QUANTIZATION OF A NON-ABELIAN TODA CHAIN

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The quantum inverse scattering method is used to study a non-Abelian Toda chain. The quantum R-matrix describing the commutation relations between the elements of the monodromy matrix of the corresponding auxiliary linear problem is computed.

Introduction

A non-Abelian Toda chain is a one-dimensional evolution model on a lattice with N nodes. The model's Lagrangian is

$$\mathcal{L} = \operatorname{tr} \sum_{k=1}^{N} \left\{ \frac{1}{2} A_k^2 - B_k \right\},$$

$$A_k = \frac{\partial g_k}{\partial t} g_k^{-1}, \quad B_k = g_{k+1} g_k^{-1}, \quad g_k \in GL(n). \tag{1}$$

This model was suggested by A. M. Polyakov as the discrete analog of the principal chiral field. Model (1) is a generalization of the Abelian Toda chain [1]. The models mentioned are completely integrable. The equations of the principal chiral field were solved in [2]. The Abelian Toda chain was investigated in [3-6]. The conservation laws for the non-Abelian Toda chain were found by Polyakov and the Lax representation for the equations of motion were found by Manakov [7]. An interesting investigation of model (1) was carried out in [8, 9].

In the present paper we examine an approach to the quantization of model (1) with the aid of the quantum inverse scattering method [10]. Let us describe the paper's contents: in Sec. 1 we compute the classical r-matrix and in Sec. 2 we compute the quantum R-matrix.

1. Classical Model

We state the properties of model (1) in a form convenient for quantization. The equations of model (1) can be presented as:

$$\dot{A}_k = B_k - B_{k-1}, \quad \dot{B}_k = A_{k+1} B_k - B_k A_k. \tag{2}$$

For example, we consider the periodic boundary conditions

$$g_{k+N} = g_k, \quad A_{k+N} = A_k. \tag{3}$$

Equations (2) can be written as compatibility conditions:

$$\Psi_{k+1} = L_k(\lambda)\Psi_k, \quad \dot{\Psi}_k = M_k(\lambda)\Psi_k$$

$$\dot{L}_k(\lambda) = M_{k+1}(\lambda) L_k(\lambda) - L_k(\lambda) M_k(\lambda). \tag{4}$$

Here ψ_k is a 2n-component column vector and L_k and M_k are $2n \times 2n$-matrices which we present in block form:

$$L_k(\lambda) = \begin{vmatrix} \lambda - A_k, & -B_{k-1} \\ I, & 0 \end{vmatrix}, \quad M_k(\lambda) = \begin{vmatrix} 0, & -B_{k-1} \\ I, & A_{k-1} - \lambda \end{vmatrix}. \tag{5}$$

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The compatibility condition in Eqs. (4) must be fulfilled identically in λ. We note that in the continuous limit model (1) turns into a principal chiral field; by formal calculations we can show that the L–M pair of (5) turns into the Zakharov–Mikhailov L–M pair [2]. The monodromy matrix

$$T(\lambda) = L_N(\lambda) \cdot \ldots \cdot L_1(\lambda) \tag{6}$$

plays an important role in the inverse scattering method. The trace of the monodromy matrix is independent of time. The coefficients in the expansion of $\ln \operatorname{tr} T(\lambda)$ in powers of λ prove to be local conservation laws among which is contained the Hamiltonian model. In what follows it happens to be convenient to make a "gauge" transformation, i.e., to replace the wave function in Eq. (4):

$$\varphi_k = Q_k^{-1}\psi_k\,, \quad Q_k = \begin{vmatrix} I\,, & 0 \\ 0\,, & g_{k-1} \end{vmatrix}. \tag{7}$$

This leads to the replacement of operator L_k by $\tilde{L}_k$:

$$\tilde{L}_k = Q_{k+1}^{-1} L_k Q_k\,, \quad \tilde{L}_k = \begin{vmatrix} \lambda - A_k\,, & -g_k \\ g_k^{-1}\,, & 0 \end{vmatrix}. \tag{8}$$

The new operator $\tilde{L}_k$ is local, i.e., depends only on the variables at the k-th node. The new monodromy matrix

$$\tilde{T}(\lambda) = \tilde{L}_N(\lambda) \cdot \ldots \cdot \tilde{L}_1(\lambda) \tag{9}$$

is connected with the original one by the formula

$$\tilde{T}(\lambda) = Q^{-1} T(\lambda) Q\,, \quad Q = Q_1 = \begin{vmatrix} 1\,, & 0 \\ 0\,, & g_N \end{vmatrix}. \tag{10}$$

The Poisson brackets between the elements of the monodromy matrix play an important role. Usually they permit us to compute the action-angle variables. We present the Hamiltonian formulation of the model:

$$\begin{gathered} H = \operatorname{tr} \sum_{k=1}^{N} \left\{ \frac{1}{2} A_k^2 + g_{k+1} g_k^{-1} \right\}, \\ \{ A_k^{ab}, A_\ell^{cd} \} = \delta_\ell^k \{ \delta_c^b A_k^{ad} - \delta_d^a A_k^{cb} \}, \\ \{ A_k^{ab}, g_\ell^{cd} \} = \delta_\ell^k \delta_c^b g_k^{ad}\,, \\ \{ A_k^{ab}, (g_\ell^{-1})^{cd} \} = -\delta_\ell^k \delta_d^a (g_k^{-1})^{cb}. \end{gathered} \tag{11}$$

It is convenient to make use of the symbol for the direct product of matrices:

$$(C \otimes D)_{ab,cd} \equiv C_{ac} D_{bd} \tag{12}$$

with whose help the Poisson brackets of elements C_{ac} and D_{bd} of matrices C and D can be written as

$$\{ C_{ac}, D_{bd} \} \equiv \{ C \overset{\otimes}{,} D \}_{ab,cd}\,. \tag{13}$$

In notation (12) the matrix product is written thus:

$$(N \cdot M)_{ab,cd} = \sum_{k\ell} N_{ab,k\ell} M_{k\ell,cd}\,. \tag{14}$$

Using such a notation, we can write formulas (11) as

$$\{A_k \overset{\otimes}{,} A_\ell\} = \delta_\ell^k \pi(I \otimes A_k) - \delta_\ell^k \pi(A_k \otimes I)$$

$$\{A_k \otimes g_\ell\} = \delta_\ell^k \pi(I \otimes g_k), \quad \{g_k \otimes A_\ell\} = -\delta_\ell^k \pi(g_k \otimes I) \tag{15}$$

$$\{A_k \otimes g_\ell^{-1}\} = -\delta_\ell^k \pi(g_k^{-1} \otimes I), \quad \{g_k^{-1} \otimes A_\ell\} = \delta_\ell^k \pi(I \otimes g_k^{-1}).$$

Here π is a permutation matrix of dimension $n^2 \times n^2$. It is defined thus:

$$\pi_{ab,cd} = \delta_d^a \delta_c^b . \tag{16}$$

We present its most important properties:

$$\pi^2 = I \ , \quad \pi(C \otimes D)\pi = D \otimes C . \tag{17}$$

Here C and D are arbitrary numerical $n \times n$-matrices. Now all has been prepared for us to compute the Poisson brackets between the elements of the monodromy matrix (9). We do this by using the classical r-matrix method [11]. We remark that if we succeed in representing the Poisson brackets between the elements of $\tilde{L}(\lambda)$ of (8) in the form

$$\{\tilde{L}_k(\lambda) \overset{\otimes}{,} \tilde{L}_k(\mu)\} = [\tilde{L}_k(\lambda) \otimes \tilde{L}_k(\mu), r(\lambda,\mu)] , \tag{18}$$

then the Poisson brackets between the elements of $\tilde{T}(\lambda)$ are given by the same formula (see Appendix 1):

$$\{\tilde{T}(\lambda) \overset{\otimes}{,} \tilde{T}(\mu)\} = [\tilde{T}(\lambda) \otimes \tilde{T}(\mu), r(\lambda,\mu)] \tag{19}$$

In these formulas the brackets on the right-hand side signify the matrix commutator of two matrices of dimension $4n^2 \times 4n^2$. The quantity $r(\lambda, \mu)$ is the classical r-matrix and its elements depend only on λ and μ. A direct calculation with the aid of (15) (see Appendix 2) shows that the r from (18) exists and is given by the expression

$$r(\lambda,\mu) = (\lambda-\mu)^{-1} P . \tag{20}$$

Here P is a permutation matrix of dimension $4n^2 \times 4n^2$:

$$P = \left[\begin{array}{cc|cc} \pi & & & \\ & & \pi & \\ \hline & \pi & & \\ & & & \pi \end{array}\right] \tag{21}$$

Thus, formulas (19) and (20) give the desired Poisson brackets between all elements of the monodromy matrix $\tilde{T}(\lambda)$ of (9). This completes the formulation of the properties of the classical model (1).

2. Quantum Model

In the quantum version the model is specified by the Hamiltonian

$$H = \mathrm{tr} \sum_{k=1}^{N} \{ \tfrac{1}{2} A_k^2 + g_{k+1} g_k^{-1} \} . \tag{22}$$

Here A and g are $n \times n$-matrices whose elements are quantum operators; their commutation relations have the form

$$A_k^{ab} A_k^{cd} = A_k^{cd} A_k^{ab} + i\hbar(\delta_d^a A_k^{cb} - \delta_c^b A_k^{ad}) ,$$

$$A_k^{ab} A_\ell^{cd} = A_\ell^{cd} A_k^{ab}, \quad g_k^{ab} A_\ell^{cd} = A_\ell^{cd} g_k^{ab} \quad \text{when } k \neq \ell ,$$

$$A_k^{ab} g_k^{cd} = g_k^{cd} A_k^{ab} - i\hbar \delta_c^b g_k^{ad} \,, \qquad (23)$$

$$A_k^{ab} (g_k^{-1})^{cd} = (g_k^{-1})^{cd} A_k^{ab} + i\hbar \delta_d^a (g_k^{-1})^{cb} .$$

Using the tensor product symbol, these relations can be rewritten as

$$A_k \otimes A_k = \pi A_k \otimes A_k \pi + i\hbar (I \otimes A_k - A_k \otimes I)\pi \,,$$

$$A_k \otimes g_k = \pi g_k \otimes A_k \pi - i\hbar (g_k \otimes I)\pi \,, \qquad (24)$$

$$A_k \otimes g_k^{-1} = \pi g_k^{-1} \otimes A_k \pi + i\hbar (I \otimes g_k^{-1})\pi .$$

It should be noted that formulas (2)-(10) are true also in the quantum case. An important role in the inverse scattering method is played by the permutation relations of the elements of $\tilde{T}(\lambda)$. In certain models such permutation relations enable us to compute all the eigenfunctions of the quantum Hamiltonian model. Let us compute the mentioned commutation relations with the aid of the tricks worked out in the quantum inverse scattering method [10]. If we manage to represent the commutation relations between the elements of $\tilde{L}(\lambda)$ in the form

$$R(\lambda,\mu)\tilde{L}(\lambda) \otimes \tilde{L}(\mu) = \tilde{L}(\mu) \otimes \tilde{L}(\lambda) R(\lambda,\mu) \,, \qquad (25)$$

then the permutation relations of the elements of $\tilde{T}(\lambda)$ are given by the same formula:

$$R(\lambda,\mu)\tilde{T}(\lambda) \otimes \tilde{T}(\mu) = \tilde{T}(\mu) \otimes \tilde{T}(\lambda) R(\lambda,\mu) \,. \qquad (26)$$

Here $R(\lambda, \mu)$ is a numerical matrix of dimension $4n^2 \times 4n^2$, whose elements depend only on λ and μ. The transition from (25) to (26) is based on the localness of operator $\tilde{L}_k(\lambda)$. Direct calculation by use of (24) (see Appendix 2) shows that

$$R(\lambda,\mu) = i\hbar E + (\mu - \lambda) P \,. \qquad (27)$$

Here E is the unit $4n^2 \times 4n^2$-matrix. We remark that the R-matrix for the Abelian Toda chain was found by Manakov. Expression (27) is the R-matrix for other models as well [12].

Using these results we can compute the commutation relations between the elements of the original nonlocal monodromy matrix $T(\lambda)$ of (6). For this it is convenient to substitute formula (10)

$$\tilde{T}(\lambda) = Q^{-1} T(\lambda) Q \qquad (28)$$

into (26). Only the first factor L_N in $T(\lambda)$ does not commute with Q in (28) (as a quantum operator). This enables us to write the commutation relations as: (see Appendix 3)

$$R(\lambda,\mu)(E - i\hbar X)(T(\lambda) \otimes I)(E - i\hbar Y)(I \otimes T(\mu)) = (E - i\hbar X)(T(\mu) \otimes I)(E - i\hbar Y)(I \otimes T(\lambda)) R(\lambda,\mu) \,. \qquad (29)$$

Here R is given by formula (27) and X and Y are numerical $4n^2 \times 4n^2$-matrices. (We present them in block form ($n^2 \times n^2$ blocks):

$$X = \left|\begin{array}{c|c} & \pi \\ \hline & \end{array}\right| \qquad Y = \left|\begin{array}{c|c} & \\ \hline & \pi \end{array}\right| \qquad (30)$$

Formulas (26), (27), (29), (30) are the main result. We have computed all the commutation relations between the elements of the monodromy matrix. We remark that these commutation relations turn out to be useful in the quantization $\vec{n}(x)$ of the field.

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Appendix 1

The operators $\tilde{L}_k(\lambda)$ of (8) are independent at the different nodes:

$$\{\tilde{L}_k(\lambda) \overset{\otimes}{,} \tilde{L}_\ell(\mu)\} = 0 \qquad \text{when } k \neq \ell . \tag{31}$$

Using this equality, it is easy to show that from (18) follows

$$\{\tilde{L}_k(\lambda)\tilde{L}_{k+1}(\lambda) \overset{\otimes}{,} \tilde{L}_k(\mu)\tilde{L}_{k+1}(\mu)\} = \left[\tilde{L}_k(\lambda)\tilde{L}_{k+1}(\lambda) \otimes \tilde{L}_k(\mu)\tilde{L}_{k+1}(\mu),\, r(\lambda,\mu)\right] . \tag{32}$$

It is obvious that we arrive at (19) by continuing this argument.

Appendix 2

To compute the r-matrix in (18) and the R-matrix in (25) it is convenient to use the following notation. The operator $\tilde{L}$ of (8) has been written in the block form

$$\tilde{L} = \left|\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right| \tag{33}$$

Here L_{ik} is an $n \times n$-matrix. The tensor product $\tilde{L} \otimes \tilde{L}'$ has the form

$$\tilde{L} \otimes \tilde{L}' = \left|\begin{array}{cc|cc} L_{11}\otimes L'_{11} & L_{11}\otimes L'_{12} & L_{12}\otimes L'_{11} & L_{12}\otimes L'_{12} \\ L_{11}\otimes L'_{21} & L_{11}\otimes L'_{22} & L_{12}\otimes L'_{21} & L_{12}\otimes L'_{22} \\ \hline L_{21}\otimes L'_{11} & L_{21}\otimes L'_{12} & L_{22}\otimes L'_{11} & L_{22}\otimes L'_{12} \\ L_{21}\otimes L'_{21} & L_{21}\otimes L'_{22} & L_{22}\otimes L'_{21} & L_{22}\otimes L'_{22} \end{array}\right| \tag{34}$$

In this basis the permutation matrix $P(4n^2 \times 4n^2)$ has the form

$$P = \left|\begin{array}{cc|cc} \pi & & & \\ & & \pi & \\ \hline & \pi & & \\ & & & \pi \end{array}\right| \tag{35}$$

Appendix 3

We go on to compute the commutation relations between the elements of $T(\lambda)$ of (6)

$$\tilde{T}(\lambda) = Q^{-1}T(\lambda)Q, \qquad T(\lambda) = L_N(\lambda)\cdot\ldots\cdot L_1(\lambda) . \tag{36}$$

We substitute this formula into (26):

$$R(\lambda,\mu)(Q^{-1}T(\lambda)Q) \otimes (Q^{-1}T(\mu)Q) = (Q^{-1}T(\mu)Q) \otimes (Q^{-1}T(\lambda)Q)\, R(\lambda,\mu) . \tag{37}$$

We take advantage of the fact that the R of (27) commutes with $Q \otimes Q$ and we multiply (37) by $Q \otimes Q$ from the left and by $Q^{-1} \otimes Q^{-1}$ from the right. We obtain the following equality:

$$R(I\otimes Q)\big((T(\lambda)Q)\otimes(Q^{-1}T(\mu))\big)(Q^{-1}\otimes I)=(I\otimes Q)\big((T(\mu)Q)\otimes(Q^{-1}T(\lambda))\big)(Q^{-1}\otimes I)\,R. \tag{38}$$

Using the commutativity of matrices $Q \otimes I$ and $I \otimes Q^{-1}$, we obtain

$$R\big[(I\otimes Q)(T(\lambda)\otimes I)(I\otimes Q^{-1})\big]\cdot\big[(Q\otimes I)(I\otimes T(\mu))(Q^{-1}\otimes I)\big]=$$

$$=\big[(I\otimes Q)(T(\mu)\otimes I)(I\otimes Q^{-1})\big]\cdot\big[(Q\otimes I)(I\otimes T(\lambda))(Q^{-1}\otimes I)\big]\,R. \tag{39}$$

We transform the expression $(I\otimes Q)(T(\lambda)\otimes I)(I\otimes Q^{-1})$. For this we note that the matrix elements Q do not commute only with the matrix elements of L_N. Consequently,

$$(I\otimes Q)(T(\lambda)\otimes I)(I\otimes Q^{-1})=(I\otimes Q)(L_N(\lambda)\otimes I)(I\otimes Q^{-1})$$
$$(L_{N-1}(\lambda)\cdot\ldots\cdot L_1(\lambda)\otimes I). \tag{40}$$

Direct computations lead to the equality

$$(I\otimes Q)(L_N(\lambda)\otimes I)(I\otimes Q^{-1})=(E-i\hbar X)(L_N(\lambda)\otimes I). \tag{41}$$

The final expression for $(I\otimes Q)(T(\lambda)\otimes I)(I\otimes Q^{-1})$ is

$$(I\otimes Q)(T(\lambda)\otimes I)(I\otimes Q^{-1})=(E-i\hbar X)(T(\lambda)\otimes I). \tag{42}$$

We note that

$$P(I\otimes Q)(T(\lambda)\otimes I)(I\otimes Q^{-1})P=(Q\otimes I)(I\otimes T(\lambda))(Q^{-1}\otimes I). \tag{43}$$

Substituting (42) and (43) into (39), we obtain (29).

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