d\beta\left[\frac{\dot{a}_\beta^+ a_\beta - a_\beta^+\dot{a}_\beta}{2i} - \tfrac{1}{2}(\tilde{\pi}^2 + \tilde{u}_x^2 + v''(u_s)\tilde{u}^2)\right]\right\}. \tag{3.6}$$

At first we shall derive the forward scattering amplitude. Remember that the integration variables at moments t'' and t' have values

$$\begin{aligned} a_\beta^+|_{t''} &= A_\beta^+\exp\{im\cosh\beta t''\}, \\ a_\beta|_{t'} &= A_\beta\exp\{-im\cosh\beta t'\}. \end{aligned} \tag{3.7}$$

The integral in (3.6) is of Gaussian type and the stationary phase evaluation provides its exact value. Let A_β^+ and A_β be localized in the region $\beta > \varphi'$. The stationary phase point turns the expression

$$\int_{t'}^{t''} d\beta\,dt\left\{\frac{\dot{a}_\beta^+ a_\beta - a_\beta^+\dot{a}_\beta}{2i} - \tfrac{1}{2}(\tilde{\pi}^2 + \tilde{u}_x^2 + v''(u_s)\tilde{u}^2)\right\} \tag{3.8}$$

to zero, and only the value of

$$\int d\beta(a_\beta^+ a_\beta|_{t''} + a_\beta^+ a_\beta|_{t'}) \tag{3.9}$$

is left to be calculated. Mention that

$$\begin{cases} a_\beta(t'') = s(\beta - \varphi)A_\beta \exp\{-imt'' \cosh \beta\}; \\ a_\beta^+(t') = s(\beta - \varphi)A_\beta^+ \exp\{imt' \cosh \beta\}, \end{cases} \tag{3.10}$$

where $s(\beta)$ is the penetration factor of the plane wave through the potential in the Schroedinger equation (see Appendix 4)

$$\left[-\frac{d^2}{dr^2} + v''(u_s(r))\right]\tilde{f}_\beta(r) = m^2 \cosh^2\beta \tilde{f}_\beta(r). \tag{3.11}$$

The representation (3.10) follows from the properties of the asymptotic $\tilde{f}_\beta(r)$, $\tilde{g}_\beta(r)$ (A.4.9) and from the localization of $\tilde{u}$ and $\tilde{\pi}$ in the left (right)-hand side of the soliton at $u_s(r)$. Finally we obtain for the amplitude (3.6):

$$\langle v'', A_\beta^+ | S | v', A_\beta \rangle = \frac{1}{M}\delta(\varphi'' - \varphi')\exp\left\{\int_{-\infty}^{\infty} d\beta A_\beta^+ A_\beta s(|\beta - \varphi|)\right\}. \tag{3.12}$$

Adding the similar treatment of the particle's reflection we come to a complete result for the S-matrix generating functional:

$$\begin{aligned} \langle v'', A_\beta^+ | S | v', A_\beta \rangle = \frac{1}{M}\delta(\varphi'' - \varphi')\exp\Bigg\{ & \int_{-\infty}^{\infty} d\beta A_\beta^+ A_\beta s(|\beta - \varphi|) \\ & + \int_{\varphi}^{\infty} d\beta r_{\beta-\varphi} A_{2\varphi-\beta}^+ A_\beta - \int_{-\infty}^{\varphi} d\beta \frac{\overline{r(\beta - \varphi)}}{\overline{s(\beta - \varphi)}} s(\beta - \varphi) A_{2\varphi-\beta}^+ A_\beta \Bigg\}. \end{aligned} \tag{3.13}$$

Here $r(\beta)$ is the reflection factor derived in the Appendix 4. In the sin φ_2 model [14]

$$s(\beta) = \frac{\sinh\beta + i}{\sinh\beta - i}, \qquad r = 0. \tag{3.14}$$

Thus in the lowest order approximation in any model the usual particles do not interact with one another and are scattered by a soliton like by a non-relativistic potential.

The scattering of a usual particle on a periodic soliton is treated analogously. To describe the S-matrix formula in the lowest order approximation we must display the solutions of homogeneous equation (see Appendix 4)

$$\begin{aligned} &H\psi = 0, \\ &H = \Box + v''\left(w\left(x, \frac{t}{T}, T\right)\right). \end{aligned} \tag{3.15}$$

In this case it is easier to describe the formula which is analogous to (3.13) than to write it down.

The Lorentz and the internal momenta of the soliton are conserved during the collision. The usual particles do not interact with one another. The S-matrix of the usual particle–soliton scattering is equivalent to that of the plane wave-potential scattering in the equation (3.15). We define

ψ_β^-, which describe the scattering as (A.4.18)

$$\psi_\beta^-(x,t)\to\begin{cases}\exp\{-im(t\cosh\beta-x\sinh\beta)\}+\sum_n r_+^f(n,\beta)\exp\{-im(t\cosh\beta_n+x\sinh\beta_n)\}, & x\to-\infty;\\ \sum_n d_+^f(n,\beta)\exp\{-im(t\cosh\beta_n-x\sinh\beta_n)\}, & x\to\infty;\\ \cosh\beta_n=\cosh\beta+2\pi n/mT. & \end{cases}\tag{3.16}$$

To learn the fundamental particle–soliton scattering means to find out all the factors d_n, r_n; $d_0(\beta)$ is the transmission coefficient without change of the energy; $d_n(\beta)$ is the transmission coefficient with the energy change by $2\pi n/T$; $r_n(\beta)$ are the analogous reflection coefficients. The scattering of a fundamental particle on a soliton that moves with the velocity v is obtained by changing x to r in (3.16).

Let us lay the following requirements on a plane wave-periodic soliton scattering matrix: an incoming negative (positive)-frequency wave must become again the negative (positive)-frequency wave after the collision. This will serve as a criterion of the theory stability. In other words,

$$d_n(\beta)\neq 0,\qquad r_n(\beta)\neq 0\tag{3.17}$$

must be only if $m\cosh\beta+2\pi n/T>m$.

Consider the fundamental particle-periodic soliton scattering (1.1.28) in the $\sin\varphi_2$ model. The equation (3.15) is solvable directly (A.4.23). It can be extracted from the formulae that describe the scattering of two periodic solitons, see Appendix 4, and it leads to (1.10):

$$r_n=0,\qquad d_n=0,\qquad n\neq 0,$$
$$d_0(\beta)=\left(\frac{\sinh\beta+\mathrm{i}\sin(k\gamma/16)}{\sinh\beta-\mathrm{i}\sin(k\gamma/16)}\right)^2.\tag{3.18}$$

4.4. The above-barrier soliton reflection

In general, the quantum theory affirms a non-zero probability of some processes that are forbidden classically. This paragraph displays a method of the amplitudes calculation for the processes that can not occur to the classical solitons.

The corresponding amplitudes are evidently exponentially small in γ. At the first sight the perturbative attempts seem hopeless, but sometimes one may succeed in playing the trick that is known in the quantum mechanics and is based on the use of the solutions of the classical equations in the complex time plane [90–93]. Consider the following example in the $\sin\varphi_2$ model. The classical scattering of the soliton

$$4\tan^{-1}\exp\{(x-q_+)\cosh\varphi_+-t\sinh\varphi_+\}\tag{4.1}$$

on the antisoliton

$$-4\tan^{-1}\exp\{(x-q_-)\cosh\varphi-t\sinh\varphi_-\}\tag{4.2}$$

occurs without any reflection. Indeed, the classical solution representing this scattering (1.1.35) looks especially simple in the center-of-mass system

$$u_{ss}(\tanh\varphi_1-\tanh\varphi)=4\tan^{-1}\left\{\coth\varphi\,\frac{\sinh[m\sinh\varphi t]}{\cosh[m\cosh\varphi\cdot x]}\right\}\tag{4.3}$$

and describes the scattering with the individual momenta conservation which obviously means the absence of reflection. But in the complex t-plane there is a solution that describes the reflection.

This classical solution has the following form. The time variable t in the function (4.3) must move along the contour (fig. 1).

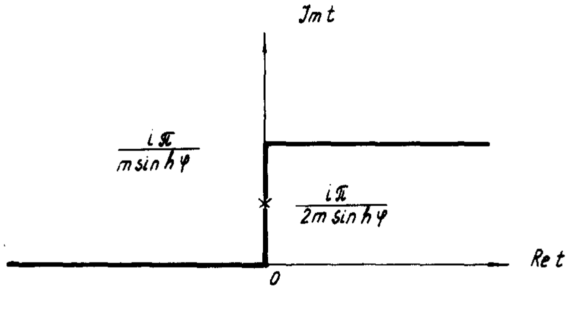


Fig. 1.

Moreover, there are two solutions of this kind, they are obtained by moving t along the contours C_1 and C_2 (fig. 2). Note that the time variable t in fig. 2 tends to infinity parallel to the real axis. The crosses denote the turning points.

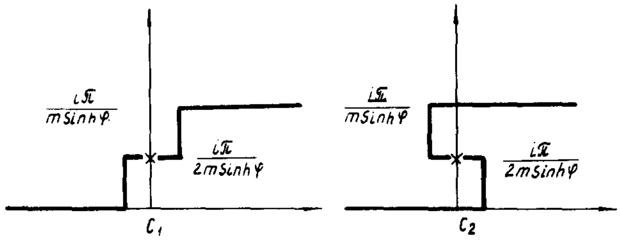


Fig. 2.

The following two remarks will show that these complex solutions can be exploited in the usual procedure as the stationary phase points of the functional integral which represents the reflection amplitude.

We write down first the reflection coefficient definition in the following form

$$\langle v''_+, v''_- | S | v'_+, v'_- \rangle = \lim_{\substack{t'' \to +\infty \\ t' \to -\infty}} \frac{G(t'', q''_+, q''_- | t', q'_+, q'_-)}{(2\pi)^2 G(t'', q''_+ | 0, 0) G(t', q''_- | 0, 0) G(0, 0 | t', q'_+) G(0, 0 | t', q'_-)}; \tag{4.4}$$

$$q''_+ = \tanh \varphi''_+ \cdot t'' + q^{0''}_+, \qquad q'_+ = \tanh \varphi'_+ \cdot t' + q^{0'}_+,$$

$$q''_- = \tanh \varphi''_- \cdot t'' + q^{0''}_-, \qquad q'_- = \tanh \varphi'_- \cdot t' + q^{0'}_-;$$

$$\varphi'_+ > 0, \qquad \varphi'_- < 0, \qquad \varphi''_+ < 0, \qquad \varphi''_- > 0.$$

1. The general definition makes possible to tend t'' to infinity not only along the real axis but also as

$$t'' \to \infty + ia. \tag{4.5}$$

2. The nominator of (4.4) is expressed by the functional integral

$$G(t'', q''_+, q''_- | t', q'_+, q'_-) = \int \exp\left\{ i \int_{t'}^{t''} \mathscr{L}(u)\, dt \right\} \prod_{x,t} du(x,t). \tag{4.6}$$

Indeed it is proved in [94] that the right-hand side of (4.6) does not depend on the form of the complex plane contour of the variable t which is the integration variable $u(x, t)$ index.

These facts authorize the calculation of the reduced S-matrix by the technique of paragraph 4.2. Our case has two points of distinction from paragraph 4.2. The first is the existence of two classical trajectories, the second is the presence of a turning point on both trajectories. Taking this into account we obtain the propagator G in the semiclassical approximation

$$G(t'', q''_+, q''_- | t', q'_+, q'_-) = -i \exp\left\{ i \int_{t'}^{t''} \mathscr{L}(u_{c_1})\, dt \right\} + i \exp\left\{ i \int_{t'}^{t''} \mathscr{L}(u_{c_2})\, dt \right\}. \tag{4.7}$$

Here $-i$ and i are the turning points contributions. The calculations similar to those in paragraph 4.2 lead us to the following reflection coefficient value [95]:

$$\langle v''_+, v''_- | S | v'_+, v'_- \rangle = \frac{1}{M^2} \delta(\varphi''_+ - \varphi'_-) \delta(\varphi''_- - \varphi'_+) \left(-2 \sin \frac{8\pi^2}{\gamma} \right) \exp\left\{ -\frac{8\pi}{\gamma} |\varphi_+ - \varphi_-| \right\}$$
$$\times \exp\left\{ i \frac{8\pi^2}{\gamma} + \frac{8}{\gamma} \int_0^{\pi} d\theta \ln \frac{\xi\, e^{-i\theta} + 1}{\xi + e^{-i\theta}} \right\}; \qquad \xi = \exp\{\varphi'_+ - \varphi'_-\} \tag{4.8}$$

which is indeed exponentially small in γ. Note that at $\gamma = 8\pi/N$ the reflection vanishes.

The soliton–antisoliton S-matrix (2.18), (4.8) in the nonrelativistic region makes possible the approximate reconstruction of these particles interaction potential. The appropriate Schroedinger equation is [95]:

$$i \frac{\partial \psi}{\partial t} = \left[-\frac{1}{M} \frac{d^2}{dR^2} - \frac{1}{M} \frac{d^2}{dr^2} - \frac{\pi^2}{4} \frac{M}{\cosh^2(\frac{1}{2}\pi \cdot Mr/N)} \right] \psi; \qquad N = \frac{8\pi}{\gamma}. \tag{4.9}$$

4.5. The ground state of the double soliton in sin φ_2 *model*

Consider the mass formula for the double soliton in the sin φ_2 model (1.10):

$$M_n = \frac{16m}{\gamma} \sin\left(\frac{n\gamma}{16} \right). \tag{5.1}$$

(We will prove in section 5 that $\gamma \to \gamma' = \gamma/(1 - \gamma/8\pi)$ [9] in the one-loop approximation.) Note that in the limit of small γ the double soliton ground state mass is identical to the usual particle mass:

$$M_1 \xrightarrow[\gamma \to 0]{} m. \tag{5.2}$$

These particles were conjectured in [9] to be the same particle. We shall support this hypothesis by the following arguments.

Consider the S-matrix of the two lowest periodic soliton's states (2.26). Retaining only the first order term of expansion, we get

$$S(s) \cong 1 + \frac{\mathrm{i}m^2\gamma}{2\sqrt{s(s-4m^2)}}, \tag{5.3}$$

what is identical to the first order term in the expansion of the S-matrix of the fundamental particles [96]. The S-matrix for the scattering of a fundamental particle on a lowest-state periodic soliton turns to (5.3) also.

Compare the matrix of the fundamental particle–soliton scattering (3.14) and the scattering that for the lowest-state periodic soliton on a simple soliton (2.24). Substituting the value $\theta_1 = \gamma/16$ and expanding the exponent in powers of γ we obtain

$$S(s) \cong \left\{\sqrt{m^2 - \left(\frac{s - m^2 - M^2}{2M}\right)^2} + m\right\} \Big/ \left\{\sqrt{m^2 - \left(\frac{s - m^2 - M^2}{2M}\right)^2} - m\right\} \tag{5.4}$$

identical to (3.14).

At last we shall mention that the nth "excited" soliton state was found to be identical to a bound state of n usual particles [9]. An analogous degeneracy that identifies the usual particle with the lowest periodic soliton state was also found in the "nonlinear Schroedinger" quantum field theory [97].

5. The quantum corrections

In this section we shall evaluate the quantum corrections to the semiclassical results derived in section 4. For the propagator of several solitons

$$G(t'', \{q_j''\}|t', \{q_i'\}) = \int \exp\left\{\mathrm{i}\int_{t'}^{t''} \mathscr{L}(u_{N_s} + \sqrt{\gamma}\varphi)\,\mathrm{d}t\right\} \prod \mathrm{d}\varphi \tag{0.1}$$

we shall use the following approximation

$$G(t'', \{q_j''\}|t', \{q_i'\}) = \det{}^{-1/2} H H_0^{-1} \exp\left\{\mathrm{i}\int_{t'}^{t''} \mathscr{L}(u_{N_s})\,\mathrm{d}t\right\}; \tag{0.2}$$

$$H = \Box + v(x,t), \qquad H_0 = \Box + m^2, \qquad m^2 = v''(0), \tag{0.3}$$

$$v(x,t) = v''(u_{N_s}).$$

This expression accounts the next perturbative term compared to the expression (4.0.2). We see that the one-loop corrections calculations require to evaluate a determinant of a Klein–Gordon operator with a potential.

The determinant $\det H$ must be expressible through the scattering matrix of a plane wave on the potential in the equation

$$H\psi_\beta = 0. \tag{0.4}$$

Indeed, the one-loop corrections arise due to the usual particles exchange, and the usual particle–soliton scattering is described by the solutions of (0.4). This idea was found to be realizable. The authors had worked out a method for the calculation of these determinants [25, 98]. The formulae written below express the derivatives of the $\ln\det H$ through the asymptotics of the solutions of the equation (0.4) at $|t| \to \infty$.

In the following the family of Jost solutions ψ_β^+ and ψ_β^- will be useful; they are uniquely defined by the requirements:

$$\psi_\beta^-(x,t) \xrightarrow[x\to-\infty]{t\to\infty} \exp\{-im(t\cosh\beta - x\sinh\beta)\}; \qquad \tilde{\psi}_\beta^+ \underset{x\to\infty}{\overset{t\to\infty}{=}} \exp\{im(t\cosh\beta - x\sinh\beta)\};$$
$$\psi_\beta^+(x,t) \xrightarrow[x\to+\infty]{t\to-\infty} \exp\{im(t\cosh\beta - x\sinh\beta)\}; \qquad \tilde{\psi}_\beta^- \xrightarrow[x\to-\infty]{t\to-\infty} \exp\{-im(t\cosh\beta - x\sinh\beta)\}. \tag{0.5}$$

The expression of $\det HH_0^{-1}$ will contain the quantities a_n, defined by

$$\psi_\beta^+(x,t) \xrightarrow[x\to-\infty]{t\to-\infty} \sum_n a_n(\beta)\exp\{im(t\cosh\beta_n - x\sinh\beta_n)\}. \tag{0.6}$$

The second important component of the $\det HH_0^{-1}$ will be obtained from the asymptotics of $\psi_\beta^-(x,t)$ at $t\to-\infty$:

$$\psi_\beta^-(x,t) \xrightarrow[x\to-\infty]{t\to-\infty} \exp\{-im(t\cosh\beta - x\sinh\beta)\} + \int c_{\beta\gamma}\exp\{im(t\cosh\gamma - x\sinh\gamma)\}\,d\gamma. \tag{0.7}$$

In the determinants evaluation we shall use the conventional definitions $m^2 \to m^2 - i0$. The ultraviolet divergence of $\det^{-1/2} HH_0^{-1}$ can be eliminated by the counter-term (3.5.5)

$$\Delta L = \frac{\gamma}{8\pi^2} D(0)\int d^2x v''(u) - m^2, \qquad D(0) = \int \frac{d^2k}{k^2+m^2}. \tag{0.8}$$

In paragraph 5.1 the general formula for $\det HH_0^{-1}$ is derived, in paragraph 5.2 the one-loop correction for the structureless soliton mass is evaluated, paragraph 5.3 contains the calculation of the periodic soliton mass correction. In paragraph 5.4 we calculate the one-loop correction to the S-matrix of two structureless solitons. All the general formulae will be illustrated by the $\sin\varphi_2$ model. In paragraph 5.5 one can find the final results for the $\sin\varphi_2$ model.

5.1. The one-loop corrections

We have to evaluate the determinant of the operator H presented above in (0.3). Note, that the potential $v''(u_{Ns})$ does not decrease along the classical solitons world lines.

It is more suitable to examine the differential of $\ln\det H$ than the determinant itself. We have

$$d\,\mathrm{Tr}\ln HH_0^{-1} = \int R(x,t,|x,t)\,dv(x,t)\,dx\,dt, \tag{1.1}$$

where $R(x_2,t_2|x_1,t_1)$ is the kernel of the resolvent of the operator H; we can express it as a bilinear combination of the functions ψ_β^+ and ψ_β^- (A.5.3)

$$R(t_2|t_1) = \begin{cases} \hat{\psi}_-(t_2)W^{-1}\hat{\psi}_+^{\mathrm{T}}(t_1), & t_2 > t_1; \\ \hat{\psi}_+(t_2)(W^{\mathrm{T}})^{-1}\hat{\psi}_-^{\mathrm{T}}(t_2), & t_2 < t_1. \end{cases} \tag{1.2}$$

Here $\hat{\psi}_+(t)$, $\hat{\psi}_-(t)$ are the positive and negative-frequency solutions of the homogeneous equation. We use short matrix notations for them, considering the spatial coordinate x and the wave number β as the matrix indices.

The Wronskian

$$W = \left(\hat{\psi}_+^{\mathrm{T}} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \psi_-\right) \tag{1.3}$$

is described in Appendix 5 for the three cases to be considered below. (See also the end of this paragraph.) Substituting (1.2) into (1.1) we get

$$\mathrm{d} \ln \det H = \mathrm{Tr}_{t'}^{t''} \hat{\psi}_- W^{-1} \hat{\psi}_+^{\mathrm{T}} \, \mathrm{d}v; \qquad \mathrm{Tr}_{t'}^{t''} A = \int_{t'}^{t''} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x A(x, t|x, t). \tag{1.4}$$

In order to transform this expression we take the derivative of equation (0.4)

$$\mathrm{d}\hat{\psi}_+^{\mathrm{T}} H + \hat{\psi}_+^{\mathrm{T}} \, \mathrm{d}v = 0 \tag{1.5}$$

and substitute into (1.4). We get

$$\mathrm{d} \ln \det H = -\, \mathrm{Tr}_{t'}^{t''} \hat{\psi}_- W^{-1} \, \mathrm{d}\hat{\psi}_+^{\mathrm{T}} H. \tag{1.6}$$

Integrating this expression by parts, recalling equation (0.4), we obtain

$$\mathrm{d} \ln \det HH_0^{-1} = \mathrm{tr}\, W^{-1}\left(\mathrm{d}\hat{\psi}_+^{\mathrm{T}} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \hat{\psi}_-\right)\Bigg|_{t'}^{t''} - \mathrm{tr}\, W_0^{-1}\left(\mathrm{d}\hat{\psi}_{0+}^{\mathrm{T}} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \hat{\psi}_{0-}\right)\Bigg|_{t'}^{t''}. \tag{1.7}$$

Here tr means an integration over x but not over t and ψ_0 means the free equation $H_0\psi_0 = 0$ solutions. Another expression for this quantity can be obtained by interchanging $\hat{\psi}_+$ and $\hat{\psi}_-$:

$$\mathrm{d} \ln \det HH_0^{-1} = -\mathrm{tr}\, W^{-1}\left(\hat{\psi}_+^{\mathrm{T}} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \mathrm{d}\hat{\psi}_-\right)\Bigg|_{t'}^{t''} + \mathrm{tr}\, W_0^{-1}\left(\hat{\psi}_{0+}^{\mathrm{T}} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \mathrm{d}\psi_{0-}\right)\Bigg|_{t'}^{t''}. \tag{1.8}$$

These formulae express the differential $\mathrm{d} \ln \det HH_0^{-1}$ through the asymptotical characteristics of the homogeneous equation solutions $\hat{\psi}_+$ and $\hat{\psi}_-$, hence they provide all the one-loop corrections calculations. Other authors [10, 99–101] evaluate the $\ln \det HH_0^{-1}$ as a sum of the Floque indexes of a system enclosed in a box but we luckily can avoid these complications.

Note that $\ln \det HH_0^{-1}$ is translation-invariant, hence

$$\mathrm{tr}\, \hat{W}^{-1}\left(\left(\frac{\mathrm{d}}{\mathrm{d}x} \hat{\psi}_+^{\mathrm{T}}\right) \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \hat{\psi}_-\right) = 0 = \mathrm{tr}\left[\hat{\psi}_+^{\mathrm{T}} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}x} \hat{\psi}_-\right] W^{-1}. \tag{1.9}$$

As an example let us calculate $\det HH_0^{-1}$ when $v(x, t)$ in (0.3) decreases in all directions. In this case we have at $t \to +\infty$

$$\psi_\beta^-(x, t) = \exp\{-im(t \cosh \beta - x \sinh \beta)\} \tag{1.10}$$

and at $t \to -\infty$

$$\psi_\beta^-(x, t) = \exp\{-im(t \cosh \beta - x \sinh \beta)\} + \int \mathrm{d}\gamma c_{\beta\gamma} \exp\{-im(t \cosh \gamma - x \sinh \gamma)\},$$

where the operator $(1 + \hat{c})$ is unitary because the following Wronskian does not depend on time:

$$\int \bar{\psi}_{\pm\beta}(x, t) \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \psi_{\pm\gamma}(x, t)\, \mathrm{d}x = \pm 4\pi \mathrm{i}\delta(\beta - \gamma). \tag{1.11}$$

The Wronskian W is then

$$\int \psi_\gamma^- \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \psi_\beta^+ \,\mathrm{d}x = 4\pi\mathrm{i}(1 + c)_{\gamma\beta}.$$

It is clear that the formula (1.8) becomes

$$\mathrm{d} \ln \det HH_0^{-1} = \frac{-1}{4\pi\mathrm{i}} \int \mathrm{d}x\, \mathrm{d}\beta\, \mathrm{d}\gamma \psi_\gamma^+ (1 + \hat{c})_{\beta\gamma}^{-1} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \mathrm{d}\hat{c}_{\beta\delta} \bar{\psi}_\delta^+ . \tag{1.12}$$

Using (1.11) and taking the integral over x we find at last

$$\ln \det HH_0^{-1} = \ln \det (1 + \hat{c}). \tag{1.13}$$

This expression is a Lorentz scalar

$$\det (\Box + m^2 + v(x \cosh \varphi - t \sinh \varphi, t \cosh \varphi - x \sinh \varphi))H_0^{-1}; \tag{1.14}$$

it does not depend on φ.

Now at the end of the paragraph we present the Wronskians for the three most important cases, see Appendix 5.

For the structureless soliton (A.4.8):

$$\int \exp \{ -imt \cosh \gamma \} f_\gamma(x) \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \exp \{imt \cosh \beta\} g_\beta(x)\, \mathrm{d}x = 4\pi \mathrm{i} a(\gamma)\delta(\gamma - \beta). \tag{1.15}$$

For the two structureless solitons scattering (A.5.24):

$$\int \mathrm{d}x \psi_{+\beta}(x, t) \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \psi_{-\gamma}(x, t) = -4\pi\mathrm{i}(1 + c)_{\gamma\beta} a_1(\beta - \varphi_1) a_2(\beta - \varphi_2). \tag{1.16}$$

For the periodic soliton we write the complete set of solutions in such a way. We define the vector $\boldsymbol{u} = \{u_n\}$

$$u_n(t) = \exp\left\{ -\mathrm{i}\left(\frac{\nu + 2\pi n}{T}\right)t \right\}, \tag{1.17}$$

and the matrix solutions (A.4.18)

$$\hat{f}_\pm^\nu(x) \to \begin{cases} \mathrm{e}^{\pm \mathrm{i}\hat{k}x} + \mathrm{e}^{\mp \mathrm{i}\hat{k}x} R_{f\pm}(\nu); & x \to -\infty; \\ \mathrm{e}^{\pm \mathrm{i}\hat{k}x} D_{f\pm}(\nu); & x \to +\infty; \end{cases} \tag{1.18}$$

$$\hat{g}_\pm^\nu(x) \to \begin{cases} \mathrm{e}^{\pm \mathrm{i}\hat{k}x} D_{g\pm}(\nu); & x \to -\infty; \\ \mathrm{e}^{\pm \mathrm{i}\hat{k}x} + \mathrm{e}^{\mp \mathrm{i}\hat{k}x} R_{g\pm}(\nu); & x \to +\infty; \end{cases}$$

and write the Wronskian of the homogeneous equation solutions (A.4.15)

$$(\boldsymbol{u}_\nu(t)\hat{g}_\pm^\nu(x))^+, \qquad (\boldsymbol{u}_\nu(t)\hat{f}_\pm^\nu(x)). \tag{1.19}$$

The Wronskian is (A.5.14)

$$\int \mathrm{d}x(\hat{g}_{\pm}^{\nu_1 +}(x)\boldsymbol{u}_{\nu_1}(t))\frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t}(\boldsymbol{u}_{\nu_2}(t)\hat{f}_{\pm}^{\nu_2}(x)) = -4\pi\mathrm{i}T\hat{k}(\nu_1)D_{f\pm}(\nu_1)\delta(\nu_1-\nu_2). \tag{1.20}$$

5.2. The one-loop mass correction for the structureless soliton

We shall calculate a one-loop correction to the mass of the structureless soliton (4.1.1) with the propagator

$$\begin{gathered} G(t'', q''|t', q') = \det{}^{-1/2}HH_0^{-1}\exp\{-\mathrm{i}M^{\mathrm{cl}}\cdot\sqrt{1-v^2}\cdot(t''-t')\}, \\ H = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - \frac{\mathrm{d}^2}{\mathrm{d}r^2} + v''(u_s(r)), \qquad v = \tanh\varphi, \end{gathered} \tag{2.1}$$

considered in paragraph 3.1. Here $u_s(r)$ is the classical structureless soliton dependent on the variables

$$r = x\cosh\varphi - t\sinh\varphi, \qquad \tau = t\cosh\varphi - x\sinh\varphi.$$

We represent the $\det^{-1/2}HH_0^{-1}$ as

$$\det{}^{-1/2}HH_0^{-1} = \exp\{-\mathrm{i}\Delta M\sqrt{1-v^2}\cdot(t''-t')\} \tag{2.2}$$

with ΔM the one-loop mass correction. For calculation of $\det HH_0^{-1}$ we have the formula (1.7). We take the functions ψ_β^+ and ψ_γ^- from Appendix 4:

$$\psi_\beta^-(x,t) = \exp\{-\mathrm{i}m\cosh\beta\cdot\tau\}f_\beta(r), \qquad \psi_\beta^+(x,t) = \exp\{\mathrm{i}m\cosh\beta\cdot\tau\}g_\beta(r). \tag{2.3}$$

For the discrete spectrum we have the solutions

$$\varphi_n^- = \exp\{-\mathrm{i}\sqrt{m^2-E_n}\tau\}\varphi_n(r), \qquad \varphi_n^+ = \exp\{\mathrm{i}\sqrt{m^2-E_n}\tau\}\varphi_n(r), \tag{2.4}$$

and

$$\tau u_s'(r),\ u_s'(r).$$

Here g_β and f_β are the same as in (A.4.8). We take the derivative of $\ln\det H$ with respect to φ and write it in the form

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} = -\left(\tau\frac{\mathrm{d}}{\mathrm{d}r} + r\frac{\mathrm{d}}{\mathrm{d}\tau}\right), \tag{2.5}$$

and so far have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varphi}\ln\det HH_0^{-1} = {} & \frac{1}{4\pi\mathrm{i}}\int \mathrm{d}x\,\frac{\mathrm{d}\beta}{}\left[\frac{\psi_\beta^-\overleftrightarrow{\partial}_t\cdot\mathrm{d}\psi_\beta^+/\mathrm{d}\varphi}{a(\beta)} - \psi_{0\beta}^-\overleftrightarrow{\partial}_t\frac{\mathrm{d}}{\mathrm{d}\varphi}\psi_{0\beta}^+\right]\Bigg|_{t'}^{t''} \\ & -\frac{\mathrm{i}}{2}\sum\frac{1}{\sqrt{m^2-E_n}}\varphi_n^-\overleftrightarrow{\partial}_t\frac{\mathrm{d}}{\mathrm{d}\varphi}\varphi_n^+\Bigg|_{t'}^{t''}. \end{aligned} \tag{2.6}$$

We present the expression in the square brackets as a sum of two terms:

$$-\psi^-\overleftrightarrow{\partial}_t\left(\tau\frac{\mathrm{d}}{\mathrm{d}r}+r\frac{\mathrm{d}}{\mathrm{d}\tau}\right)\psi^+ = -\frac{t}{\cosh\varphi}\psi^-\overleftrightarrow{\partial}_t\frac{\mathrm{d}\psi^+}{\mathrm{d}r} - \left\{\frac{1}{\cosh\varphi}\psi^-\frac{\mathrm{d}\psi^+}{\mathrm{d}r} - \psi^-\overleftrightarrow{\partial}_t\left[r\frac{\mathrm{d}}{\mathrm{d}\tau}\psi^+ - \tanh\varphi r\frac{\mathrm{d}}{\mathrm{d}r}\psi^+\right]\right\}. \tag{2.7}$$

The curled bracket does not depend on t and is cancelled in (2.6) so it becomes

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\ln\det HH_0^{-1} = -\frac{(t''-t')}{\cosh\varphi}\left\{\frac{1}{4\pi\mathrm{i}}\int\mathrm{d}x\,\mathrm{d}\beta\left[\frac{\psi_\beta^-\overleftrightarrow{\partial}_t\,\mathrm{d}\psi_\beta^+/\mathrm{d}r}{a(\beta)} - \psi_{0\beta}^-\overleftrightarrow{\partial}_t\frac{\mathrm{d}}{\mathrm{d}r}\psi_{0\beta}^+\right] - \frac{\mathrm{i}}{2}\sum\frac{1}{\sqrt{m^2-E_n}}\varphi_n^-\overleftrightarrow{\partial}_t\frac{\mathrm{d}}{\mathrm{d}r}\varphi_n^+\right\}. \tag{2.8}$$

Demonstrating the derivative in details

$$\frac{\mathrm{d}}{\mathrm{d}r}\psi_\beta^+ = \exp\{im\tau\cosh\beta\}\frac{\mathrm{d}}{\mathrm{d}r}g_\beta(r) = \frac{1}{\cosh\varphi}\exp\{im\tau\cosh\beta\}\frac{\mathrm{d}}{\mathrm{d}x}g_\beta(r) = im\cosh\beta\cdot\tanh\varphi\cdot\psi_\beta^+ + \frac{1}{\cosh\varphi}\frac{\mathrm{d}}{\mathrm{d}x}\psi_\beta^+, \tag{2.9}$$

we see the second term of it to give no contribution into (2.8) according to (1.9). So (2.8) turns to

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\mathrm{Tr}_{t'}^{t''}\ln HH_0^{-1} = -\frac{im(t''-t')}{\cosh^2\varphi}\sinh\varphi\left\{\frac{1}{4\pi\mathrm{i}}\int\mathrm{d}\beta\cosh\beta\,\mathrm{d}x\left[\frac{\psi_\beta^-\overleftrightarrow{\partial}_+\psi_\beta^+}{a(\beta)} - \psi_{0\beta}^-\overleftrightarrow{\partial}_t\psi_{0\beta}^+\right] + \frac{1}{2}\sum\varphi_n^-\overleftrightarrow{\partial}_t\varphi_n^+\right\}. \tag{2.10}$$

We can use the formulae (A.6.2), (A.6.14) to evaluate the integral over x and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\mathrm{Tr}_{t'}^{t''}\ln HH_0^{-1} = -2\mathrm{i}\frac{(t''-t')}{\cosh^2\varphi}\sinh\varphi\left\{\frac{im}{8\pi}\int_0^\infty\mathrm{d}\beta\sinh\beta\ln\frac{a^2(\beta)}{a^2(-\beta)} + \frac{1}{2}\sum_n\sqrt{m^2-E_n}\right\}. \tag{2.11}$$

This can be easily integrated over φ

$$-\frac{1}{2}\mathrm{Tr}_{t'}^{t''}\ln HH_0^{-1} = -\mathrm{i}\frac{(t''-t')}{\cosh\varphi}\left\{\frac{im}{8\pi}\int_0^\infty\mathrm{d}\beta\sinh\beta\ln\frac{a^2(\beta)}{a^2(-\beta)} + \frac{1}{2}\sum_n\sqrt{m^2-E_n}\right\} \tag{2.12}$$

and the mass correction ΔM is equal to

$$\Delta M = \frac{im}{8\pi}\int_0^\infty\mathrm{d}\beta\sinh\beta\ln\frac{a^2(\beta)}{a^2(-\beta)} + \frac{1}{2}\sum_n\sqrt{m^2-E_n} = \frac{ima^\infty}{2\pi} - \frac{im}{8\pi}\int_0^\infty\mathrm{d}\beta\cosh\beta\frac{\mathrm{d}}{\mathrm{d}\beta}\ln\frac{a^2(\beta)}{a^2(-\beta)} + \frac{1}{2}\sum_n\sqrt{m^2-E_n}, \tag{2.13}$$

$$\ln a(\beta) \to \frac{a^{\infty}}{\sinh \beta}, \qquad a^{\infty} = \frac{1}{2im} \int_{-\infty}^{\infty} dx(v''(u_s(x)) - m^2).$$

The right side of this equality diverges in the ultraviolet region. Taking the counter-term (3.5.5) into account we arrive at the following correction to the classical soliton mass [9, 14, 16, 17, 25]

$$\Delta M = \frac{1}{2}\sum_{n} \sqrt{m^2 - E_n} + \frac{ima^{\infty}}{2\pi} + \frac{m}{8\pi i} \int_{0}^{\infty} d\beta \cosh \beta \left[\frac{d}{d\beta} \ln \frac{a^2(\beta)}{a^2(-\beta)} + \frac{4a^{\infty}}{\cosh \beta}\right]. \tag{2.14}$$

The convergence of the right-hand side is provided by the a^{∞} definition (2.13).

Within the sin φ_2 model all these general formulae become

$$H = \Box + m^2 - \frac{2m^2}{\cosh^2 mr}; \tag{2.15}$$

$$\psi_{\beta}^{-} = \exp\{-im(\tau \cosh \beta - r \sinh \beta)\}\left(\frac{\sinh \beta + i \tanh(mr)}{\sinh \beta - i}\right),$$

so that

$$a(\beta) = \frac{\sinh \beta + i}{\sinh \beta - i}, \qquad \ln a(\beta) \xrightarrow[\beta \to \infty]{} \frac{2i}{\sinh \beta}, \qquad a^{\infty} = 2i, \tag{2.16}$$

see [13]. The expression (2.14) has only one non-zero term, the second one, and the mass correction is

$$\Delta M = -m/\pi, \qquad M = 8m/\gamma - m/\pi. \tag{2.17}$$

So the sin φ_2 model soliton mass in the one-loop approximation is found to be

$$M = 8m/\gamma', \qquad 8\pi/\gamma' = 8\pi/\gamma - 1, \tag{2.18}$$

cf. [9].

5.3. The one-loop correction to the periodic soliton mass

In this paragraph we are going to calculate the correction for the classical value of the periodic soliton mass, by means of

$$S = \lim_{n \to \infty} \frac{-i \ln nG(nT, 0, n|0, 0, 0)}{n} = S^{cl} + \Delta S(T); \tag{3.1}$$

$$S^{cl}(T) = \frac{1}{\gamma} \int_{0}^{T} d^2x[\tfrac{1}{2}(\partial_{\mu} w)^2 - v(w)],$$

discussed in (4.1.7). We shall express the corrections ΔM_k and ΔT_k through $\Delta S(T)$ by solving the system (2.2.9)

$$\frac{\mathrm{d}}{\mathrm{d}T}(S^{\mathrm{cl}} + \Delta S + TM) = 0,$$
$$S^{\mathrm{cl}} + \Delta S + TM = 2\pi k \tag{3.2}$$

by a perturbative method. It leads to

$$\Delta M_k = -\Delta S(T_k)/T_k;$$
$$\Delta T_k = -T_k\left(\frac{\mathrm{d}\Delta M(T)}{\mathrm{d}T}\right)\left(\frac{\mathrm{d}M^{\mathrm{cl}}(T)}{\mathrm{d}T}\right)^{-1}\Bigg|_{T=T_k}, \tag{3.3}$$

and now $\Delta S(T)$ is left to be found. One may differentiate $\mathrm{Tr}\ln HH_0^{-1}$ with respect to φ once again and make clear the Lorentz invariance of the perturbation theory. But we shall use some more convenient method. Let us differentiate the $\mathrm{Tr}\ln HH_0^{-1}$ with respect to t'' keeping the $(t'' - t')/T$ fixed and assuming $\varphi = 0$. This means to keep fixed the initial and the final internal coordinate values and to vary the time of transition from the initial state into the final state. It is easy to show the mass correction to be

$$\Delta M_k = -\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}t''}(\ln\det HH_0^{-1})|_{(t''-t')/T=\mathrm{const}}. \tag{3.4}$$

For the evaluation of $\ln\det HH_0^{-1}$ we have to describe all solutions of the equation

$$H\psi = 0. \tag{3.5}$$

This is done in Appendix 4. These solutions are expressed through the solutions of the Schroedinger matrix equation (4.3.17)

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \hat{k}_v^2 + \hat{v}\right]f = 0,$$
$$v_{nl} = \frac{1}{T}\int_0^T \mathrm{d}t\exp\left\{\mathrm{i}\frac{2\pi t}{T}(n-l)\right\}v''\left(w\left(x,\frac{t}{T},T\right)\right) - \delta_{ln}m^2;$$
$$k_{ln} = \sqrt{\left(\frac{v+2\pi n}{T}\right)^2 - m^2};$$
$$l > \frac{mT - v}{2\pi}, \qquad n > \frac{mT - v}{2\pi}. \tag{3.6}$$

The spectral parameter v belongs to the interval $mT < v < 2\pi + mT$. Consider the matrix solutions of this equation. The Schroedinger operator acts on the first index of the matrix solution, the second index is a vector solution number. The scattering solutions are

$$\hat{f}_\pm^v \to \begin{cases} \mathrm{e}^{\pm\mathrm{i}\hat{k}x} + \mathrm{e}^{\mp\mathrm{i}\hat{k}x}R_{f\pm}, & x\to-\infty,\\ \mathrm{e}^{\pm\mathrm{i}\hat{k}x}D_{f\pm}, & x\to+\infty;\end{cases}$$
$$\hat{g}_\pm^v \to \begin{cases} \mathrm{e}^{\pm\mathrm{i}\hat{k}x}D_{g\pm}, & x\to-\infty,\\ \mathrm{e}^{\pm\mathrm{i}\hat{k}x} + \mathrm{e}^{\mp\mathrm{i}\hat{k}x}R_{g\pm}, & x\to\infty.\end{cases} \tag{3.7}$$

The correction ln det HH_0^{-1} can be expressed through the S-matrix of a plane wave on the potential of eq. (3.6). In Appendix 6 we show that (A.6.23)

$$-\mathrm{i}\Delta M = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t''}\mathrm{Tr}_{t'}^{t''}\ln HH_0^{-1} = \frac{1}{4\pi\mathrm{i}}D^\infty - \frac{\mathrm{i}}{2T}\sum_n \nu_n - \frac{1}{8\pi}\int\limits_{mT}^{mT+2\pi}\mathrm{d}\nu\,\mathrm{Sp}\sqrt{\hat{k}^2+m^2}$$

$$\times\left\{D_{f+}^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{f+} + D_{g+}^{+\,-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{g+}^{+} - D_{f-}^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{f-} - D_{g-}^{+\,-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{g-}^{+} - \frac{2\mathrm{i}}{T}\frac{D^\infty}{k\sqrt{k^2+m^2}}\right\}; \tag{3.8}$$

$$(D_+^f)_{ln} = (D_-^g)_{ln} = \overline{(D_+^g)_{ln}} = \overline{(D_-^f)_{ln}} = \delta_{ln} + \frac{D^\infty}{2\mathrm{i}k_{ln}}; \qquad l\to\infty,\ n\to\infty$$

$$D^\infty = \frac{1}{T}\int\limits_0^T \mathrm{d}t\int\limits_{-\infty}^{\infty}\mathrm{d}x\left[v''\left(w\left(x,\frac{t}{T},T\right)\right) - m^2\right].$$

The trace of the curled bracket converges due to the counter term (0.8) account. The corrections of the mass, period and action are to be calculated by the formulae (3.3).

Within the sin φ_2 model the classical solution is (1.1.28). The S-matrix is reduced to a perfectly diagonal form, the potential is reflectionless and the transmission coefficient is (see (A.4.24))

$$R_g^\pm = R_f^\pm = 0, \qquad (D_+^f)_{ln} = (D^g)_{ln} = \overline{(D_+^g)_{ln}} = \overline{(D_-^f)_{ln}} = \delta_{ln}a(\nu+2\pi n), \quad \nu+2\pi n = mT\cosh\beta. \tag{3.9}$$

The general formula for the correction of the action becomes

$$-\Delta S(T) = \mathrm{i}T\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t''}(\ln\det HH_0^{-1})|_{(t''-t')/T} \tag{3.10}$$

$$= \frac{\mathrm{i}mT}{2\pi}a^\infty - \frac{\mathrm{i}mT}{8\pi}\int\limits_0^\infty \mathrm{d}\beta\cosh\beta\left[\frac{\mathrm{d}}{\mathrm{d}\beta}\ln\frac{a^2(\beta)}{a^2(-\beta)} + \frac{4a^\infty}{\cosh\beta}\right];$$

$$a^\infty = \lim_{\beta\to\infty}\sinh\beta\ln a(\beta),$$

$$a(\beta) = \left(\frac{\sinh\beta + \mathrm{i}\sin\theta}{\sinh\beta - \mathrm{i}\sin\theta}\right)^2.$$

The integral is easy to evaluate and we find

$$\Delta S = -[-2\pi + 4(\theta - \tan\theta)]. \tag{3.11}$$

We shall drop the 2π because ΔS is always to be placed into exponents. So the one-loop correction for the action is found to be reduced to the replacement (4.1.9), (4.1.10)

$$\gamma\to\gamma', \qquad 8\pi/\gamma' = 8\pi/\gamma - 1. \tag{3.12}$$

Finally the periodic soliton spectrum is [9]

$$M_n = (16m/\gamma')\sin(\gamma' n/16). \tag{3.14}$$

Note that the second equation of (3.2) leads to the following Bohr–Sommerfeld rule:

$$\theta_n = (\gamma'/16)n. \tag{3.15}$$

The classical Bohr–Sommerfeld rule was $\theta_n = (\gamma/16)n$. It is clear that passing from the semiclassical to the one-loop description of the nth periodic soliton state we must replace

$$\theta \to \theta + \delta\theta, \qquad \delta\theta = \frac{\gamma}{8\pi}\theta. \tag{3.16}$$

This will help us to choose the right sort of variables for the S-matrix corrections evaluation.

5.4. *The one-loop correction to the structureless solitons S-matrix*

To calculate this correction we shall exploit the definitions (3.2.7) and the approximation (0.2) for the propagator. So we get for the one loop correction to the S-matrix

$$S_0(\varphi_1 - \varphi_2) = \left[\frac{\det H(u_{ss})H_0^{-1}}{\det H(u_{1s})H_0^{-1}\cdot\det H(u_{2s})H_0^{-1}}\right]^{-1/2}, \tag{4.1}$$

$$H(u) \equiv \Box + v''(u); \qquad S(\varphi_1 - \varphi_2) = S_{-1}(\varphi_1 - \varphi_2)S_0(\varphi_1 - \varphi_2),$$

recollecting also (4.2.1), (4.2.2), (4.2.3), (4.2.5).

We have to describe all the solutions of the equation

$$H(u_{ss})\psi_\beta = 0 \tag{4.2}$$

for calculation of the (4.1) by the expression (1.7). But previously we shall write in a more appropriate way the solutions of the homogeneous equation when only one of the two solitons is present. To make the formulae more readable we consider here the simplest case when both the soliton potentials are reflectionless,

$$\begin{aligned}\psi^-_{1,2\beta}(x,t) &= \exp\{-im(t\cosh\beta - x\sinh\beta)\}a_{1,2}(r_{1,2}, \beta - \varphi_{1,2});\\ \psi^+_{1,2}(x,t) &= \exp\{im(t\cosh\beta - x\sinh\beta)\}a_{1,2}(\beta - \varphi_{1,2})\alpha_{1,2}(r_{1,2}; \varphi_{1,2} - \beta).\end{aligned} \tag{4.3}$$

These expressions define the functions $a_{1,2}(r_{1,2}, \beta)$ uniquely. They have the following asymptotics:

$$\begin{aligned}&a_{1,2}(r_{1,2}, \beta) \xrightarrow[r_{1,2}\to-\infty]{} 1, \qquad a_{1,2}(r_{1,2}, \beta) \xrightarrow[r_{1,2}\to\infty]{} a_{1,2}(\beta),\\ &\bar{a}_{1,2}(\beta) = a_{1,2}(-\beta) = a^{-1}_{1,2}(\beta).\end{aligned} \tag{4.4}$$

We write all the formulae in the coordinate system with its origin at the soliton center of mass. The quantities $a_{1,2}(r_{1,2}, \beta)$ are non-constant only in their soliton's vicinity. Now look at the functions $\psi^\pm_\beta(x,t)$ for both solitons present. At $t = t' \to -\infty$ the first soliton was situated very far on the left side of the x-axis, and the second very far on the right, at the points $t'\tanh\varphi_1$ and $t'\tanh\varphi_2$. At $t = t'' \to \infty$ the picture will be the same but the solitons will have changed their places and their coordinates will be

$$t''\tanh\varphi_1 + \frac{1}{M_1\cosh\varphi_1}\Delta(\varphi_1 - \varphi_2)$$

and

$$t'' \tanh \varphi_2 - \frac{1}{M_2 \cosh \varphi_2} \Delta(\varphi_1 - \varphi_2); \qquad \varphi_1 > 0, \varphi_2 < 0.$$

When the solitons are far apart the plane wave ψ_β scatters on them independently, and the S-matrix is, roughly speaking, factorised. But when they are close to one another, the scattering on the potentials overlap region makes the plane wave become at $t \to -\infty$ the sum of the plane wave and of the wave packet

$$\psi_\beta^- \xrightarrow[x\to\infty]{t\to-\infty} \exp\{-\mathrm{i}m(t\cosh\beta - x\sinh\beta)\} + \int C_{\beta\gamma} \exp\{-\mathrm{i}m(t\cosh\gamma - x\sinh\gamma)\}\,\mathrm{d}\gamma. \tag{4.5}$$

Now we have all necessary notations to present the final result of this paragraph before starting its actual derivation. We shall find the two-soliton elastic scattering matrix in the one-loop approximation to be

$$S(\varphi_1 - \varphi_2) = Z\exp\Big\{-\mathrm{i}\int_0^{\varphi_1-\varphi_2} \Delta(\beta)\,\mathrm{d}\beta + \frac{1}{4\pi\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{d}\beta \frac{\mathrm{d}}{\mathrm{d}\beta} \ln a_1(\beta - \varphi_1)\ln a_2(\beta-\varphi_2)$$
$$- \mathrm{i}\Delta(\varphi_1 - \varphi_2)\omega(\varphi_1 - \varphi_2) - \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\beta[\ln(1+\hat{c})_{\beta\beta} - c^\infty]\Big\}, \qquad \varphi_1 > \varphi_2. \tag{4.6}$$

Here the $\ln(1+\hat{c})_{\alpha\beta}$ is the kernel of the operator $\ln(1+\hat{c})$ and

$$c^\infty = \lim_{\beta\to\infty} c_{\beta\beta}. \tag{4.7}$$

All the terms in the exponent are convergent. The very last term is the transformed counter-term (0.8). The constant factor Z should be found from some supplementary requirements, e.g. from the crossing symmetry. The function $\omega(\varphi_1 - \varphi_2)$ is arbitrary, because the correction is dependent on the type of variables it is calculated in.

As a matter of fact, the relation between the variables that describe the two-body process is changed by the passing from the semiclassics to the one-loop approximation due to the masses renormalization $M^{\mathrm{cl}} \to M^{\mathrm{cl}} + \Delta M$. Our correction is calculated with the fixed solitons' velocities, i.e. as a function of $\varphi_1 - \varphi_2$. A quantity another than $\varphi_1 - \varphi_2$ being kept fixed would have compelled us to replace $\varphi_1 - \varphi_2 \to \varphi_1 - \varphi_2 + \omega(\varphi_1 - \varphi_2)$ in the one-loop approximation. The function $\omega(\varphi_1 - \varphi_2)$ must compensate the fixed quantity change caused by the mass renormalization. For example, if one wants to fix the Mandelstam variable

$$s = \left(2M\sinh\frac{\varphi_1 - \varphi_2}{2}\right)^2,$$

then he ought to assume

$$\omega(\varphi_1 - \varphi_2) = -2\frac{\Delta M}{M^{\mathrm{cl}}}\tanh\frac{\varphi_1 - \varphi_2}{2}.$$

All this depicts the ununiqueness of the expansion in powers of γ, but the exact quantum S-matrix is unique.

Let us learn at last how the expression (4.6) is derived. We rewrite in details the asymptotics of the equation (4.2) solution which is of a plane-wave type at $t \to -\infty$. The negative-frequency solutions become:

$$\tilde{\psi}_\beta^-(x,t) = \exp\{-im(t\cosh\beta - x\sinh\beta)\}a_1(r_1, \beta - \varphi_1)a_2(r_2, \beta - \varphi_2). \tag{4.8}$$

The coefficients $a_{1,2}(r_{1,2})$ are not constant here only in the vicinity of their soliton, and the positive-frequency solutions are

$$\psi_\beta^+(x,t) = \exp\{im(t\cosh\beta - x\sinh\beta)\}a_1(\beta - \varphi_1)a_1(r_1, \varphi_1 - \beta)a_2(\beta - \varphi_2)a_2(r_2, \varphi_2 - \beta). \tag{4.9}$$

We represent (4.1) as a product of the two factors:

$$S_{10} = \frac{\exp\{-\frac{1}{2}\mathrm{Tr}_{t'}^{t''}\ln H(u_{ss}(x,t|q_1^0, q_2^0, \tanh\varphi_1\tanh\varphi_2))\}}{\exp\left\{-\frac{1}{2}\mathrm{Tr}_{t_0}^{t''}\ln H\left(u_{1s}\left(q_1^0 + \frac{\Delta(\varphi_1-\varphi_2)}{M_1\cosh\varphi_1};\varphi_1\right)\right) - \frac{1}{2}\mathrm{Tr}_{t_0}^{t''}\ln H\left(u_{2s}\left(q_2^0 - \frac{\Delta(\varphi_1-\varphi_2)}{M_2\cosh\varphi_2};\varphi_2\right)\right)\right\}}$$
$$\times \frac{1}{\exp\{-\frac{1}{2}\mathrm{Tr}_{t_0}^{t''}\ln H(u_{1s}(q_1^0;\varphi_1)) - \frac{1}{2}\mathrm{Tr}_{t_0}^{t''}\ln H(u_{2s}(q_2^0;\varphi_2))\}}; \tag{4.10}$$

$$S_{20} = \frac{\exp\left\{-\frac{1}{2}\mathrm{Tr}_{t_0}^{t''}\ln H\left(u_{1s}\left(q_1^0 + \frac{\Delta(\varphi_1-\varphi_2)}{M_1\cosh\varphi_1};\varphi_1\right)\right) - \frac{1}{2}\mathrm{Tr}_{t'}^{t_0}\ln H(u_{1s}(q_1^0;\varphi_1))\right\}}{\exp\left\{-\frac{1}{2}\mathrm{Tr}_{t'}^{t''}\ln H\left(u_{1s}\left(q_1^0 - \frac{t'}{(t''-t')}\frac{\Delta(\varphi_1-\varphi_2)}{M_1\cosh\varphi_1};\varphi_1 + \frac{\Delta(\varphi_1-\varphi_2)}{M_1}\cdot\frac{\cosh\varphi_1}{(t''-t')}\right)\right)\right\}}$$
$$\times \frac{\exp\left\{-\frac{1}{2}\mathrm{Tr}_{t_0}^{t''}\ln H\left(u_{2s}\left(q_2^0 - \frac{\Delta(\varphi_1-\varphi_2)}{M_2\cosh\varphi_2};\varphi_2\right)\right) - \frac{1}{2}\mathrm{Tr}_{t'}^{t_0}\ln H(u_{2s}(q_2^0,\varphi_2))\right\}}{\exp\left\{-\frac{1}{2}\mathrm{Tr}_{t'}^{t''}\ln H\left(u_{2s}\left(q_2^0 + \frac{t'}{(t''-t')}\frac{\Delta(\varphi_1-\varphi_2)}{M_2\cosh\varphi_2};\varphi_2 - \frac{\Delta(\varphi_1-\varphi_2)}{M_2}\cdot\frac{\cosh\varphi_2}{(t''-t')}\right)\right)\right\}} \tag{4.11}$$

$$u_s(x,t|\tanh\varphi, q) \equiv u_s(q,\varphi);$$
$$S_0(\varphi_1 - \varphi_2) = S_{10}(\varphi_1 - \varphi_2)S_{20}(\varphi_1 - \varphi_2). \tag{4.12}$$

Figure 3 pictures "the trajectories of the solitons at the nominator and in the denominator" of the first factor.

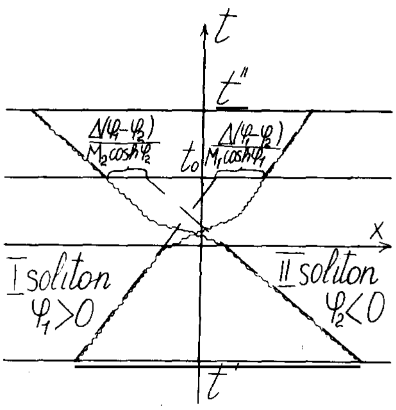


Fig. 3.

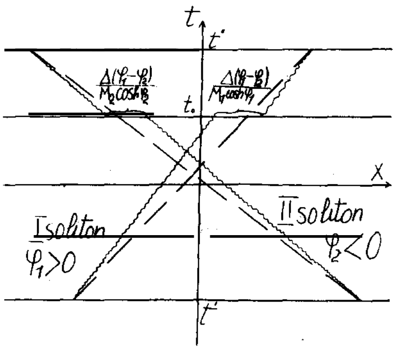


Fig. 4.

The wavy lines and the broken lines on fig. 3 represent the "solitons' trajectories" in the nominator and in the denominator of (4.10) correspondingly. Figure 4 represents the same for the second factor (4.11) of S_0, the wavy lines depict the nominator's "trajectories" and the broken lines the denominator's, as above.

To evaluate S_{20} (4.11) we need just the single-particle propagators in the one-loop approximation. A trivial calculation leads to

$$S_{20}(\varphi_1 - \varphi_2) = \exp\left\{-\mathrm{i}\Delta(\varphi_1 - \varphi_2)\left(\frac{\Delta M_1}{M_1^{\mathrm{cl}}}\tanh\varphi_1 - \frac{\Delta M_2}{M_2^{\mathrm{cl}}}\tanh\varphi_2\right)\right\}. \tag{4.13}$$

To calculate S_{10} we can use (1.8):

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\mathrm{Tr}_{t'}^{t''}\ln HH_0^{-1} = \frac{1}{4\pi\mathrm{i}}\int \mathrm{d}x\,\mathrm{d}\beta\,\mathrm{d}\gamma\left[\frac{\psi_{+\beta}\vec{\partial}_t(\mathrm{d}/\mathrm{d}\varphi_1)\psi_\gamma^-}{a_1(\beta-\varphi_1)a_2(\beta-\varphi_2)}(1+c)_{\beta\gamma}^{-1} - \psi_{0\beta}^+\vec{\partial}_t\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\psi_{0\gamma}^-\right]\Bigg|_{t'}^{t''}. \tag{4.14}$$

Let us assume for a while the discrete spectrum to be absent. For the single-soliton determinants in the denominator of (4.10) we can employ an analogue of (4.14) and get

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\mathrm{Tr}_{t_0}^{t''}\ln H(u_{1\mathrm{s}})H_0^{-1} = \frac{1}{4\pi\mathrm{i}}\int \mathrm{d}x\,\mathrm{d}\beta\left[\frac{\psi_{+\beta}\vec{\partial}_t(\mathrm{d}/\mathrm{d}\varphi_1)\psi_{-\beta}}{a_1(\beta-\varphi_1)} - \psi_{+0\beta}\vec{\partial}_t\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\psi_{0-\beta}\right]\Bigg|_{t_0}^{t''}. \tag{4.15}$$

Let us express $(\mathrm{d}/\mathrm{d}\varphi_1)\ln S_{10}$ with help of (4.14) and (4.15). First of all we ought to take into account the contribution at t_0. It is "single-particle reducible", i.e. it looks like the formulae of paragraph 5.2. The contribution to the term $(\mathrm{d}/\mathrm{d}\varphi_1)\ln S_{10}$ at the moment t_0 is equal to

$$-\frac{1}{2}\frac{1}{4\pi\mathrm{i}}\int\frac{\mathrm{d}x\,\mathrm{d}\beta}{a_1(\beta-\varphi_1)}\left[\psi_{1\beta}^+\left(\tau, r-\frac{\Delta}{M}\right)\vec{\partial}_t\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\psi_{1\beta}^-\left(\tau, r-\frac{\Delta}{M_1}\right) - \psi_{1\beta}^+(\tau,r)\vec{\partial}_t\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\psi_{1\beta}^-(r,\tau)\right]\Bigg|_{t_0}. \tag{4.16}$$

We have written out only the term which is localized in the region of the first soliton. The functions $\psi_{1\beta}^\pm$ are the single-soliton plane waves (4.3). The derivative $(\mathrm{d}/\mathrm{d}\varphi_1)\psi_\beta^-(\tau, r-\Delta/M_1)$ can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\psi_{1\beta}^-\left(\tau, r-\frac{\Delta}{M_1}\right) = \frac{\partial}{\partial\varphi_1}\psi_{1\beta}^-\left(\tau, r-\frac{\Delta}{M_1}\right) - \frac{\Delta'}{M_1}\frac{\mathrm{d}}{\mathrm{d}r}\psi_{1,3}^-\left(\tau, r-\frac{\Delta}{M_1}\right) \tag{4.17}$$

and (4.16) we rewrite as

$$\begin{aligned}&-\frac{1}{2}\frac{1}{4\pi\mathrm{i}}\int\frac{\mathrm{d}x\,\mathrm{d}\beta}{a_1(\beta-\varphi_1)}\left[\psi_{1\beta}^+\left(\tau, r-\frac{\Delta}{M_1}\right)\vec{\partial}_t\frac{\partial}{\partial\varphi_1}\psi_{1\beta}^-\left(\tau, r-\frac{\Delta}{M_1}\right) - \psi_{1\beta}^+(\tau,r)\vec{\partial}_t\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\psi_{1\beta}^-(\tau,r)\right]\\ &\quad+\frac{\Delta'}{M_1}\frac{1}{8\pi\mathrm{i}}\int\frac{\mathrm{d}x\,\mathrm{d}\beta}{a_1(\beta-\varphi_1)}\psi_{1\beta}^+\left(\tau, r-\frac{\Delta}{M_1}\right)\vec{\partial}_t\frac{\mathrm{d}}{\mathrm{d}r}\psi_{1\beta}^-\left(\tau, r-\frac{\Delta}{M_1}\right).\end{aligned} \tag{4.18}$$

Its first term can be calculated similarly to (2.6) and appears to be

$$\left[\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\left(-\mathrm{i}\frac{\Delta M}{\cosh\varphi_1}\Delta t\right)\right]\Bigg|_{\Delta t = \Delta/M_1\sinh\varphi_1} = \mathrm{i}\frac{\Delta M_1}{M_1}\frac{\Delta(\varphi_1-\varphi_2)}{\cosh^2\varphi_1}. \tag{4.19}$$

The second term is calculable like (2.8), it is

$$\mathrm{i}\frac{\Delta M_1}{M_1}\tanh\varphi_1\cdot\Delta'(\varphi_1-\varphi_2). \tag{4.20}$$

The analogous calculations of the terms localized in the region of the second soliton allow to express the complete contribution of $t=t_0$ terms to the derivative $(\mathrm{d}/\mathrm{d}\varphi_1)\ln S_{10}$ as follows

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\left[\mathrm{i}\frac{\Delta M_1}{M_1^{\mathrm{cl}}}\tanh\varphi_1\cdot\Delta(\varphi_1-\varphi_2)-\mathrm{i}\frac{\Delta M_2}{M_2^{\mathrm{cl}}}\tanh\varphi_2\cdot\Delta(\varphi_1-\varphi_2)\right]. \tag{4.21}$$

Comparing (4.13) and (4.21) we see that they cancel one another, and we can forget them. All the correction to the S-matrix turns out to be the lower limit contribution $t=t'$ at (4.14). The upper limit contribution into (4.14) is zero. This is not too difficult to understand noting that $(\mathrm{d}/\mathrm{d}\varphi_1)\times\psi_\beta^-(x,t)$ is non-zero only in the region of the first soliton and at the right-hand side of it. But at the right-hand side of the first soliton the factor at the $(\mathrm{d}/\mathrm{d}\varphi_1)\psi_\beta^-(x,t)$ becomes a plane wave $\psi_{0\beta}^+$. Hence the contribution of $(\mathrm{d}/\mathrm{d}\varphi_1)\psi_\beta^-(x,t)$ on the upper limit is cancelled by the denominator of (4.10) contribution.

So far we have found the upper limit $t=t''$ contribution to the $(\mathrm{d}/\mathrm{d}\varphi')\ln S_{10}$ to be zero. Now we have to examine the lower limit $t=t'$ contribution, which is the sum of two terms. The first term originates from the $\mathrm{d}\hat{c}$:

$$\frac{+1}{8\pi\mathrm{i}}\int\frac{\mathrm{d}x\,\mathrm{d}\beta\,\mathrm{d}\gamma(\psi_{+\beta}(1+c)^{-1}_{\beta\gamma}\overleftrightarrow{\partial}_t(\mathrm{d}/\mathrm{d}\varphi_1)\hat{c}_{\gamma\delta}\tilde{\psi}_{-\delta})}{a_1(\beta-\varphi_1)a_2(\beta-\varphi_2)}. \tag{4.22}$$

Remembering that (A.5.23)

$$\int\mathrm{d}x\,\psi_{+\beta}\frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t}\tilde{\psi}_{-\delta}=-4\pi\mathrm{i}\delta(\beta-\delta)a_1(\beta-\varphi_1)a_2(\beta-\varphi_2), \tag{4.23}$$

we can find analogously to (1.13) the contribution to $(\mathrm{d}/\mathrm{d}\varphi_1)\ln S_{10}$ to be

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\left[-\tfrac{1}{2}\ln\det(1+\hat{c})\right]. \tag{4.24}$$

Another term from the lower limit t' into the (4.14) arises from $(\mathrm{d}/\mathrm{d}\varphi_1)a_1(r_1;\beta-\varphi_1)$ and after all cancellations turns out to be

$$\frac{1}{8\pi\mathrm{i}}\int\mathrm{d}\beta\,\mathrm{d}x\frac{\mathrm{d}\ln a_1(\beta-\varphi_1)}{\mathrm{d}\varphi_1}\left[\frac{\psi_{2\beta}^+\overleftrightarrow{\partial}_t\psi_{2\beta}^-}{a_2(\beta-\varphi_2)}-\psi_{0\beta}^+\overleftrightarrow{\partial}_t\psi_{0\beta}^-\right]. \tag{4.25}$$

The functions $\psi_{2\beta}^\pm$ are the same as in (4.3). The integral x can be evaluated as it is shown in Appendix 6 (see (A.6.2))

$$\int\mathrm{d}x\left[\frac{\psi_{2\beta}^+\overleftrightarrow{\partial}_t\psi_{2\beta}^-}{a_2(\beta-\varphi_2)}-\psi_{0\beta}^+\overleftrightarrow{\partial}_t\psi_{0\beta}^-\right]=-2\frac{\mathrm{d}}{\mathrm{d}\beta}\ln a_2(\beta-\varphi_2), \tag{4.26}$$

so that (4.25) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_1}\left[-\frac{1}{4\pi\mathrm{i}}\int\mathrm{d}\beta\ln a_1(\beta-\varphi_1)\frac{\mathrm{d}}{\mathrm{d}\beta}\ln a_2(\beta-\varphi_2)\right]. \tag{4.27}$$

Collecting all the terms for $S_0(\varphi_1 - \varphi_2)$ we get the final expression

$$S_0(\varphi_1 - \varphi_2) = Z \exp\left\{ -\tfrac{1}{2} \ln \det(1 + \hat{c}) + \frac{1}{4\pi i} \int d\beta \frac{d}{d\beta} \ln a_1(\beta - \varphi_1) \ln a_2(\beta - \varphi_2) \right\}, \quad (4.28)$$

which was presented above in (4.6). It will not be altered by an account of the discrete spectrum.

It is clear from the examples discussed above that we can calculate a one-loop correction to any process that can occur to solitons by means of the formulae (1.7) and (1.8), thus needing neither finite boxes nor Floquet indexes in the scattering problems and obtaining the results in the manifest Lorentz-invariant form.

Consider now the sin φ_2 model. Dashen et al. had noted in [9] the following. The periodic soliton scattering on the system of two structureless solitons describes in the limit $\theta \to 0$ the scattering of a plane wave on the same system, whereas the periodic soliton scattering is known explicitly [35], and we can easily see that $C = 0$.

The transmission coefficients $a(\beta)$ appear as (2.16)

$$a_1(\beta) = a_2(\beta) = \frac{\sinh \beta + i}{\sinh \beta - i}. \quad (4.29)$$

The S-matrix correction is obtained from (4.6)

$$S_0 = i \exp\left\{ -\left[\frac{1}{\pi} + \left(\frac{\varphi_+ - \varphi_-}{\pi} - \frac{8}{\gamma} \omega(\varphi_+ - \varphi_-) \right) \frac{d}{d(\varphi_+ - \varphi_-)} \right] \times \int_0^\pi d\theta \ln \left(\frac{e^{\varphi_+ - \varphi_-} \cdot e^{-i\theta} + 1}{(e^{\varphi_+ - \varphi_-} + e^{-i\theta}} \right) \right\}. \quad (4.30)$$

Let us discuss now what variables are the most natural in the quantum sin φ_2 model. Note first of all that two classical solutions, the periodic soliton (1.1.28) and the soliton–antisoliton solution (1.1.35) are related by the analytic continuation

$$\tfrac{1}{2}(\varphi_+ - \varphi_-) \to i\left(\frac{\pi}{2} - \theta \right). \quad (4.31)$$

It seems natural to preserve this relation in the quantum domain as well. Passing from the semi-classics to the one-loop approximation we ought to change θ in order to retain the same quantum number of the periodic soliton n in spite of the renormalization $\gamma \to \gamma'$ (3.16). Hence it is natural to suggest

$$\omega = \frac{\gamma}{8\pi} (\varphi_+ - \varphi_-). \quad (4.32)$$

This will cancel the last term in (4.30). So we obtain the soliton–soliton S-matrix in the one-loop approximation

$$S(\varphi_+ - \varphi_-) = -i \exp\left\{ i \frac{8\pi^2}{\gamma'} + \frac{8}{\gamma'} \int_0^\pi d\theta \ln \left(\frac{\exp\{\varphi_+ - \varphi_- - i\theta\} + 1}{\exp\{\varphi_+ - \varphi_-\} + \exp\{-i\theta\}} \right) \right\}; \quad (4.33)$$

$$8\pi/\gamma' = 8\pi/\gamma - 1.$$

The soliton–soliton S-matrix in the one-loop approximation is calculable just in the same manner as the soliton–antisoliton S-matrix. The one-loop correction is reduced to the replacement $\gamma \to \gamma'$ in the semiclassical result (4.2.20).

5.5. The survey of known results for the sin φ_2 model

It has been conjectured in [9] that all the one-loop corrections in the sin φ_2 model are reduced to the replacement $\gamma \to \gamma'$, $8\pi/\gamma' = 8\pi/\gamma - 1$. This was proved by direct calculations. The mass corrections for the structureless and periodic solitons were calculated in [9, 14]. The one-loop correction of the two structureless solitons' S-matrix was derived in [98, 99, 100]. The same for the periodic solitons was done in [100], and the N-soliton case was considered in [101]. The one-loop correction for the above-barrier soliton reflection was calculated in [102].

In the paper [9] it was also conjectured that the one-loop mass spectrum of the sin φ_2 model is exact

$$M_n/M_l = \sin\left(\frac{\gamma'}{16}n\right)\Big/\sin\left(\frac{\gamma'}{16}l\right). \tag{5.1}$$

This was demonstrated in [103] to be true. Later it was found out that this form of the mass spectrum is a consequence of the infinite number of conservation laws (1.1.44) of the model. In ref. [74] it was demonstrated for the first time that these classical conservation laws lay down some principal restrictions on the S-matrix, e.g. they prohibit the multiple production. Next it was shown in [104] that in the exact quantum theory all the physical consequences of the conservation laws are the same. It is also clear from [105] that these conservation laws make the N-particle S-matrix factorizable into the two-body ones. The paper [106] proves that in the sin φ_2 model the form of the mass spectrum is a consequence of the S-matrix factorization.

Consider now the two-soliton S-matrix. The one-loop expression of the S-matrix (4.33) makes possible the following hypothesis about the form of the S-matrix at $\gamma' = 8\pi/N$ where N is some integer number. It was conjectured in [13, 14] that for such γ'

$$\langle v''_+, v''_- | S | v'_+, v'_- \rangle = \delta(\varphi''_+ - \varphi'_+)\delta(\varphi''_- - \varphi'_-) \prod_{k=1}^{N-1} \frac{\exp\{\varphi_+ - \varphi_-\}\exp\{-i\pi k/N\} + 1}{\exp\{\varphi_+ - \varphi_-\} + \exp\{-i\pi k/N\}},$$

$$\varphi_+ > \varphi_-, \tag{5.2}$$

for the soliton–antisoliton scattering. We just replace the integral in (4.33) by its integral sum:

$$\frac{N}{\pi}\int^{\pi} d\theta \ln\left(\frac{\exp\{\varphi_+ - \varphi_-\}\exp\{-i\theta\} + 1}{\exp\{\varphi_+ - \varphi_-\} + \exp\{-i\theta\}}\right) \Rightarrow \prod_{k=1}^{N-1} \ln\left(\frac{\exp\{\varphi_+ - \varphi_-\}\exp\{-i\pi k/N\} + 1}{\exp\{\varphi_+ - \varphi_-\} + \exp\{-i\pi k/N\}}\right). \tag{5.3}$$

The above-barrier reflection factor expression (4.4.8) and the explicit form of the S-matrix at $\gamma' = 8\pi/N$, i.e. at the values of γ' which nullify the reflection lead us to the idea that the solitons scattering in this model is a relativistic generalization of the plane wave scattering on the potential $u_0/\cosh^2 x$. The values $\gamma = 8\pi/N$ are the analogues of the values $u_0 = N(N+1)$ which also turn the reflection into zero. This observation and the conjecture about the meromorphic dependence of the S-matrix on the rapidity propose the hypothesis of the two-soliton S-matrix exact form at any value of the coupling constant [107]:

$$\langle v''_+, v''_- | S | v'_+, v'_- \rangle = \delta(\varphi''_+ - \varphi'_+)\delta(\varphi''_- - \varphi'_-)D(\varphi''_+ - \varphi''_-) + \delta(\varphi''_+ - \varphi'_-)\delta(\varphi''_- - \varphi'_+)R(\varphi''_+ - \varphi''_-)D(\varphi''_+ - \varphi''_-); \tag{5.4}$$

$$D(\theta) = -\frac{i}{\pi}\sinh\left(\frac{8\pi}{\gamma'}\theta\right)\Gamma\left(\frac{8\pi}{\gamma'}\right)\Gamma\left(1 + i\frac{8\theta}{\gamma'}\right)\Gamma\left(1 - \frac{8\pi}{\gamma'} - i\frac{8\theta}{\gamma'}\right)\prod_{l=1}^{\infty}\frac{\Phi_l(\theta)\Phi_l(i\pi - \theta)}{\Phi_l(0)\Phi_l(i\pi)},$$

with notations

$$R(\theta) = -i\frac{\sin(8\pi^2/\gamma')}{\sinh(8\pi\theta/\gamma')};$$

$$\Phi_l(\theta) = \frac{\Gamma(16l\pi/\gamma' + i8\theta/\gamma')\Gamma(1 + 16l\pi/\gamma' + i8\theta/\gamma')}{\Gamma((2l+1)8\pi/\gamma' + i8\theta/\gamma')\Gamma(1 + (2l-1)8\pi/\gamma' + i8\theta/\gamma')}. \tag{5.5}$$

The soliton–soliton scattering matrix can be obtained by the analytical continuation into the cross-channel. This hypothetical S-matrix is unitary and analytic, its expansion reproduces all the known results obtained by the perturbations theory. The most compact form for this S-matrix was given in [108],

$$D(\theta) = \exp\left\{ i \int_0^{\infty} \frac{dx}{x} \frac{\sin[x(i - \theta/\pi)]\sinh[x(\gamma'/16\pi - \frac{1}{2})]}{\sinh(\gamma' x/16\pi)\cosh(x/2)} \right\}. \tag{5.6}$$

The N-soliton S-matrix factorization enables to construct it of the two-soliton ones, this was performed in [109]. So the scattering matrix for the processes with periodic solitons appears to be derivable trivially by tending the Mandelstam variable of two structureless solitons in the N-soliton matrix to the pole which represents the nth state of periodic soliton, because a periodic soliton is a bound state of two structureless ones.

This procedure was carried out in [109] and the matrix of the structureless-periodic solitons scattering was found to be

$$S^{(n)}(\theta) = \frac{\sinh\theta + i\cos(n\gamma'/16)}{\sinh\theta - i\cos(n\gamma'/16)}\prod_{l=1}^{n-1}\frac{\sin^2((n-2l)\gamma'/32 - \pi/4 + i\theta/2)}{\sin^2((n-2l)\gamma'/32 - \pi/4 - i\theta/2)}. \tag{5.7}$$

The periodic–periodic soliton S-matrix similarly is

$$S^{(n,m)}(\theta) = \frac{\sinh\theta + i\sin((n+m)\gamma'/16)}{\sinh\theta - i\sin((n+m)\gamma'/16)}\cdot\frac{\sinh\theta + i\sin((m-n)\gamma'/16)}{\sinh\theta - i\sin((m-n)\gamma'/16)} \times \prod_{l=1}^{n-1}\frac{\sin^2((n-m-2l)\gamma'/32 + i\theta/2)}{\sin^2((n-m-2l)\gamma'/32 - i\theta/2)}\cdot\frac{\cos^2((n+m-2l)\gamma'/32 + i\theta/2)}{\cos^2((n+m-2l)\gamma'/32 - i\theta/2)}; \tag{5.8}$$

see also [110].

Note the lowest states $n = m = 1$ periodic solitons S-matrix to be identical to the S-matrix of the usual particles [96]. So we arrive at the complete S-matrix form in the $\sin\varphi_2$ model, keeping in mind that the usual particles states are the excited states of a periodic soliton (paragraph 4.5).

The same statement is true in the Thirring model, because the Thirring model with the Lagrange function

$$\mathscr{L} = \int dx\{i\bar{\psi}\hat{\partial}\psi - \mu\bar{\psi}\psi - \tfrac{1}{2}g(\bar{\psi}\gamma_\mu\psi)^2\} \tag{5.9}$$

was demonstrated [111–113] to be equivalent to the $\sin\varphi_2$ with the coupling constants being related by

$$1 + 2g/\pi = 8\pi/\gamma'. \tag{5.10}$$

Conclusions

We hope to have convinced the reader that the quantum theory of solitons has some attractive properties.

1. The Lagrange function of the theory contains few fields but produces a rich particle spectrum. In the weak coupling approximation the solitons interact strongly.
2. The solitons possess a new quantum number of a topological origin which can be regarded as the charge.
3. In the weak coupling approximation there exists a well developed perturbation theory. The quantum corrections are small at the small coupling and all the non-analytic contribution to the observables is derived from the semiclassical approximation.

Appendix 1

We give here a more general derivation of the quantum mechanical S-matrix. Consider the scattering of several particles in the laboratory system. Let $G(t'', \{q''\}|t', \{q'\})$ be their propagator. It does not change after the simultaneous shifts of coordinates

$$G(t'', \{q''\}|t', \{q'\}) = G(t'' - t_0, \{q'' - q_0\}|t' - t_0, \{q' - q_0\}). \tag{A.1.1}$$

The wave packet

$$\psi_j(x, t) = \frac{1}{\sqrt{2\pi}} \int c_j(k) \exp\{ikx\} \exp\{-i(k^2/2m)t\} \, dk \tag{A.1.2}$$

describes the free motion of the particle number j with the momentum distribution $c_j(k)$. By definition the S-matrix element between in and out wave packets is given by

$$\prod_{i''} \prod_{i'} \int c^*_{i''}(k''_{i''}) S(\{k''\}, \{k'\}) c_{i'}(k'_{i'}) \, dk'' \, dk' = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} \int \psi^*_{i''}(q''_{i''}) G(t'', \{q''\}|t', \{q'\}) \psi_{i'}(q'_{i'}) \, dq'' \, dq'. \tag{A.1.3}$$

We evaluate the integral (A.1.2) by the stationary phase method

$$\int c(k) \, e^{ikq} \, e^{-i(k^2/2m)t} \, dk \cong \left(\frac{-2\pi i m}{t}\right)^{1/2} \exp\{imq^2/2t\} c(mq/t). \tag{A.1.4}$$

The expression (A.1.3) can then be rewritten as

$$\begin{aligned} &\int \prod_{i'',i'} dp''_{i''} dp'_{i'} c^*_{i''}\left(p''_{i''} + m\frac{q''_{0i''}}{t''}\right) \left[\left(\frac{t''}{im}\right)^{1/2} \exp\left\{i\frac{m}{2t''}\left(\frac{p''_{i''}}{m}t'' + q''_{0i''}\right)^2\right\}\right] \\ &\times G\left(t'' - t_0, \left\{\frac{p''_{i''}}{m}t'' + q''_{0i''} - q_0\right\}\middle| t' - t_0, \left\{\frac{p'_{i'}}{m}t' + q'_{0i'} - q_0\right\}\right) \\ &\times \left(\frac{it'}{m}\right)^{1/2} \exp\left\{-i\frac{m}{2t'}\left(\frac{p'_{i'}}{m}t' + q'_{0i'}\right)^2\right\} c_{i'}\left(p'_{i'} + m\frac{q'_{0i'}}{t'}\right); \quad p''_{i''} = m\frac{q''_{i''} - q''_{0i''}}{t''}; \quad p'_{i'} = m\frac{q'_{i'} - q'_{0i'}}{t'}. \end{aligned} \tag{A.1.5}$$

At the first step we assume $q'_{0i'} = q''_{0i''} = 0$. Taking into account that

$$G(t'', q''|t', q') = \left(\frac{m}{2\pi i(t'' - t')}\right)^{1/2} \exp\left\{im\frac{(q'' - q')^2}{2(t'' - t')}\right\}, \tag{A.1.6}$$

and differentiating (A.1.5) with respect to c we come to the expression of the S-matrix

$$S(\{p''\}, \{p'\}) = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \frac{G(t'', \{(p''/m)t''\}|t', \{(p'/m)t'\})}{\prod_{i''} \sqrt{2\pi}G(t'', (p''_{i''}/m)t''|t_0, q_0) \prod_{i'} \sqrt{2\pi}G(t_0, q_0|t', (p'_{i'}/m)t')}. \tag{A.1.7}$$

Next we consider the case of $q_j^{0''} \neq 0$, $q_i^{0'} \neq 0$. We expand the expressions of type $c(p''_j + mq''_{0j}/t'')$ into the series of the mq''_{0j}/t'' powers

$$c\left(p'' + m\frac{q''_0}{t''}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(m\frac{q''_0}{t''}\right)^n c^{(n)}(p''). \tag{A.1.8}$$

The existence of the finite limit of the operator kernel (A.1.7) makes sure that all the terms of this series except of the zero term shall not contribute to the (A.1.5). This leads us to

$$S(\{p''\}, \{p'\}) = \lim_{\substack{t''\to\infty \\ t'\to-\infty}} \frac{G(t'', \{q''\}|t', \{q'\})}{\prod_{i'',i'} \sqrt{2\pi}G(t'', q''_{i''}|t_0, q_0)G(t_0, q_0|t', q'_{i'})\sqrt{2\pi}}; \tag{A.1.9}$$

$$q'' = \frac{p''}{m}t'' + q''_0; \qquad q' = \frac{p'}{m}t' + q'_0,$$

which is a result we have referred to in the main text.

Appendix 2

Consider a number of particles with both translational $\{q_i\}$ and internal $\{\alpha_i\}$ degrees of freedom (2.1.19). Let their interaction potential depend on q and α. We shall write down the matrix element of the scattering of N incoming particles with momenta $\{p'_i\}$ and quantum numbers $\{k'_i\}$ forming N'' outgoing particles with momenta $\{p''_j\}$ and quantum numbers $\{k''_j\}$

$$\langle\{p''_j\}, \{k''_j\}|S|\{p'_i\}, \{k'_i\}\rangle = \prod_{j=1}^{N''}\prod_{i=1}^{N'} \int_{-\infty}^{\infty} dq''_j\, dq'_i \int_0^L d\alpha''_j\, d\alpha'_i$$
$$\times \bar{\psi}_{\{p''_j\},\{k''_j\}}(\{q''_j\}, \{\alpha''_j\}, t'')G(t'', \{q''_j\}, \{\alpha''_j\}|t', \{q'_i\}, \{\alpha'_i\})\psi_{\{p'_i\},\{k'_i\}}(\{q'_i\}, \{\alpha'_i\}, t'). \tag{A.2.1}$$

We represent the propagator on the circle (2.1.20) as a sum of the propagators on an axis (2.1.23) in a periodic potential and exploit the periodic properties of ψ with respect to α. Then (A.2.1) can be written as

$$\prod_{i,j} \int_{-\infty}^{\infty} dq''_j\, dq'_i \int_{-\infty}^{\infty} d\alpha''_j\, d\alpha'_i \bar{\psi}_{\{p''_j\},\{k''_j\}}(\{q''_j\}, \{\alpha''_j\}, t'')\tilde{G}(t'', \{q''_j\}, \{\alpha''_j\}|t', \{q'_i\}, \{\alpha'_i\})\psi_{\{p'_i\},\{k'_i\}}(\{q'_i\}, \{\alpha'_i\}, t'). \tag{A.2.2}$$

The operator kernel $\tilde{G}$ is defined by the contribution of only one stationary phase path in the functional integral, it is the propagator of the corresponding Schroedinger equation with all the variables on an axis. So we have reduced our problem to the scattering problem with all the variables defined on the real axis. There is an appropriate definition for the S-matrices of this kind in the Appendix 1. We shall use corresponding quantum-mechanical formula. Consider the Schroedinger equation for a number of particles in a periodic potential $V(\alpha)$ interacting with each other by some short-range potentials dependent only on the difference of the particles' coordinates. The corresponding S-matrix will be

$$\langle\{k_j''\}|S|\{k_i'\}\rangle = \lim_{\substack{t''\to\infty\\ t'\to-\infty}} \frac{\tilde{G}(t'',\{\alpha_j''\}|t',\{\alpha_i'\})\prod_{j,i}\bar{\psi}_j(0)\psi_i(0)}{\prod_j\sqrt{2\pi\tilde{G}(t'',\alpha_j''|t^0,0)}\prod_i\sqrt{2\pi\tilde{G}(t^0,0|t',\alpha_i')}};$$
$$\alpha_j'' = Lt''/T_{k_j''}, \qquad \alpha_i' = Lt'/T_{k_i'}. \tag{A.2.3}$$

It can be derived in direct analogy with (A.1.8).

The denominator contains the product of the single-particle propagators. At the moments t'', t' their coordinates α are the same as the interacting particles' coordinates in the nominator.

Appendix 3

For the diagram technique with the propagator (3.1.1) we are going to demonstrate the possibility of the replacement of the Green function (3.3.8) by the Green function (3.3.26) on the structureless soliton example. Consider for simplicity a motionless $v = 0$ soliton's propagator (3.0.6)

$$G(t'',0|t',0) = \int \exp\left\{i\int_{t'}^{t''}\mathscr{L}(u_s(x)+\sqrt{\gamma}\varphi)\,dt\right\}\prod d\varphi$$
$$= \exp\{-iM^{cl}(t''-t')\}\int\exp\left\{-\frac{i}{2}\int_{t'}^{t''}d^2x\varphi H\varphi - i\sum_{n=3}^{\infty}\frac{1}{n!}\gamma^{n/2-1}\int v^{(n)}(u_s(x))\varphi^n\,d^2x\right\}\prod d\varphi \tag{A.3.1}$$

$$= \exp\{-iM^{cl}(t''-t')\}\det{}^{-1/2}H\exp\left\{-\frac{i}{2}\int\frac{\delta}{\delta A(x_1,t_1)}R(x_1,t_1|x_2,t_2)\frac{\delta}{\delta A(x_2,t_2)}d^2x_1\,d^2x_2\right\}$$
$$\times\exp\left\{-\frac{i}{\gamma}\int d^2x\left[v(u_s+\sqrt{\gamma}A)-v(u_s)-\sqrt{\gamma}Av'(u_s)-\frac{\gamma}{2}A^2v''(u_s)\right]\right\}\Bigg|_{A=0} \tag{A.3.2}$$

The R is here the same as (3.3.8).

We start by proving the diagrams sum to be kept unchanged by the replacement $R\to R + \alpha u_s'(x_1)u_s'(x_2)$. We shall use the translation invariance for this. The propagator (A.3.1) does not depend on q_0

$$G(t'',-q^0|t',-q^0) = \int\exp\left\{i\int_{t'}^{t''}\mathscr{L}(u_s(x+q^0)+\sqrt{\gamma}\varphi)\right\} = G(t'',0|t',0). \tag{A.3.3}$$

We can expand the $u_s(x+q^0)$ in powers of q^0,

$$u_s(x+q^0) = u_s(x) + \sum_{n=2}^{\infty} \frac{1}{n!} q_0^n u_s^{(n)}(x) + q_0 u_s'(x). \tag{A.3.4}$$

If this expression contained only the last term we would have reached our aim immediately, but the sum $\sum_{n=2}^{\infty} q_0^n u_s^{(n)}(x)/n!$ hampers. Though we can make use of the zero mode orthogonality to the meson modes. Changing the variable φ in (A.3.3) we can transform the second term of (A.3.4) into

$$u_s'(x)\cdot\frac{1}{\|u_s'\|^2}\int \mathrm{d}y u_s'(y) \sum_{n=2}^{\infty}\frac{1}{n!} q_0^n u_s^{(n)}(y) \tag{A.3.5}$$

and obtain for (A.3.3) the expression

$$G(t'',0|t',0) = \int \exp\left\{\mathrm{i}\int_{t'}^{t''} \mathscr{L}(u_s(x) + f(q_0)u_s'(x) + \sqrt{\gamma}\varphi)\,\mathrm{d}t\right\}\prod \mathrm{d}\varphi \tag{A.3.6}$$

$$\|u'\|^2\cdot f(q) = \int \mathrm{d}x u_s'(x)[u_s(x+q) - u_s(x)].$$

Let us pass on to the new independent variable y, assuming $y = f(q^0)/\sqrt{\gamma}$, and rewrite

$$G(t'',0|t',0) = \int\prod\mathrm{d}\varphi \exp\left\{\mathrm{i}\int_{t'}^{t''}\mathscr{L}(u_s)\,\mathrm{d}t - \frac{\mathrm{i}}{2}\int_{t'}^{t''}\varphi H\varphi\,\mathrm{d}^2x - \frac{\mathrm{i}}{\gamma}\int\left[v(u_s + \sqrt{\gamma}(yu_s' + \varphi))\right.\right.$$

$$\left.\left. - v(u_s) - \sqrt{\gamma}(yu_s' + \varphi)\cdot v'(u_s) - \frac{\gamma}{2}(yu_s' + \varphi)v''(u_s)\right]\mathrm{d}^2x\right\} = \exp\{-\mathrm{i}M^{\mathrm{cl}}(t'' - t')\}$$

$$\times \det{}^{-1/2}H \exp\left\{y\int \mathrm{d}^2x u_s'(x)\frac{\delta}{\delta A(x,t)} - \frac{\mathrm{i}}{2}\int\frac{\delta}{\delta A(x_1,t_1)}R(x_1,t_1|x_2,t_2)\cdot\frac{\delta}{\delta A(x_2,t_2)}\mathrm{d}^2x_1\mathrm{d}^2x_2\right\}$$

$$\times\exp\left\{-\frac{\mathrm{i}}{\gamma}\int\mathrm{d}^2x\left[v(u_s(x) + \sqrt{\gamma}A) - v(u_s) - \sqrt{\gamma}Av'(u_s) - \frac{\gamma}{2}A^2v''(u_s)\right]\right\}\Bigg|_{A=0}. \tag{A.3.7}$$

We have got a supplementary factor in (A.3.7) compared to (A.3.2):

$$\exp\left\{y\int\mathrm{d}^2xu_s'(x)\frac{\delta}{\delta A(x,t)}\right\}. \tag{A.3.8}$$

The left-hand side of (A.3.7) does not depend on it. Let us multiply both sides of (A.3.7) by $\exp\{-\mathrm{i}y^2/\alpha\}$ and integrate over y; this will produce an expression for G different from (A.3.2) just by the term

$$\alpha u_s'(x_1)u_s'(x_2). \tag{A.3.9}$$

Passing from (A.3.3) to (A.3.6) we have used the following observations. At t'', t' we have $\varphi = 0$ (3.0.7). So we can separate the variables in two parts: those proportional to the zero mode and a linear combination of the meson modes $\psi_\beta^\pm$ (A.4.5). The meson modes depend on time as

$$\exp\{\pm\, \mathrm{i}m\cosh\beta t\} f_\beta(x). \tag{A.3.10}$$

At $t \to t'(t'')$ the frequency of the integration variables proportional to the meson modes become positive (negative). Being exponentially decreasing at t (remember that $m \to m - \mathrm{i}0$), they are equivalent to zero boundary conditions. This enables us to shift the integration variable passing from (A.3.3) to (A.3.6). Note that the term $f(q_0)u_s'(x)$ can not be removed by the shift of the integration variable. The integration variable proportional to the zero mode is

$$\chi_0(t)u_s'(x), \qquad \chi_0(t'') = \chi_0(t') = 0, \tag{A.3.11}$$

and $f(q_0)$ does not depend on time.

These considerations are correct in general. For a moving soliton we can replace $t \to \tau$, $x \to r$ and repeat these reasons word by word. In the case of two interacting solitons the operator H will have two zero modes u_{ss}' and $\dot u_{ss}$, and to prove the possibility of adding $\alpha\dot u_{ss}\cdot \dot u_{ss} + \beta u_{ss}' u_{ss}' + \gamma \dot u_{ss} u_{ss}'$ to the inverse operator R we have to exploit the translational invariance in space and time. The functional integral calculated in the region of $u_{ss}(x-q, t-t_0)$ does not depend on q_0, t_0. We expand this solution in powers of t^0, q^0 and shift the meson part of the integration variable in order to make all the Taylor series proportional to $f_1(q^0, t^0)\dot u_{ss} + f_2(q^0, t^0)u_{ss}'$. This means that the terms

$$\exp\left\{y_1 \int \dot u_{ss}(x_1, t_1)\frac{\delta}{\delta A(x_1, t_1)}\,\mathrm{d}^2x_1 + y_2\int u_{ss}'(x_2, t_2)\frac{\delta}{\delta A(x_2, t_2)}\,\mathrm{d}^2x_2\right\} \tag{A.3.12}$$

emerge in a formula analogous to (A.3.7). Having integrated this formula with an appropriate quadratic form we shall obtain the desired supplement to the Green function.

Now we shall prove that adding $\beta[u_s'(x_1)u_\varphi^s(x_2t_2) + u_\varphi^s(x_1t_1)u_s'(x_2)]$ (3.3.23) to the Green function (3.3.8) will not alter the sum of the Feynman graphs. Here $u_\varphi^s(x, t) = -tu_s'(x)$. We write $G(t'', 0|t', 0)$ in the form

$$\begin{aligned} G(t'', 0|t', 0) &= \exp\{-\mathrm{i}M^{\mathrm{cl}}\cdot(t''-t')\}\det{}^{-1/2}H \\ &\times \exp\left\{-\frac{\mathrm{i}}{2}\int \mathrm{d}^2x_1\,\mathrm{d}^2x_2\,\frac{\delta^2}{\delta A(1)\delta A(2)}(R(1|2) + \beta[u_s'(1)u_\varphi^s(2) + u_\varphi^s(1)u_s'(2)])\right\} \\ &\times \exp\left\{\frac{\mathrm{i}}{\gamma}\int \mathrm{d}^2x\left[v(u_s(x) + \sqrt{\gamma}A) - \sqrt{\gamma}Av'(u_s) - \frac{\gamma}{2}v''(u_s(x))A^2\right]\right\}. \end{aligned} \tag{A.3.13}$$

This expression does not depend on β. One can make it clear by expanding it in powers of β and comparing this expansion to the expansion (A.3.7) in powers of y.

Appendix 4

In this appendix we shall discuss the plane wave scattering on potential within the Klein–Gordon equation

$$[\Box + v(x, t)]\psi_\beta(x, t) = 0. \tag{A.4.1}$$

We shall assume the potential to satisfy the condition

$$v(x, t) \xrightarrow[|x|\to\infty]{} m^2. \tag{A.4.2}$$

Note, that the Wronskian of two solutions does not depend on time

$$\int \mathrm{d}x \psi_\beta(x,t) \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \psi_\gamma(x,t) = \text{const.} \tag{A.4.3}$$

Let us examine a number of particular types of potentials. At first we consider a time-independent potential

$$[\Box + v(x)]\psi_\beta(x,t) = 0, \qquad v(x) = v''(u_s(x)). \tag{A.4.4}$$

In this case the variables can be separated and the solutions can be found in the form

$$\psi_\beta^\pm(x,t) = \exp\{\pm\, im \cosh \beta t\} f_\beta(x). \tag{A.4.5}$$

The function f satisfies the equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x) - m^2\right] f_\beta(x) = m^2 \sinh^2 \beta f_\beta(x). \tag{A.4.6}$$

All the Schroedinger equation solutions can be easily classified. We denote the discrete spectrum eigenfunctions by

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x) - m^2\right] \varphi_n(x) = -E_n \varphi_n(x); \qquad \int \varphi_n^2\, \mathrm{d}x = 1, \qquad 0 < E_n < m^2. \tag{A.4.7}$$

The last inequality is the stability condition. The continuous spectrum eigenfunctions can always be represented in the form

$$f_\beta(x) \to \begin{cases} \exp\{imx \sinh \beta\}, & x \to -\infty, \\ a(\beta) \exp\{imx \sinh\beta\} + b(\beta)\exp\{-imx \sinh\beta\}, & x \to \infty, \end{cases}$$

$$g_\beta(x) \to \begin{cases} a(\beta) \exp\{-imx \sinh\beta\} - \bar{b}(\beta) \exp\{imx \sinh\beta\}, & x \to -\infty, \\ \exp\{-imx\sinh\beta\}, & x \to +\infty, \end{cases} \tag{A.4.8}$$

$$\bar{a}(\beta) = a(-\beta), \qquad \bar{b}(\beta) = b(-\beta), \qquad |a(\beta)|^2 - |b(\beta)|^2 = 1.$$

These solutions $f_\beta(x)$ and $g_\beta(x)$ are called the Jost functions. Another family of solutions is more appropriate for the description of particles scattering on the potential; these solutions have another asymptotical form:

$$\tilde{f}_\beta(x) \to \begin{cases} \exp\{im \sinh\beta x\} + r_f(\beta)\exp\{-im\sinh\beta x\}, & x \to -\infty, \\ s(\beta)\exp\{im\sinh\beta x\}, & x \to +\infty, \end{cases}$$

$$\tilde{g}_\beta(x) \to \begin{cases} s(\beta)\exp\{-im\sinh\beta x\}, & x \to -\infty \\ \exp\{-im\sinh\beta x\} + r_g(\beta)\exp\{im\sinh\beta x\}, & x \to +\infty, \end{cases} \tag{A.4.9}$$

$$s(\beta) = \frac{1}{a(-\beta)}, \qquad r_f(\beta) = -\frac{\overline{b(\beta)}}{a(-\beta)}, \qquad r_g(\beta) = \frac{b(-\beta)}{a(-\beta)}.$$

Here $s(\beta)$ is the transmission coefficient and $r(\beta)$ is the reflection coefficient.

The S-matrix of the plane wave scattering on the potential

$$S(\beta) = \begin{pmatrix} s(\beta) & r(\beta) \\ -\{s(\beta)/s(-\beta)\}r(-\beta) & s(\beta) \end{pmatrix} \tag{A.4.10}$$

(i.e. the usual particle on a soliton) is unitary, $S^*S = SS^* = I$. Finally, the full list of solutions of the equation (A.4.4) can be written as

$$\begin{aligned} &\exp\{\pm it(m^2 - E_n)^{1/2}\}\varphi_n(x); \qquad tu_s'(x), \qquad u_s'(x), \\ &\psi_\beta^{(-)} = \exp\{-im\cosh\beta t\}f_\beta(x); \qquad \psi_\beta^{(+)} = \exp\{im\cosh\beta t\}g_\beta(x). \end{aligned} \tag{A.4.11}$$

The potential which appears in the calculations of the quantum corrections for the moving soliton depends not on x but on $r = x\cosh\varphi - t\sinh\varphi$ and the equation (A.4.1) with this potential has the same solutions as (A.4.8) but with x and t replaced by $r = x\cosh\varphi - t\sinh\varphi$ and $\tau = t\cosh\varphi - x\sinh\varphi$ respectively.

Consider now another type of potential, the periodic one:

$$[\Box + v(x,t)]\psi_\beta = 0; \qquad v(x,t) = v''(w(x,t)); \qquad v(x,t+T) = v(x,t). \tag{A.4.12}$$

We choose the Floquet solutions of the equation (A.4.12):

$$\psi_\nu(x, t+T) = \exp\{-i\nu\}\psi_\nu(x,t). \tag{A.4.13}$$

The Floquet indexes may form a discrete and a continuous spectrum. The discrete solutions and their indexes we denote by $\psi_n(x,t)$ and ν_n respectively,

$$0 < \nu_n < mT; \qquad \int |\psi_n(x,t)|^2\,\mathrm{d}x < \infty. \tag{A.4.14}$$

To describe the continuous spectrum Floquet solutions we write ψ_ν in the form:

$$\psi_\nu(x,t) = \sum_{n=-\infty}^{\infty} \exp\left\{-i\frac{\nu + 2\pi n}{T}t\right\} f_{\nu,n}(x). \tag{A.4.15}$$

Substituting this into the equation (A.4.12), we obtain an equation for the vector $f(x) = \{f_n(x)\}$

$$\begin{aligned} &\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \hat{k}_\nu^2 + \hat{v}\right)f = 0; \\ &v_{nl}(x) = \frac{1}{T}\int_0^T \mathrm{d}t \exp\left\{i\frac{2\pi t}{T}(n-l)\right\} v(x,t) - \delta_{ln}m^2; \\ &k_{ln} = \left(\left(\frac{\nu + 2\pi n}{T}\right)^2 - m^2\right)^{1/2} \delta_{ln}. \end{aligned} \tag{A.4.16}$$

We can suppose that $mT < \nu < mT + 2\pi$.

Let us combine the complete set of the vector solutions into one matrix solution f. The second index of this matrix is just a sequential number of a vector solution f. The solutions of the equation (A.4.12) can be expressed through this matrix solution. We define the vector $\boldsymbol{u}(t)$ with the components

$$u_n(t) = \exp\left\{-\mathrm{i}\frac{\nu + 2\pi n}{T}t\right\}. \tag{A.4.17}$$

The solution (A.4.15) can be rewritten in these notations as uf. Note that the spectral parameter ν enters the equation (A.4.16) in a non-trivial manner.

The complete set of solutions which describes the scattering for this equation looks as follows

$$\begin{aligned} &\hat{f}^{\nu}_{\pm}(x) \to \begin{cases} \exp\{\pm\mathrm{i}\hat{k}x\} + \exp\{\mp\mathrm{i}\hat{k}x\}\cdot R_{f\pm}(\nu), & x \to -\infty, \\ \exp\{\pm\mathrm{i}\hat{k}x\}D_{f\pm}(\nu), & x \to +\infty, \end{cases} \\ &\hat{g}^{\nu}_{\pm}(x) \to \begin{cases} \exp\{\pm\mathrm{i}\hat{k}x\}D_{g\pm}(\nu), & x \to -\infty, \\ \exp\{\pm\mathrm{i}\hat{k}x\} + \exp\{\mp\mathrm{i}\hat{k}x\}R_{g\pm}(\nu), & x \to \infty. \end{cases} \end{aligned} \tag{A.4.18}$$

It was required in section 4 (4.3.17) that in the scattering of a positive-frequency wave on a periodic soliton only positive frequency waves can outcome. This means that the matrix elements R_{ln}, D_{ln} are non-zero only when $l > (mT - \nu)/2\pi$ and $n > (mT - \nu)/2\pi$. We shall call the matrix

$$\hat{S} = \begin{pmatrix} D_{f+} & R_{g-} \\ R_{f+} & D_{g-} \end{pmatrix} \tag{A.4.19}$$

the scattering matrix of a plane wave on a periodic potential. The negative-frequency scattering waves deserve no special attention; they are obtained simply by complex conjugation of the positive-frequency solutions.

The equation (A.4.16) can be supplied with its Wronskian which is independent of x:

$$f_1^+(\nu)\frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}x}f_2(\nu) = \text{const.} \tag{A.4.20}$$

Here the matrices f_1 and f_2 are the solutions of the equation (A.4.16) with the same index ν. Substituting (A.4.18) into (A.4.20) we can see the restrictions on the S-matrix elements

$$\begin{cases} D^+_{f\pm}\hat{k}D_{f\pm} + R^+_{f\pm}\hat{k}R_{f\pm} = \hat{k}; \\ D^+_{g\pm}\hat{k}D_{g\pm} + R_{g\pm}\hat{k}R_{g\pm} = \hat{k}; \\ R^+_{f\mp}\hat{k} = \hat{k}R_{f\pm};\ R^+_{g\mp}\hat{k} = \hat{k}R_{g\pm}; \\ \hat{k}D_{g\pm} = D^+_{f\pm}\hat{k};\ D^+_{f\pm}\hat{k}R_{g\mp} + R^+_{f\pm}\hat{k}D_{g\mp} = 0; \end{cases} \tag{A.4.21}$$

$$\Downarrow$$

$$\begin{cases} D_{g+}D_{f+} + R_{f-}R_{f+} = I, \\ D_{g-}D_{f-} + R_{f+}R_{f-} = I. \end{cases}$$

To describe the homogeneous equation (A.4.1) solutions for a moving periodic soliton, i.e. when the potential

$$v(x,t) = v''(w(r,\tau)), \qquad \begin{aligned} r &= x\cosh\varphi - t\sinh\varphi; \\ \tau &= t\cosh\varphi - x\sinh\varphi; \end{aligned} \tag{A.4.22}$$

is periodic with respect to τ and tends rapidly to a constant at $r \to \infty$, we have to replace $x \to r$ and $t \to \tau$ in the solutions written above.

In the $\sin\varphi_2$ model all the solutions for the plane wave-periodic soliton scattering are known due to the comment in [9]

$$\psi_\beta^- = \frac{\frac{1}{2}\exp\{-\mathrm{i}m(t\cosh\beta - x\sinh\beta)\}}{\cosh^2(mx\sin\theta) + \tan^2\theta\cdot\sin^2(mt\cos\theta)}\Bigg\{\cosh(xm\sin\theta)\cdot\Bigg[\exp\{-mx\sin\theta\} + \left(\frac{\sinh\beta + \mathrm{i}\sin\theta}{\sinh\beta - \mathrm{i}\sin\theta}\right)^2\exp\{mx\sin\theta\}\Bigg] - \mathrm{i}\tan^2\theta\cdot\sin(mt\cos\theta) \Bigg[\left(\frac{\mathrm{e}^{\beta-\mathrm{i}\theta}+1}{\mathrm{e}^{\beta-\mathrm{i}\theta}-1}\right)^2\cdot\exp\{\mathrm{i}mt\cos\theta\} - \left(\frac{\mathrm{e}^{\beta+\mathrm{i}\theta}-1}{\mathrm{e}^{\beta+\mathrm{i}\theta}+1}\right)^2\cdot\exp\{-\mathrm{i}mt\cos\theta\}\Bigg]\Bigg\}. \tag{A.4.23}$$

This makes clear that the plane wave-periodic soliton S-matrix is diagonal,

$$R = 0, \qquad [D_{f+}(v)]_{ln} = [\bar{D}_{f-}(v)]_{ln} = \delta_{ln}a(\beta)_n;$$
$$mT\cosh\beta_n = v + 2\pi n > mT \tag{A.4.24}$$
$$a(\beta) = \left(\frac{\sinh\beta + \mathrm{i}\sin\theta}{\sinh\beta - \mathrm{i}\sin\theta}\right)^2.$$

In the end of this appendix we write down the solutions of the homogeneous equation (A.4.1) in the sin φ_2 case of soliton–antisoliton scattering, whit rapidity $\varphi_+ - \varphi_- = 2\varphi$,

$$\psi_\beta^-(x,t) = \frac{\frac{1}{2}\exp\{-\mathrm{i}m(t\cosh\beta - x\sinh\beta)\}}{\cosh^2(mx\cosh\varphi) + \sinh^2(mt\sinh\varphi)/\tanh^2\varphi}\cdot\Bigg\{\cosh(m\cosh\varphi x) \times\Bigg[\exp\{-m\cosh\varphi x\} + \left(\frac{\sinh(\beta-\varphi)+\mathrm{i}}{\sinh(\beta-\varphi)-\mathrm{i}}\right)\left(\frac{\sinh(\beta+\varphi)+\mathrm{i}}{\sinh(\beta+\varphi)-\mathrm{i}}\right)\exp\{m\cosh\varphi x\}\Bigg] + \frac{1}{\tanh^2\varphi}\sinh(mt\sinh\varphi)\Bigg[\left(\frac{\sinh(\beta+\varphi)+\mathrm{i}}{\sinh(\beta+\varphi)-\mathrm{i}}\right)\exp\{mt\sinh\varphi\} - \left(\frac{\sinh(\beta-\varphi)+\mathrm{i}}{\sinh(\beta-\varphi)-\mathrm{i}}\right)\exp\{-mt\sinh\varphi\}\Bigg]\Bigg\}. \tag{A.4.25}$$

Appendix 5

In this appendix we construct an operator inverse to

$$H = \Box + v(x,t); \qquad v(x,t)\underset{|x|\to\infty}{\to} m^2. \tag{A.5.1}$$

The resolvent $R(t_2, x_2|t_1, x_1)$, i.e. the kernel of the operator R, inverse to H will be supposed to be hermitian, $R(1,2) = R^*(2,1)$. It is uniquely defined by the boundary conditions.

We are interested in two types of such boundary conditions:

1. the zero boundary conditions for finite times

$$R(t_2, x_2|t_1, x_1)|_{t_2=t',t''} = 0.$$

2. the Feynman boundary conditions which we interprete as the decrease at the infinity with time taking into account the prescription $m^2 \to m^2 - \mathrm{i}0$; the resolvent R at $t_2 \to \infty$ $(-\infty)$ ought to contain only negative (positive) frequencies.

We shall express the resolvent through the two sets of solutions of the homogeneous equation

$$H\psi_\beta^{(+)} = 0; \qquad H\psi_\beta^{(-)} = 0. \tag{A.5.2}$$

These complete sets of solutions ψ_β^+ and ψ_β^- are either positive (negative) frequency solutions that satisfy the boundary conditions of the second type at large positive (negative) time or they are the functions that turn into zero at t'', t' for the first type of the boundary conditions.

We shall call each of these sets the matrix solution. The first index of a matrix is the spatial coordinate x, the second is the wave number of the solution β which may vary continuously or "discretely" (with the corresponding solution square integrable in x). In terms of two matrix solutions the resolvent can be expressed as follows

$$\hat{R}(t_2|t_1) = \begin{cases} \hat{\psi}^-(t_2)\cdot \hat{W}^{-1}\hat{\psi}^{+\mathrm{T}}(t_1), & t_2 > t_1, \\ \hat{\psi}^+(t_2)\cdot \hat{W}^{\mathrm{T}^{-1}}\cdot \hat{\psi}^{-\mathrm{T}}(t_1) & t_2 < t_1. \end{cases} \tag{A.5.3}$$

It satisfies the equation

$$HR = \delta(t_2 - t_1)\cdot \hat{I}, \qquad \hat{I} = \delta(x_2 - x_1). \tag{A.5.4}$$

We have used the Wronskian of the solutions $\hat{\psi}^+$ and $\hat{\psi}^-$

$$W = \left[\psi_+^{\mathrm{T}} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t}\hat{\psi}^-\right]. \tag{A.5.5}$$

It does not depend on time due to the equations (A.5.2).

The formula (A.5.3) is well known in the mathematical physics, it can be elucidated in such a way: at $t_1 \neq t_2$ the function $\hat{R}$ satisfies the equation (A.5.2), and the boundary conditions are satisfied. The calculation of the first derivative jump at $t_1 = t_2$

$$\partial_{t_2}\hat{R}(t_2|t_1)\Big|_{t_2=t_1-0}^{t_2=t_1+0} = \delta(x_1 - x_2) \tag{A.5.6}$$

makes out an unite operator, i.e. R does really satisfy the equation

$$HR = I \tag{A.5.7}$$

with the correct boundary conditions. To find the resolvent explicitly it is sufficient to calculate the Wronskian. We shall do this for three important cases. First of all we take the structureless soliton. The corresponding complete set of the homogeneous equation solutions one can find in the Appendix 4.

Let us construct the resolvent which is zero at t'' and t'. The solutions which are asymptotically zero at t'' are

$$\hat{\psi}^-(t) = \{\exp\{\mathrm{i}m\cosh\beta\cdot t\}\tilde{f}_\beta(x), \exp\{-\mathrm{i}t\sqrt{m^2 - E_n}\}\,\varphi_n(x), (t - t'')u_s'(x)\}, \tag{A.5.8}$$

and the solutions which are zero at t' are

$$\hat{\psi}^+(t) = \{\exp\{\mathrm{i}m\cosh\beta\cdot t\}\cdot\tilde{g}_\beta(x), \exp\{\mathrm{i}t\sqrt{m^2 - E_n}\}\,\varphi_n(x), (t - t')u_s'(x)\}. \tag{A.5.9}$$

We have accounted for the fact that $m^2 \to m^2 - \mathrm{i}0$ and hence we can regard the oscillating solutions to be exponentially decreasing. The Wronskian of these two sets can be easily calculated, and the non-zero elements are:

$$\int \mathrm{d}x(\exp\{imt\cosh\gamma\}\,\tilde{g}_\gamma(x))\overleftrightarrow{\partial}_t(\exp\{-imt\cosh\beta\}\,\tilde{f}_\beta(x)) = -4\pi i s(\beta)\delta(\gamma-\beta);$$

$$\int \mathrm{d}x(\exp\{-i\sqrt{m^2-E_n}\cdot t\}\,\varphi_n(x))\overleftrightarrow{\partial}_t(\exp\{i\sqrt{m^2-E_n}\cdot t\}\,\varphi_n(x)) = 2i\sqrt{m^2-E_n}; \tag{A.5.10}$$

$$\int \mathrm{d}x(t-t'')u_s'(x)\overleftrightarrow{\partial}_t(t-t')u_s'(x) = -(t''-t')\|u_s'\|^2.$$

Finally the resolvent expression is:

$$R(x_2,t_2|x_1,t_1) = \frac{u_s'(x_2)u_s'(x_1)}{2\|u_s'\|^2}\left\{\frac{t_2t_1}{\Delta} - \frac{\Sigma}{\Delta}(t_2+t_1) + \frac{\Sigma^2}{\Delta} - \Delta + |t_2-t_1|\right\}$$
$$+\frac{i}{8\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\beta}{s(\beta)}\cdot\exp\{-im\cosh\beta|r_2-t_1|\}\cdot\{\tilde{f}_\beta(x_2)\tilde{g}_\beta(x_1)+\tilde{g}_\beta(x^2)\tilde{f}_\beta(x_1)\}$$
$$+\frac{i}{2}\sum_n\frac{\exp\{i\sqrt{m^2-E_n}|t_2-t_1|\}}{\sqrt{m^2-E_n}}\varphi_n(x_2)\varphi_n(x_1). \tag{A.5.11}$$

Here we denote $\Delta = \frac{1}{2}(t''-t')$, $\Sigma = \frac{1}{2}(t''+t')$.

In the case when the potential in (A.5.1) arises from the calculations of corrections for the moving soliton and depends on $r = x\cosh\varphi - t\sinh\varphi$, the resolvent ought to be calculated from this by the replacement $x \to r$, $t \to \tau$.

We turn now to another particular potential type, the periodic potential, which is to be considered in the calculation of corrections for the periodic soliton. We shall write down the resolvent for the motionless soliton; the resolvent for the moving soliton can be obtained by a Lorentz transformation.

Let us calculate the Wronskian of the positive- and negative-frequency sets of solutions. Note that only a determinant of those solutions may be different from zero which have the opposite Floquet indexes values. The positive-frequency solutions are; see Appendix 4 (A.4.18)

$$(\boldsymbol{u}_\nu(t)\hat{g}_{+\nu}(x))^+, \qquad (\boldsymbol{u}_\nu(t)\cdot\hat{g}_{-\nu}(x))^+ \tag{A.5.12}$$

and the negative-frequency solutions are given by

$$(\boldsymbol{u}_\nu(t)\hat{f}_{+\nu}(x)), \qquad (\boldsymbol{u}_\nu(t)\hat{f}_{-\nu}(x)), \tag{A.5.13}$$

The Wronskian of the continuous spectrum solution is equal to

$$\int \mathrm{d}x(\hat{g}^+_{\pm\nu_1}\cdot\boldsymbol{u}_{\nu_1})\overleftrightarrow{\partial}_t(\boldsymbol{u}_{\nu_2}\hat{f}_{\pm\nu_2}) = -4\pi i T\hat{k}(\nu_1)D_{f\pm}(\nu_1)\delta(\nu_1-\nu_2). \tag{A.5.14}$$

To derive this result we write first of all:

$$(\boldsymbol{u}_\nu(t))_n = \exp\{-i(\sqrt{k_\nu^2+m^2})_{nn}t\}. \tag{A.5.15}$$

We can replace in the integral (A.5.14)

$$(u^+_{v_1})_l \vec{\partial}_t (u_{v_2})_n = -\mathrm{i}\left(\sqrt{k_1^2+m^2}+\sqrt{k_2^2+m^2}\right)_{ln}. \tag{A.5.16}$$

We use here the fact that (A.5.14) does not depend on time (A.4.3). To evaluate the integral

$$-\mathrm{i}\int \mathrm{d}x \hat{g}^+_{\pm v_1}\left(\sqrt{\hat{k}_1^2+m^2}+\sqrt{\hat{k}_2^2+m^2}\right)\hat{f}_{\pm v_2} \tag{A.5.17}$$

we subtract the two equalities

$$\begin{aligned} &\hat{g}_1^+\left[-\frac{\overrightarrow{\mathrm{d}^2}}{\mathrm{d}x^2}-\hat{k}_2^2+\hat{v}\right]\hat{f}_2 = 0;\\ &\hat{g}_1^+\left[-\frac{\overleftarrow{\mathrm{d}^2}}{\mathrm{d}x^2}-\hat{k}_1^2+\hat{v}\right]\hat{f}_2 = 0, \end{aligned} \tag{A.5.18}$$

$$\int \mathrm{d}x \hat{g}_1^+\left(\sqrt{\hat{k}_1^2+m^2}+\sqrt{\hat{k}_2^2+m^2}\right)\hat{f}_2 = \frac{T}{(v_1-v_2)}\left(\hat{g}_1^+ \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}x}\hat{f}_2\right)\Big|_{-\infty}^{\infty}. \tag{A.5.19}$$

Tending $x \to \infty$, we make use of the equality

$$\lim_{x\to\infty}\frac{1}{(v_1-v_2)}\exp\{\mathrm{i}x(v_1-v_2)\} = \mathrm{i}\pi\delta(v_1-v_2) \tag{A.5.20}$$

and of the identities (A.4.21). We arrive at the right-hand side of (A.5.14). Recollecting the zero modes and the discrete spectrum we can write the final expression for the resolvent:

$$\begin{aligned} R(t_2,x_2|t_1,x_1) = {} & \frac{\mathrm{i}}{4\pi T}\int_{mT}^{mT+2\pi} \mathrm{d}v\, \boldsymbol{u}_v(t_2)\hat{f}_{+v}(x_2)D_{f+}^{-1}(v)\hat{k}^{-1}(v)\hat{g}^+_{+v}(x_1)\boldsymbol{u}_v^+(t_1)\\ & + \frac{\mathrm{i}}{4\pi T}\int_{mT}^{mT+2\pi} \mathrm{d}v\, \boldsymbol{u}_v(t_2)\hat{f}_{-v}(x_2)D_{f-}^{-1}(v)\hat{k}^{-1}(v)\hat{g}_v(x_1)\boldsymbol{u}_v^+(t_1)\\ & + \sum_n \frac{\psi_n^-(t_2,x_2)\psi_n^+(t_1,x_1)}{[\psi_n^+\vec{\partial}_t\psi_n^-]} + \frac{w_\varphi(x_2,t_2)w'(x_1t_1)-w'(x_2t_2)w_\varphi(x_1t_1)}{2[w'\vec{\partial}_t w_\varphi]}\\ & + \frac{w_T(x_2,t_2)\dot{w}(x_1,t_1)-\dot{w}(x_2,t_2)w_T(x_1,t_1)}{2[\dot{w}\vec{\partial}w_T]}; \qquad t_2 > t_1. \end{aligned} \tag{A.5.21}$$

The square brackets mean

$$[f\vec{\partial}_t g] = \int \mathrm{d}x f\vec{\partial}_t g. \tag{A.5.22}$$

The function w is the classical periodic soliton, its derivatives $w' = \mathrm{d}w/\mathrm{d}x$, $\dot{w} = \mathrm{d}w/\mathrm{d}t$ are the zero modes and the growing zero modes are $w_\varphi = \mathrm{d}w/\mathrm{d}\varphi$, $w_T = \mathrm{d}w/\mathrm{d}T$.

Consider now the third case of the potential $v''(u_{ss})$ in the H operator. Here u_{ss} is the two soliton scattering solution.

The continuous spectrum solutions have been examined in Appendix 4. The discrete spectrum in an asymptotical state can be localized only in the neighbourhood of each soliton. The Wronskian of two discrete spectrum solutions which are localized near different solitons is equal to zero; if the solutions are concentrated near the same soliton, the Wronskian is the same as in the single-particle case.

The non-zero Wronskians of all solutions are (5.05)

$$\int \mathrm{d}x \psi_{+\beta} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \tilde{\psi}_{-\gamma} = -4\pi \mathrm{i} \delta(\beta - \gamma) a_1(\beta - \varphi_1) a_2(\beta - \varphi_2), \tag{A.5.23}$$

$$\int \mathrm{d}x \psi_{+\beta}(x,t) \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}t} \psi_{\gamma-}(x,t) = -4\pi \mathrm{i}(1 + c)_{\gamma\beta} a_1(\beta - \varphi_1) a_2(\beta - \varphi_2). \tag{A.5.24}$$

We consider here the simplest case when both the soliton potentials are reflectionless $b_1 = b_2 = 0$. The solutions which are square integrable over x are localized in the region of each soliton and do not overlap,

$$\begin{aligned} &\psi_{1n}(x,t) \underset{t\to-\infty}{\to} \exp\{-\mathrm{i}\sqrt{m^2 - E_n}\tau_1\} \varphi_{1n}(r_1), \\ &\psi_{2n}(x,t) \underset{t\to-\infty}{\to} \exp\{-\mathrm{i}\sqrt{m^2 - E_n}\tau_2\} \varphi_{2n}(r_2), \\ &\tau_{1,2} = t\cosh\varphi_{1,2} - x\sinh\varphi_{1,2}, \qquad \tau_{1,2} = x\cosh\varphi_{1,2} - t\sinh\varphi_{1,2}. \end{aligned} \tag{A.5.25}$$

The final expression of the resolvent with an account of zero modes and the notations φ_1, φ_2 being the solitons, rapidities, is (4.2.3)

$$\begin{aligned} R(t'', x''|t', x') = {} & \frac{\mathrm{i}}{4\pi} \int \frac{\mathrm{d}\beta\,\mathrm{d}\gamma}{a_1(\beta - \varphi_1) a_2(\beta - \varphi_2)} \psi_\gamma^-(x'', t'')(1 + c)_{\beta\gamma}^{-1} \psi_\beta^+(x't') \\ & + \frac{\mathrm{i}}{2} \sum_n \frac{\exp\{-\mathrm{i}\sqrt{m^2 - E_{1n}}(\tau_1'' - \tau_1')\}}{\sqrt{m^2 - E_{1n}}} \varphi_{n_1}(r_1'')\varphi_{n_2}(r_1') \\ & + \frac{\mathrm{i}}{2} \sum_n \frac{\exp\{-\mathrm{i}\sqrt{m^2 - E_{2n}}(\tau_2'' - \tau_2')\}}{\sqrt{m^2 - E_{2n}}} \varphi_{2n}(r_2'')\varphi_{2n}(r_2') \\ & + \frac{[\cosh\varphi_1 \dot{u}_{ss}(x'',t'') + \sinh\varphi_1 u'_{ss}(x_1''t'')] u^{ss}_{\varphi 2}(x',t')}{2\|u'_{2s}\|^2 \sinh(\varphi_1 - \varphi_2)} - \frac{u^{ss}_{\varphi 2}(x'',t'')}{2\|u'_{2s}\|^2 \sinh(\varphi_1 - \varphi_2)} \\ & \times [\cosh\varphi_1 u_{ss}(x',t') + \sinh\varphi_1 u'_{ss}(x',t')] - \frac{\cosh\varphi_2 \cdot u_{ss}(x'',t'') + \sinh\varphi_2 \cdot u'_{ss}(x'',t'')}{2\|u'_{1s}\|^2 \cdot \sinh(\varphi_1 - \varphi_2)} \\ & \times u^{ss}_{\varphi_1}(x',t') + \frac{u^{ss}_{\varphi_1}(x'',t'')[\cosh\varphi_2 \dot{u}_{ss}(x',t') + \sinh\varphi_2 u'_{ss}(x',t')]}{2\|u'_{1s}\|^2 \sinh(\varphi_1 - \varphi_2)}, \quad t'' > t', \end{aligned} \tag{A.5.26}$$

$$\|u'_{1s}\|^2 = \int [u'_{1s}(x)]^2\,\mathrm{d}x, \qquad \|u'_{2s}\|^2 = \int [u'_{2s}(x)]^2\,\mathrm{d}x,$$

$$\dot{u} = \mathrm{d}u/\mathrm{d}t, \qquad u' = \mathrm{d}u/\mathrm{d}x, \qquad u_{\varphi_1} = \mathrm{d}u/\mathrm{d}\varphi_1, \qquad u_{\varphi_2} = \mathrm{d}u/\mathrm{d}\varphi_2.$$

Appendix 6

Let us calculate the spectral density for the structureless soliton case:

$$\rho(\beta) = \int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{\psi_\beta^+ \overleftrightarrow{\partial}_t \psi_\beta^-}{a(\beta)} - \psi_{0\beta}^+ \overleftrightarrow{\partial}_t \psi_{0\beta}^- \right] \tag{A.6.1}$$

We shall find it to be [114]

$$\rho(\beta) = -2\, \mathrm{d} \ln a(\beta)/\mathrm{d}\beta. \tag{A.6.2}$$

We start from

$$\rho(\beta) = -2\mathrm{i}m \cosh\beta \int_{-\infty}^{\infty} \mathrm{d}x \left[\frac{f_\beta(x) g_\beta(x)}{a(\beta)} - 1 \right] \tag{A.6.3}$$

The functions $f_\beta(x)$ and $g_\beta(x)$ satisfy the Schroedinger equation of Appendix 4, (A.4.6), (A.4.8)

$$\begin{aligned} &\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x) - m^2 \right] f_\beta = \lambda f_\beta; \\ &\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x) - m^2 \right] g_\beta = \lambda g_\beta; \qquad \lambda = m^2 \sinh^2\beta. \end{aligned} \tag{A.6.4}$$

We differentiate the first equation with respect to λ and multiply the result by g_β from the left-hand side and subtract from it the second equation multiplied by $\dot{f}_\beta = \mathrm{d}f_\beta/\mathrm{d}\lambda$ from the right-hand side:

$$f_\beta(x) g_\beta(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left(\dot{f}_\beta \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}x} g_\beta \right). \tag{A.6.5}$$

After integration of this equality and subtraction from it of the analogous combination constructed of the free equation solutions (i.e. with $v(x) = m^2$), we get

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{f_\beta(x) g_\beta(x)}{a(\beta)} - 1 \right) &= \frac{1}{a(\beta)} \left(\dot{f}_\beta(x) \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}x} g_\beta(x) \right) \Bigg|_{-\infty}^{\infty} \\ &\quad - \left(\frac{\mathrm{i}x}{2m \sinh\beta} \exp\{\mathrm{i}m \sinh\beta x\} \frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}x} \exp\{-\mathrm{i}m \sinh\beta x\} \right) \Bigg|_{-\infty}^{\infty}. \end{aligned} \tag{A.6.6}$$

Next we substitute the asymptotics of f_β and g_β (A.4.8) and get

$$\int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{f_\beta(x) g_\beta(x)}{a(\beta)} - 1 \right) = -\frac{\mathrm{i}}{m \cosh\beta} \frac{\mathrm{d} \ln a(\beta)}{\mathrm{d}\beta}. \tag{A.6.7}$$

The substitution of this expression into (A.6.3) leads to the result (A.6.2).

The spectral density is needed for evaluation of the traces of the functions of the Schroedinger operator, i.e. of the integrals of type

$$\int_{-\infty}^{\infty} F(\beta)\rho(\beta)\,\mathrm{d}\beta; \qquad F(\beta) = F(-\beta). \tag{A.6.8}$$

This integral is equal to

$$\int_{0}^{\infty} \mathrm{d}\beta F(\beta)\,[\rho(\beta) + \rho(-\beta)]. \tag{A.6.8}$$

Since β and $-\beta$ correspond to the same spectral point λ, the full spectral density is

$$\rho(\beta) + \rho(-\beta) = -2\frac{\mathrm{d}}{\mathrm{d}\beta}\ln\frac{a(\beta)}{a(-\beta)} = -2\frac{\mathrm{d}}{\mathrm{d}\beta}\ln\det\hat{S}_\beta. \tag{A.6.10}$$

Here $\hat{S}_\beta$ is the scattering matrix (A.4.10).

But this expression for the spectral density is to be used with care. For example, calculating the structureless soliton mass corrections we have to evaluate the expressions of type

$$\operatorname{tr}\hat{F} = \int_{-\infty}^{\infty} \mathrm{d}\beta F(\beta) \int \mathrm{d}x \left[\frac{\psi_\beta^+ \overleftrightarrow{\partial}_+ \psi_\beta^-}{a(\beta)} - \psi_{0\beta}^+ \overleftrightarrow{\partial}_t \psi_{0\beta}^-\right]. \tag{A.6.11}$$

The first term here is the sum over the homogeneous equation solutions $H\psi = 0$ and the second is the sum over the solutions of the free (i.e. with $v(x) = m^2$) homogeneous equation $H_0\psi = 0$ (the vacuum oscillations).

To evaluate (A.6.11) we have to sum over the same number of solutions of the free equation and of the equation with the potential. In order to perform this accurately and in details we shall use a cut-off in the integration over β. Then the expression (A.6.11) will be rewritten so:

$$\operatorname{tr} F_\Lambda = \int_0^\Lambda \mathrm{d}\beta F(\beta)\left[-2\frac{\mathrm{d}}{\mathrm{d}\beta}\ln\frac{a(\beta)}{a(-\beta)}\right]. \tag{A.6.12}$$

However it is obvious that if we shall tend now $\Lambda \to \infty$ we shall sum over the different numbers of solutions of the free equation and the equation with a potential. To find the difference of these solutions numbers we assume $F = 1$ and derive this difference to be

$$-2\ln\frac{a(\Lambda)}{a(-\Lambda)}. \tag{A.6.13}$$

This makes clear that the formula (A.6.12) must be improved as follows

$$\operatorname{tr} F_0 = -2\int_0^\Lambda \mathrm{d}\beta F(\beta)\frac{\mathrm{d}}{\mathrm{d}\beta}\ln\frac{\alpha(\beta)}{a(-\beta)} + 2F(\Lambda)\ln\frac{a(\Lambda)}{a(-\Lambda)} = 2\int_0^\Lambda \mathrm{d}\beta\frac{\mathrm{d}F(\beta)}{\mathrm{d}\beta}\ln\frac{a(\beta)}{a(-\beta)}. \tag{A.6.14}$$

The last term makes a finite contribution only for $F(\beta)\xrightarrow[\beta\to\infty]{} e^\beta$ and this is just the case to be considered at (5.2.10). As a matter of fact, at $\beta \to \infty$ the quantity $\ln \alpha(\beta)$ tends to

$$\ln a(\beta) \to \frac{a^\infty}{\sinh\beta}, \qquad a^\infty = \frac{1}{2im}\int_{-\infty}^{\infty} \mathrm{d}x(v''(u_s(x)) - m^2). \tag{A.6.15}$$

For the periodic soliton in its center of mass system we calculate (5.1.8), (5.3.4)

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t''}\left[\mathrm{Tr}_{t'}^{t''}\ \ln H\left(w\left(x,\frac{t}{T},T\right)\right)\right]\Bigg|_{\frac{t''-t'}{T}} = \\ = \frac{1}{8\pi T}\int_{mT}^{mT+2\pi}\mathrm{d}\nu\int_{-\infty}^{\infty}\mathrm{d}x\sum_{j=\pm}\mathrm{Sp}\,[(\hat g_j^+(x)\boldsymbol{u}^+(t)\overleftrightarrow{\partial}_t\boldsymbol{u}(t)\hat f_j(x))\sqrt{\hat k_\nu^2+m^2}\,D_{fj}^{-1}(\nu)\cdot\hat k^{-1}]. \tag{A.6.16}$$

Let us differentiate $\boldsymbol{u}(t)$ in detail. It is clear that

$$\boldsymbol{u}_{\nu l}^+\overleftrightarrow{\partial}_t\boldsymbol{u}_{\nu n} = -2\mathrm{i}(\sqrt{k^2+m^2})_{ln}\,. \tag{A.6.17}$$

We use here the fact that $(\mathrm{d}/\mathrm{d}t)\,[g_j^+(x)u^+\overleftrightarrow{\partial}_t uf_j] = 0$ (A.4.3), (A.5.12), (A.5.13).

The formula analogous to (A.6.16) for the free operator H_0 is

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t''}\left[\mathrm{Tr}_{t'}^{t''}\ \ln H_0\right] = \frac{1}{8\pi T}\int_{mT}^{mT+2\pi}\mathrm{d}\nu\int_{-\infty}^{\infty}\mathrm{d}x\sum_{j=\pm}\mathrm{Sp}\,[(\exp\{-\mathrm{i}\hat k x\cdot j\}\boldsymbol{u}^+\overleftrightarrow{\partial}\boldsymbol{u} \\ \times\exp\{\mathrm{i}\hat k x\cdot j\})\sqrt{\hat k_\nu^2+m^2}\cdot\hat k^{-1}]. \tag{A.6.18}$$

So, we have to evaluate the integral

$$\int\mathrm{d}x[D_{f\pm}^{-1}\cdot\hat k^{-1}\cdot\hat g_\pm^+(x)\sqrt{\hat k^2+m^2}\cdot f_\pm - \hat k^{-1}\sqrt{\hat k^2+m^2}] = \hat\rho(\nu). \tag{A.6.19}$$

We consider the equality (A.5.19). Previously we have extracted an infinite term from the right-hand side of this equality, but it also contains a finite term that we need now. We expand the matrices $D_{f(g)}$ in the right-hand side of (A.5.19) into the series of $(\nu_1-\nu_2)$ powers:

$$\int\mathrm{d}x\hat g_\pm^+(\nu_1)(\sqrt{\hat k_1^2+m^2}+\sqrt{\hat k_2^2+m^2})f_\pm(\nu_2) = \frac{T}{\nu_1-\nu_2}\Bigg[(\exp\{\mp\mathrm{i}\hat k_1 x\} \\ + R_{g\,\pm}^+(\nu_1)\exp\{\pm\mathrm{i}\hat k_1 x\})\frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}x}\left(\exp\{\pm\mathrm{i}\hat k_2 x\}D_{f+}\left(\frac{\nu_1+\nu_2}{2}-\left(\frac{\nu_1-\nu_2}{2}\right)\right)\right) \\ -\left(D_{\pm g}^+\left(\frac{\nu_1+\nu_2}{2}+\left(\frac{\nu_1-\nu_2}{2}\right)\right)\times\exp\{\mp\mathrm{i}\hat k_1 y\}\right)\frac{\overleftrightarrow{\mathrm{d}}}{\mathrm{d}y}(\exp\{\pm\mathrm{i}\hat k_2 y\}+\exp\{\mp\mathrm{i}\hat k_2 y\}R_{f\pm}(\nu_2))\Bigg], \\ x\to\infty,\ y\to-\infty. \tag{A.6.20}$$

At last we obtain the following expression of (A.6.19):

$$\hat\rho_\pm(\nu) = \mp\frac{\mathrm{i}T}{2}\left\{D_{f\pm}^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{f\pm} + \hat k^{-1}(D_{\pm g}^+)^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{\pm g}^+\hat k\right\}. \tag{A.6.21}$$

Substituting it into the (A.6.16) we arrive at

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t''}\mathrm{Tr}_{t'}^{t''}\ \ln H\cdot H_0^{-1}\Big|_{(t''-t')/T} = -\frac{1}{8\pi}\int_{mT}^{mT+2\pi}\mathrm{d}\nu\,\mathrm{Sp}\Bigg[\Bigg\{D_{f\pm}^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{f+} \\ + D_{+g}^{\mp 1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{+g}^+ - D_{f-}^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{f-} - D_{g-}^{\mp 1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{g-}^+\Bigg\}\sqrt{\hat k^2+m^2}\Bigg]. \tag{A.6.22}$$

These formulae should be accompanied by the same deliberations as above on the structureless soliton case about the summing over the equal number of solutions of the free and the interacting equations

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t''}\,\mathrm{Tr}_{t'}^{t''}\ln H\cdot H_0^{-1}\Big|_{(t''-t')/T}=\frac{1}{4\pi\mathrm{i}}\frac{1}{T}\int_0^T \mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\Big(v''\Big(w\Big(x,\frac{t}{T},T\Big)\Big)-m^2\Big)-\frac{\mathrm{i}}{2T}\sum\nu_n$$

$$-\frac{1}{8\pi}\int_{mT}^{mT+2\pi}\mathrm{d}\nu\,\mathrm{Sp}\Big[\Big\{D_{f+}^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{f+}+D_{g+}^{\mp 1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{g+}^{+}-D_{f-}^{-1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{f-}-D_{g-}^{\mp 1}\frac{\mathrm{d}}{\mathrm{d}\nu}D_{g-}^{+}\Big\}\sqrt{\hat{k}^2+m^2}\Big],\tag{A.6.23}$$

$$(D_{+}^{f})_{ln}=(D_{-}^{g})_{ln}=\overline{(D_{-}^{f})_{ln}}=\overline{(D_{+}^{g})_{ln}}\underset{\substack{l\to\infty\\ n\to\infty;\,l=n}}{=}\delta_{ln}+\frac{1}{2\mathrm{i}k_{ln}}\frac{1}{T}\int_0^T\mathrm{d}^2x(v''(w)-m^2).\tag{A.6.24}$$

Here we have taken into account also the discrete spectrum Floque indices (A.4.14). The expression in the curved brackets may be replaced by $S^{-1}\cdot \mathrm{d}S/\mathrm{d}\nu$, where S is the scattering matrix. This is quite general fact. The spectral density is always equal to $S^{-1}\,\mathrm{d}S/\mathrm{d}\nu$.

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