A THREE-INSTANTON SOLUTION

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Abstract. The Yang-Mills field is studied in the case of the SU(2) algebra. An explicit three-instanton solution is constructed. This solution is a rational function of free real parameters which vary in $\mathbb{R}^n + \mathbb{R}^4$.

Bibliography: 12 notes.

Introduction

A duality equation was derived in [1] and [2], and the physical interpretation of its solutions (instantons) was clarified and the simplest one-instanton solution was obtained. The general $N$-instanton solution was obtained in [3]–[7].

A quaternion formulation of such a solution was given in the lecture notes [8]. We recall that an $N$-instanton solution for the SU(2) algebra depends on $8N - 3$ independent real parameters (see [9] and [10]). The explicit form of a three-instanton solution was obtained in [11], but the corresponding parametrization was not rational. The general three-instanton solution constructed in the present paper is a rational function of free real parameters which vary in $\mathbb{R}^8 + \mathbb{R}^4$.

We note that the notation of [12] is most convenient in the present context.

The present paper consists of two sections. The algebraic-geometric construction of instantons is briefly explained in the first section. The explicit parametrization of the three-instanton solution is described in the second section.

§ 1. The construction of Atiyah, Drinfel'd, Manin, Hitchin and Ward

We shall follow the exposition of [12]. We consider the duality equation

$$F_{\mu
\nu}(x) = -i \sum F_{\rho \lambda \nu}F_{\mu \lambda}(x)$$

(summation over repeated indices is assumed unless stated otherwise). Here

$$F_{\mu \nu}(x) = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x) + [A_{\mu}(x), A_{\nu}(x)].$$

It follows from the conformal invariance that the field $A_{\mu}(x)$ can be regarded as defined on the sphere $S^4$ rather than $\mathbb{R}^4$; the quantities $x_{\mu}$ are the stereographic coordinates on the sphere.

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We consider the Yang-Mills field in the case of SU(2). Let us pass from $S^4$ to $C^4$. It is convenient to use quaternions $q$ in writing explicit formulas. They will be regarded as $2 \times 2$ matrices defined by

$$q = a_x + i a_y + a_z = \begin{pmatrix} a_x & a_y \\ a_y & -a_x \end{pmatrix},$$

Here $a_x, a_y, a_z$ are real numbers, $i$ is the imaginary unit, and $a_j$ are the Pauli matrices:

$$a_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The quantity $a_x$ will be called the real part of a quaternion. We assign each point in $S^4$ a quaternion

$$z = a_x + i \sum_{j=1}^3 a_j a_j.$$

A point in $C^4$ is assigned two quaternions:

$$q_1 = \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} z_3 & z_4 \\ -z_4 & z_3 \end{pmatrix}.$$

The quantities $z_j$ are the coordinates of a point in $C^4$. Each point of $C^4$ is assigned a point of $S^4$ by

$$q_j \rightarrow \bar{q}_j q_1 q_2.$$

To clarify which object corresponds to a point in $S^4$, it is convenient to define an involution $\sigma$ in $C^4$ which acts as follows:

$$(z_1, z_2, z_3, z_4) \rightarrow (-z_2, z_1, -z_4, z_3).$$

It can be shown that a $\sigma$-invariant two-dimensional plane in $C^4$ passing through the origin corresponds to a point in $S^4$. The equation of such a plane is given by (1).

Since multiplication of each $z_j$ by a common complex factor does not change $x_a$ in (1), equation (1) is simply the projection of a three-dimensional projective complex space $CP^3$ onto $S^4$.

We define a field $C_i(x, \bar{z})$ in $C^4$:

$$C_i(x, \bar{z}) = \sum \frac{\partial x_i}{\partial z_j} A_j(x).$$

The strength of the field

$$F_a(x, \bar{z}) = \frac{\partial C_\bar{z}}{\partial z_a} - \frac{\partial C_z}{\partial \bar{z}_a} + [C_z, C_\bar{z}]$$

can be expressed in terms of $F_a(x)$:

$$F_a(x, \bar{z}) = \sum \frac{\partial x_i}{\partial z_j} \frac{\partial x_j}{\partial \bar{z}_a} F_a(x) = \sum \frac{D(x_i, \bar{z}_a)}{D(z_j, \bar{z}_a)} F_a(x),$$

where

$$\frac{D(x_i, \bar{z}_a)}{D(z_j, \bar{z}_a)} = \frac{1}{2} \left( \frac{\partial x_i}{\partial z_j} + \frac{\partial x_j}{\partial \bar{z}_a} \right).$$

This quantity has the following important property:

$$\frac{D(x_i, \bar{z}_a)}{D(z_j, \bar{z}_a)} = \frac{1}{2} \sum_{j=1}^3 \frac{D(x_j, \bar{z}_a)}{D(z_j, \bar{z}_a)}.$$
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For self-dual solutions $F_\mu(x) = -F_\mu^*(x)$, we obtain

$$F_\mu(z, \bar{z}) = \frac{1}{2} \sum \frac{\Lambda(x, \bar{x})}{\Lambda(z, \bar{z})} \left( F_\mu(x) + F_\mu^*(x) \right) = 0.$$  \hspace{1cm} (3)

Equation (3) expresses the content of the Ajoyah-Ward theorem, i.e., fields with $F_\mu = 0$ in $CP^3$ correspond to self-dual fields in $S^4$.

The solution of (3) has the form

$$C(x, \bar{z}) = \psi(x, \bar{z}) \frac{\partial}{\partial z} \psi^*(x, \bar{z}).$$

However, we are interested in $C$, which can be represented in the form (2), i.e., the above solution should be projected into $S^4$.

The problem of projecting the solution of the equation $F_\mu = 0$ in $CP^3$ into self-dual fields in $S^4$ can be easily solved using the Ajoyah-Dinknich-Maia-Hitchin construction. It is necessary in this construction to consider a vector space $L$ of dimension $2N + 2$ in order to obtain the solution of the equality equation with a topological charge $N$ for the group $SU(2)$. The inner product $(L_1, L_2)$ of two vectors $L_1$ and $L_2$ in $L$ is linear in $L_1$ and antilinear in $L_2$:

$$(L_1, L_2) = (\overline{L_2}, L_1),$$

$$c(L_1, L_2) = c(L_1, \overline{L_2}),$$

$$c(L_1, L_2) = \overline{c(L_2, L_1)},$$

$$c(L_1, L_2) \geq 0.$$

Choosing two orthonormal vectors $E_a(x, \bar{z})$, $a = 1, 2$ in $L$, $(E_a, E_b) = \delta_{ab}$, we define $C_a^b(z, \bar{z})$ by

$$C_a^b(z, \bar{z}) = -\left( E_a, \frac{\partial}{\partial z} E_b \right).$$

It is clear that (5) can be projected provided $E_a(x, \bar{z})$ is constant on $\sigma$-invariant planes in $C^*(i.e.,$ if $E_a$ depends only on $x$). In that case, we obtain

$$C_a^b = -\left( \sum \frac{\partial x_k}{\partial z} E_a(x), \frac{\partial}{\partial x_k} E_b(x) \right),$$

$$A_a^b = -\left( E_a(x), \frac{\partial}{\partial x_k} E_b(x) \right).$$

We first construct the orthogonal complement to the vectors $E_a$ in $L$. We consider two sequences of linearly independent vectors in $L$: $v_\mu(x, \bar{z})$ and $v_\mu^*(x, \bar{z})$, $a = 1, \ldots, N$. We assume that they have the following properties:

1) $v_\mu(x, \bar{z})$ is a linear function of $z$, and $v_\mu^*(x, \bar{z})$ is a linear function of $\bar{z}$.

2) Each vector of the first sequence transforms to a vector of the second sequence under the involution $\sigma$:

$$v_\mu(\sigma(x)) = v_\mu^*(x).$$

3) Any vector from the first sequence is orthogonal to any vector from the second sequence:

$$\langle v_a, v_b \rangle = 0.$$

Since the dimension of $L$ is equal to $2N + 2$, we can find two vectors $E_a$ such that

$$(E_a, E_b) = \delta_{ab},$$

$$E_a^\dagger E_a = 0,$$

$$E_a^\dagger E_b - (E_a, E_b^*) = 0.$$  \hspace{1cm} (6)
We shall now verify that the two vectors $E_1$ and $E_2$ thus obtained are constant on $o$-invariant planes; it follows from requirements 1), 2), and 3) that

$$
\eta_1 = a_1 e_{x_1} + a_2 e_{x_2} + a_3 e_{x_3}, \quad \eta_2 = -a_2 e_{x_1} + a_1 e_{x_2} + a_3 e_{x_3},
$$

(7)

where the $e_{x_i}$ are constant vectors. We shall combine the vectors $\eta_1$ and $\eta_2$ into pairs

$$
\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.
$$

Hence

$$
\eta = q_1 \eta_1 + q_2 \eta_2,
$$

$$
\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.
$$

Multiplying $\eta$ from the left by $q_1^*$, we obtain a pair of vectors $h_1, h_2$ which are linear combinations of $\eta_1$ and $\eta_2$:

$$
\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} bt_1^* \\ bt_2^* \end{pmatrix} = x_1 t_1^* + x_2 t_2^*
$$

(8)

and depend only on $x$.

The vectors $E_1$ are orthogonal to $h_1$ and $h_2$, and therefore they also depend only on $x$ and are constant on $o$-invariant planes in $C^4$.

Finally, we shall show that the fields $C(z, \bar{z})$ defined by (5) constructed from such vectors $E_1$ are solutions of the equation $F_{\eta} = 0$.

For $\eta E$, we obtain

$$
\delta \eta E = -\sum C_{n}^{m} E_{n} + \sum X_{n}^{m} q_{n}(z) + \sum Y_{n}^{m} \eta_{n}(z).
$$

(9)

Since $n$ depends only on $x$, we have

$$
\langle \eta_{n}(z), \delta \eta E \rangle = \frac{\delta}{\delta i_{n}} \langle \eta_{n}(z), E \rangle = 0.
$$

We have made use of the antilinearity property of the inner product defined by (4).

Multiplying (9) from the left by $\eta_{n}(z)$, we obtain $X_{n}^{m} = 0$. Differentiating (9) with respect to $z$, we obtain

$$
\delta \eta \eta E = -\sum \delta_{n} C_{n}^{m} E_{n} + \sum C_{n}^{m} C_{n}^{m} E_{n} + \sum Y_{n}^{m} \eta_{n}(z).
$$

Making use of the symmetry $\delta_{n} \eta_{n} E = \delta_{n} \eta_{n} E$ and the orthogonality of $E_{n}$ and $\eta_{n}$, we obtain

$$
\langle \eta, \delta_{n} \eta E - \delta_{n} \eta E \rangle = (F_{n})_{i} = 0.
$$

It follows that $C_{n}$ is a solution of the equation $F_{\eta} = 0$ and it can be projected onto $S^4$; the Atiyah-Ward theorem then implies that $A_{n}$ is a solution of the duality equation. By the Atiyah-Driver-Maier-Mumford theorem, the fields $A_{n}$ constructed by this method represent the complete solution of the duality equation. We shall not prove the completeness of the solutions but merely show that they depend on $8N - 3$ parameters.

We choose an orthonormal basis $e_{i}$ in $L$, where $a = 1, 2$ and $i = 1, \ldots, N + 1$. The pair of vectors $\eta_{n}$ can then be written as

$$
\eta = \sum_{n=1}^{N+1} (q_{1} A_{n} + q_{2} B_{n}) e_{n}, \quad e_{n} = \begin{pmatrix} e_{n}^{1} \\ \vdots \\ e_{n}^{N+1} \end{pmatrix}.
$$

(10)
Here, $A$ and $B$ are $N \times (N + 1)$ matrices with respect to the indices $a$ and $i$ and their matrix elements $A_{ai}$ and $B_{ai}$ are quaternions. The form of (10) ensures that conditions 1) and 2) are satisfied. Condition 3) is satisfied for $a = b$ since $A_{ai}$ and $B_{ai}$ are quaternions. The condition $(\eta, \eta_2) = 0$ for $a < b$ imposes the following restrictions:

$$
\sum_{a=1}^{N+1} A_{ai} \tilde{a}_i = \lambda_a \delta_i, \quad \sum_{a=1}^{N+1} B_{ai} \tilde{b}_i = \mu_a \delta_i, \quad \sum_{a=1}^{N+1} A_{ai} \tilde{b}_i = \sum_{a=1}^{N+1} A_{ai} \tilde{b}_i.
$$

where $a < b$; $\lambda_a$ and $\mu_a$ are real numbers.

We note that not all the parameters in the matrices $A_{ai}$ and $B_{ai}$ have an effect on $E_a$ and $F_a$. The structure of (10) remains unchanged upon the substitution:

$$
A_{ai} \rightarrow A_{ai}' = \sum_{a=1}^{N+1} G_{ai} A_{ai} \gamma_a, \quad G \in GL(N),
$$

$$
B_{ai} \rightarrow B_{ai}' = \sum_{a=1}^{N+1} G_{ai} B_{ai} \gamma_a, \quad \gamma \epsilon \text{ Sp}(N+1),
$$

where $G$ is a numerical matrix and $\gamma$ is an $(N + 1) \times (N + 1)$ matrix whose matrix elements are quaternions and $\gamma^* \gamma = I$. The group $GL(N)$ corresponds to linear transformations of the vectors $\tilde{v}_a$ and $\tilde{w}_a$ and the group $\text{Sp}(N + 1)$ is the group of rotations of the orthonormal basis $\tilde{e}_a$.

The number of parameters in the matrices $A$ and $B$ is equal to $2 \cdot 4 \cdot N \cdot (N + 1)^2 = 8N^2 + 8N$, the number of parameters of the group $GL(N)$ is $N^2$, and the number of parameters of $\text{Sp}(N + 1)$ is $2(N + 1)^2 - (N + 1) = 2N^2 + 5N = 2N - 3$, the number of independent parameters in the solution (10) is equal to

$$
8N^2 + 8N - N^2 - 2(N + 1)^2 - (N + 1) = 5N^2 + 5N = 8N - 3,
$$

i.e., it follows from the number of parameters that the solution (10) corresponds to the complete solution of the duality equation.

§2. Explicit form of the three-instanton solution

Using the transformations from $GL(N)$ and $\text{Sp}(N + 1)$, we can simplify (11).

Let us consider a quantity $u_a$ defined by

$$
u_a = \left( \begin{array}{c} u_1^a \\ u_2^a \end{array} \right) = \sum_{i=1}^{N+1} A_{ai} \tilde{e}_i.
$$

It is clearly possible to find a matrix $G \in GL(N)$ such that

$$
u_a \rightarrow \delta_a = \sum_{a=1}^{N+1} G_{ai} u_a \quad \text{and} \quad (\delta_a, \delta_b) = \delta_{ab} \delta_a \delta_b.
$$

We then obtain for the pair $\eta_a$

$$
\eta_a \rightarrow \tilde{\eta}_a = \sigma_1 \tilde{\delta}_a + q_1 \sum_{a=1}^{N+1} B_{ai} \tilde{a}_i.
$$

We note that the matrix $\lambda_{ab}$ in (12) now reduces to $\delta_{ab}$.

We introduce a basis vector $\tilde{\delta}_{a+1}$ normalized to unity (i.e., $\langle \tilde{\delta}_{a+1}, \tilde{\delta}_{a+1} \rangle = \delta_{a+1}$) and orthogonal to all the $\tilde{\delta}_a$. We can then expand the second term in (12) in terms of the orthonormal basis $\tilde{\delta}_a$, which yields

$$
\eta_a = q_1 \tilde{\delta}_a + q_1 \sum_{i=1}^{N+1} \tilde{B}_{ai} \tilde{a}_i.
$$
where the \( \hat{b}_a \) are new quaternions. For \( \epsilon_a \) expressed in the notation of (10), we obtain
\[
\hat{A}_a = \hat{b}_a.
\]
(14)

For \( \hat{A}_a \) defined by (14), the first equation of (11) is satisfied trivially with \( \lambda_{a\beta} = 0 \) for \( \alpha < \beta \), the third equation in (11) yields
\[
\hat{b}_{a\beta} = \hat{b}_{\beta a}.
\]
and the second yields
\[
\sum_{\beta=1}^3 \hat{b}_{a\beta} \bar{b}_{\beta\eta} = \mu_{a\eta} l.
\]
(13)

We omit \( \alpha < \beta \) since (15) becomes an identity for \( \alpha = \beta \) (for \( \epsilon_a \)), the left-hand side of (13) is proportional to a unit matrix, the proportionality factor is denoted by \( \mu_{a\eta} \). The matrix \( \mu \) is a real symmetric \( N \times N \) matrix with numerical matrix elements. We note that this is the form of the equations satisfied by the parameters of the \( N \)-instanton solution which was quoted in [8].

Let us now discuss the remaining symmetry groups. Firstly, the group SL(2) has been reduced to O(\( N \)), but, to preserve the form of \( \hat{b}_a \) given by (13), it is necessary not only to rotate \( \hat{b}_a \) but also to perform the same rotations of the basis (the group Sp(\( N + 1 \)) contains O(\( N \)) as a subgroup):
\[
\hat{b}_a - \epsilon_a = \sum_{\beta=1}^3 V_{a\beta} \hat{b}_\beta, \quad \forall \in O(\( N \)).
\]
(16.1)

As a result of such rotations, we obtain
\[
\hat{b}_a = \epsilon_a - \sum_{\beta=1}^3 V_{a\beta} \hat{b}_\beta + \sum_{\beta=1}^3 B_{a\beta} \epsilon_\beta.
\]
(16.2)

It follows from (15) that the new parameters \( B_{a\beta} \) satisfy
\[
\sum_{\beta=1}^3 B_{a\beta} B_{\beta\eta} = \mu_{a\eta} l, \quad \mu = \nu \nu^{-1}.
\]

We now choose the matrix \( \nu \). Let \( \nu \) be a matrix which diagonalizes the real symmetric matrix \( \tilde{\mu} : \)
\[
\mu = \nu \tilde{\mu} \nu^{-1} = \text{diag}(\mu_1, ..., \mu_m).
\]
Such a choice reduces O(\( N \)) to the group of reflections, i.e., the transformations defined by (16) with a diagonal matrix
\[
\nu_{a\beta} = \delta_{a\beta} (-1)^{b_{a\beta}},
\]
(17)
are admissible and the diagonal matrix elements of the matrix \( \nu \) are powers of (−1). We shall employ the group of reflections to transform the real parts of certain quaternions into positive quantities.
In the basis defined by (16), the group $SO(N + 1)$ reduces to the group of rotations of the $(N + 1)$th axis:

$$G = I, \quad r = \begin{pmatrix} I & 0 \\ 0 & m \end{pmatrix}, \quad e_{N+1} \rightarrow e_{N+1} = me_{N+1},$$

where $m$ is a quaternion whose modulus is equal to 1, $mm^* = I$. The aforementioned group acts on the matrix $B_\alpha$ as follows:

$$B_\alpha \rightarrow B_\alpha' = \begin{cases} B_{\alpha_{N+1}m^{-1}}, & i = N + 1, \\ B_{\alpha_i}, & i = 1, \ldots, N. \end{cases}$$

(18)

we choose $m$ from the condition

$$B_{1,N+1} = b_{1,N+1} I,$$

where $b_{1,N+1}$ is a real positive number. Having eliminated the arbitrariness due to symmetry groups, we find that the system (11) assumes the form

$$\sum_{i=1}^{N+1} B_{i} B_{j} = 0, \quad \alpha < \beta, \quad B_{\alpha} = B_{\alpha}^*, \quad B_{1,N+1} = b_{1,N+1} I.$$  

(19)

Explicit parametrization of instantsions can be obtained from the solution of (15). It is then necessary to take care that the vectors $v_1^0$ and $v_2^0$ defined by (7) are indeed linearly independent. The dimension of the space spanned by these vectors should be $2N$.

Let us first consider the case of two instantsions: $N = 2$. The system (19) assumes the form

$$B_{11} B_{12}^* + B_{12} B_{12}^* + b_{13} B_{13}^* = 0.$$  

(20)

The quaternions $B_{11}$, $B_{12}$, and $B_{13}$ and the number $b_{13}$ can be regarded as free parameters. The quaternion $B_{11}$ can be obtained explicitly from (20):

$$B_{11} = -(B_{12} B_{12}^* + b_{13} B_{13}^*) (B_{11}^*)^{-1}.$$  

An explicit expression for $A_n(x)$ is discussed at the end of this section. We have thus constructed the complete two-instanton solution depending on 13 real parameters. It is shown in Appendix 1 that the corresponding vectors $v_1^0$ and $v_2^0$ are linearly independent in the most general situation. The dimension of the space of parameters for which the vectors $v_1$ are linearly dependent is equal to 9. It follows that the corresponding codimension is equal to 4, which is a sufficient condition for the set of parameters to be doubly connected.

We now discuss the true three-instanton case. The system (19) assumes the form

$$B_{11} B_{12}^* + B_{12} B_{12}^* + B_{13} B_{13}^* + b_{14} B_{14}^* = 0.$$  

(21)

$$B_{11} B_{13}^* + B_{12} B_{13}^* + B_{17} B_{17}^* + b_{14} B_{14}^* = 0.$$  

(22)

$$B_{12} B_{13}^* + B_{13} B_{13}^* + B_{17} B_{17}^* + b_{14} B_{14}^* = 0.$$  

(23)

We choose the quaternions

$$B_{12}, \quad B_{13}, \quad B_{13}, \quad B_{13}, \quad B_{13}$$  

(24)

and a number

$$b_{14} > 0.$$  

(25)
as independent parameters. The remaining quaternions $b_{ij}$, $b_{ij}$, and $b_{ij}$ can be determined from the above system of equations. The quaternion $b_{ij}$ can be taken from (21)

$$b_{ij} = -\left( b_{1j}b_{2j} + b_{1j}b_{2j} + b_{1j}b_{2j} \right)(b_{1j})^{-1}.$$ 

We note that the remaining equations (22) and (23) are linear in the quaternions $b_{ij}$, $b_{ij}$, and also that these two quaternions are multiplied by other quaternions only from the left and, therefore, can be readily determined:

$$b_{ij} = \left( b_{ij}b_{1j} + b_{ij}b_{1j} \right)(b_{1j})^{-1},$$

$$b_{ij} = -\frac{1}{b_{1j}}(b_{ij}b_{1j} + b_{ij}b_{1j} + b_{ij}b_{1j}).$$

We note that the group of reflections defined by (17) and (16) can be used to achieve $\text{Re}b_{ij} > 0, \text{Re}b_{ij} < 0$.

Since each quaternion depends on 4 real parameters, we find that the total number of real parameters is equal to 21, where three parameters assume values on the real axis and the remaining 18 parameters assume values in the complex plane. Equations (24) represent a parametrization in the general situation. Degenerate cases are not applicable (for example, $b_{ij} = 0$) are discussed in Appendix, and it can be shown that the dimension of such a manifold of parameters is equal to 17, the corresponding codimension is equal to 4. Therefore we can claim that the manifold of parameters is smoothly connected.

It remains to verify that the corresponding vectors $\eta_{ij}$ defined by (7) are independent. We show in Appendix 1 that the vectors $\eta_{ij}$ are linearly independent in general situation. The dimension of the manifold of parameters for which the vectors $\eta_{ij}$ are linearly independent is equal to 17. It follows that the corresponding codimension is equal to 4, which implies that the manifold of parameters remains smoothly connected.

Finally, following the general method of constructing instanton solutions, we get explicit form of their coordinate representation:

$$u^2(x) = \frac{1}{2}\sum_{k=1}^{4} (A_k(x)q_k)^{\alpha} = -(E_{ij}q_iq_j), \quad k = 1, 2, 3; \alpha, \beta = 1, 2,$$

where $E_{ij}$ is given by (6).

Let us now construct the vectors $h_{\alpha}$ using (8)

$$h_{\alpha} = q_{\alpha} - i\gamma_{\alpha} = \sum_{\gamma=1}^{4} B_{\alpha\gamma}q_{\gamma}.$$ 

We consider a pair of vectors $E = (E_{ij})$:

$$E = \frac{1}{\sqrt{1 + \sum_{\gamma=1}^{4} |B_{\alpha\gamma}|^2}}.$$

Here the $K_{\alpha}$ are quaternions.

We assume that $E_{ij}$ and $E_{ij}$ are orthogonal to the vectors $h_{\alpha}$ and $h_{\beta}$. This following equation for the matrix $K_{\alpha}$:

$$\sum_{\beta=1}^{4} (E_{ij}q_{\beta} + B_{\alpha\beta})K_{\beta} = -B_{\alpha\beta}.$$
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The orthogonality of $E_1$ and $E_2$ is trivial since the $K_\mu$ are quaternions. An explicit solution of (31) can be obtained; clearly such a solution is a rational function of independent parameters. The Yang-Mills fields are given by

$$A^\mu_i(x) = \frac{2\sum q_{\lambda\rho} \overline{K_{\lambda}} \cdot K_{\rho}}{1 + \sum K_{\alpha} K_{\beta}}.$$  (32)

Here, $\eta_{\lambda\rho}$ is the 't Hooft tensor [13] given by

$$\eta_{\lambda\rho} = \begin{cases} 
\delta_{\lambda\rho}, & \lambda, \rho = 1, 2, 3, \\
-\delta_{\lambda4}, & \lambda = 4, \\
0, & \lambda, \rho = 4.
\end{cases}$$

Equations (26)--(28) and (30)--(32) define the general form of the three-instanton solution in terms of independent parameters defined by (24), (25), and (29). The construction of the 't Hooft solution is given in Appendix 3. We note that the general solution $A^\mu_i(x)$ is a rational function of independent real parameters.

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Appendix 1

The dimension of the space spanned by the vectors $\eta_\lambda$ should be $2N$. However, the vectors $\eta_\lambda$ become linearly dependent for certain values of the free parameters defined by (26) and (25). We now determine the dimension of such a space of parameters.

First we discuss general problems. Since the vectors $\eta_\lambda$ of the first series are orthogonal to the vectors $\eta_4^a$ of the second series, vectors belonging to different series cannot be linearly dependent. We shall now state the condition of linear dependence (we consider the case of $n$-fold degeneracy, i.e., a space spanned by the vectors $\eta_\lambda$ is $(N - n)$-dimensional):

$$\sum_{a=1}^n a^{(a)} \eta_4^a = 0, \quad k = 1, \ldots, N.$$

There are only $n$ different sets of the coefficients $a^{(a)}$, and they are labelled by the upper index $k$. We can bring the coefficients $a^{(a)}$ to the canonical form $a^{(a)} = a^{(a)}_k$ for $k = 1, \ldots, n$. Such a normalization uniquely determines all the other coefficients $a^{(a)}$. We can obtain other sets of $a^{(a)}$ from the canonical set using arbitrary linear combinations $\sum c^a \eta_4^a$ (the $c^a$ are complex numbers).

It can be easily shown that the vectors $\eta_4^a$ also become linearly dependent:

$$\sum_{a=1}^n a^{(a)} \eta_4^a = 0.$$

Here the $a^{(a)}$ are complex numbers, and $a^{(a)}$ are their complex conjugates.

The above conditions can be written in the form

$$\sum_{a=1}^n a^{(a)} \eta_4^a = 0.$$  (33)

Here the $a^{(a)}$ are quaternions of the form

$$a^{(a)} = \begin{pmatrix} a^{(a)} & 0 \\
0 & \overline{a^{(a)}} \end{pmatrix}.$$  (34)
It is easy to show that the linear dependence condition expressed in terms of the vectors $h_\alpha$ is analogous:

\[ \sum_{\alpha=1}^{N} b^{(\alpha)} h_\alpha = 0. \]  

(35)

Here the $b^{(\alpha)}$ are quaternions of the form (34). In fact, we can easily show that all the first components of the vectors $h_\alpha$ are orthogonal to all the second components of $h_\beta$. We also note that the vectors $h_\alpha$ are $n$ times degenerate, i.e., there are only $n$ sets of $h_\alpha$ such that the condition (35) is satisfied together with the normalization condition $b^{(\alpha)}_\alpha = b_\alpha$ (here, $\alpha$ assumes $n$ different values).

We now show that the quaternions $\tilde{a}_\alpha^{(\alpha)}$ and $\tilde{b}_\alpha^{(\alpha)}$ are real. Equation (33) represents conditions on $a^{(\alpha)}$ and $b^{(\alpha)}_\alpha$. Since we require coefficients $a^{(\alpha)}_\alpha$ and points $z_j$ in $\mathbb{C}^*$ satisfy (33). Let us consider all the points in $\mathbb{C}^*$ belonging to the same $\sigma$-invariant plane. If the coefficients $a^{(\alpha)}_\alpha$ can be found for at least one point of the plane, they exist for all the other points since all such points are projected onto the same point in $\mathbb{C}^*$.

Let us consider a $\sigma$-invariant plane defined by

\[ k = q_2^{-1} q_1. \]

Here, $k$ is fixed. We shall parametrize the points of such a plane by means of the quaternion $q_2$. We use the relation $h_\alpha = q_2^{-1} q_\alpha$ to rewrite (33) in the form

\[ \sum_{\alpha=1}^{N} q_2^{-1} b^{(\alpha)} q_2 h_\alpha = 0. \]  

(36)

Comparing (36) and (25), we obtain $\tilde{a}_\alpha^{(\alpha)} = \tilde{b}_\alpha^{(\alpha)}$ for $\alpha = 1, \ldots, n$, and

\[ \tilde{a}_\alpha^{(\alpha)} = q_2^{-1} a^{(\alpha)} q_2. \]

Here $a^{(\alpha)}_\alpha$ and $\tilde{b}(k)$ are fixed quaternions of the form (34), and $q_2$ is an arbitrary quaternion. Such a condition can be satisfied only if the quaternions $\tilde{a}_\alpha^{(\alpha)}$ and $\tilde{b}_\alpha^{(\alpha)}$ are real.

Finally, we find that the condition of linear dependence assumes the form

\[ \sum_{\alpha=1}^{N} a_\alpha h_\alpha = 0, \]

where the $a_\alpha$ are real numbers (we shall omit the index $k$).

Using the relation

\[ h_\alpha = \bar{a}_\alpha e_1 + \sum_{j-1}^{N} B_{\alpha j} e_j, \]

we obtain

\[ a_\alpha = \sum_{\alpha=1}^{N} a_\alpha B_{\alpha j} = 0, \quad \sum_{\alpha=1}^{N} a_\alpha B_{\alpha j} = 0. \]  

(37)

Let us first consider the case of two instantons. The system (37) assumes the form

\[ a_1 (\bar{k} + \bar{b}_{11}) + a_2 b_{12} = 0, \quad a_1 b_{12} + a_2 (\bar{k} + \bar{b}_{12}) = 0, \quad a_1 b_{12} + a_2 b_{22} = 0. \]  

(38)

We first consider the case $a_{1} \neq 0$. We introduce a variable $c = a_1 a_2^{-1}$. Equation (38) yields

\[ \bar{c} = -b_{12} - c b_{12}, \quad b_{12} = -b_{12}, \quad b_{22} = b_{11} + (c^{-1} - c) b_{12}. \]
Substituting these expressions in (20), we obtain

\[ B_{11} B_{11}^c + B_{12} B_{12}^c + B_{13} B_{13}^c (c - c) - b_{12}^2 c = 0. \]

Regarding this as an equation for \( c \), we find that it has two real solutions. We have thus found the solution of the orthogonality condition (20) and of the linear dependence condition (38): we choose the quaternions \( B_{11}, B_{12} \) and the number \( b_{11} \) as free parameters.

It can be seen that these equations have a 9-parameter family of solutions. The complete three-instanton solution depends on 13 parameters. It follows that the corresponding codimension is equal to 4. The cases \( a_1 = 0 \) or \( a_2 = 0 \) can be discussed separately, and it can be shown that the corresponding codimension is also equal to 4.

We shall now consider the case of three instantons. Then (37) assumes the form

\[ \begin{align*}
    a_1 (x + b_{11}) + a_2 B_{12} + a_3 B_{13} &= 0, \\
    a_1 B_{12} + a_2 (x + b_{12}) + a_3 B_{13} &= 0, \\
    a_1 B_{13} + a_2 B_{12} + a_3 (x + b_{13}) &= 0, \\
    a_1 b_{14} + a_2 B_{14} + a_3 B_{15} &= 0.
\end{align*} \]

We shall seek the solution of this system together with the orthogonality conditions (21)-(23). We shall first consider the case \( a_1 a_2 a_3 \neq 0 \). Let us rewrite (39)-(41) in the form

\[ x = B_{11} + \frac{a_2}{a_1} B_{12} + \frac{a_3}{a_1} B_{13} = B_{22} + \frac{a_1}{a_2} B_{12} + \frac{a_3}{a_2} B_{13} = B_{33} + \frac{a_1}{a_3} B_{11} + \frac{a_2}{a_3} B_{13}. \]  

We now transform (22) and (23); we multiply (22) by \( a_2/a_3 \) and add it to (21); using (42), (23), and (22), we obtain

\[ B_{11} + |B_{12}|^2 + |B_{13}|^2 + |B_{14}|^2 = |x|^2. \]  

Similarly, we multiply (21) by \( a_1/a_3 \) and add it to the equation conjugate to (23), which yields

\[ |B_{11}|^2 + |B_{12}|^2 + |B_{13}|^2 + |B_{14}|^2 = |x|^2. \]

We now require the solution of (43), (42), (21), (44), and (45). We shall show that such a system of equations is equivalent to a single real equation involving 18 real parameters. We shall choose such parameters to be the four quaternions

\[ B_{11}, B_{12}, B_{13}, B_{14} \]

and two real numbers

\[ a_1, a_2, a_3. \]

We can determine \( B_{12} \) and \( B_{13} \) from (43). The quantity \( b_{14} \) can be calculated from (44). We evaluate \( B_{14} \) from (21). Using (42), we determine \( B_{14} \). It follows that we have determined all the required quantities, a real equation, namely (45), still remains. This procedure proves that the solution of the system (39)-(42), (21)-(23) depends on 17 real parameters. The general three-instanton solution depends on 21 parameters. It follows that the corresponding codimension is equal to 4. The case when several \( a_n \) vanish can be treated separately. Such situations do not alter the value of the codimension.
Appendix 2

As already discussed, equations (26)-(23) determine the parametrization of the problem in the general situation. They contain singularities in the cases when any of the quaternions \( B_{11}, B_{23} \) or \( B_{32}, b_{14}J \) are equal to zero. It is therefore necessary to investigate all the degenerate cases.

We shall discuss in detail the case when \( B_{12} = 0 \) and \( B_{34} = 0 \). Then (21)–(23) assume the form

\[
\begin{align*}
B_{13}B_{13}^* + b_{14}B_{14}^* &= 0, \\
B_{11}B_{11}^* + b_{14}B_{14}^* &= 0, \\
B_{22}B_{22}^* + b_{14}B_{14}^* &= 0.
\end{align*}
\]

It is easy to solve the above system: when \( B_{32} = 0 \), we can write

\[
\begin{align*}
B_{13}^* &= \frac{1}{b_{14}}(B_{11}B_{11}^* + B_{13}B_{23}^*), \\
B_{22}^* &= -(B_{12}B_{12}^* + B_{23}B_{23}^*), \\
B_{34}^* &= \frac{1}{b_{14}}B_{14}B_{14}^*.
\end{align*}
\]  

(46)

If \( B_{32} = 0 \), then

\[
\begin{align*}
b_{14} &= 0, \\
B_{13}^* &= \frac{1}{b_{14}}(B_{11}B_{11}^* + B_{13}B_{23}^*).
\end{align*}
\]  

(47)

The quaternions

\[
B_{11}, B_{13}, B_{23}, B_{34}, b_{14}J
\]

represent independent parameters for the solution defined by (46), and the quaternions

\[
B_{11}, B_{13}, B_{22}, B_{23}, b_{14}J
\]

are independent parameters for the solution (47). In both cases, the number of free real parameters is equal to 17.

A similar approach can be used to study the case \( b_{14} = 0 \). It is possible to make the quaternion \( B_{32} \) (or \( B_{34} \)) real, \( B_{32} = b_{14}J \), using the method that was applied in (18) to make \( B_{31} \) real; the number of parameters then reduces by 3. Solving (21)–(23) with \( b_{14} = 0 \) and \( B_{32} = b_{14}J \), we find that the number of free parameters is then at most 17.

Finally, if

\[
B_{13} = \frac{B_{32}B_{31}}{b_{14}},
\]

we can determine from (21) and (22) the quaternions \( B_{22} \) and \( B_{34} \):

\[
\begin{align*}
B_{22} &= \left[ B_{22} \left( \frac{1}{b_{14}}B_{14}B_{14}^* \right) + B_{12}B_{12}^* \right] (B_{12})^{-1}, \\
B_{34} &= \frac{1}{b_{14}} \left[ B_{32}B_{32}^* + B_{13}B_{13}^* + \frac{1}{b_{14}}B_{31}B_{31}^* \right].
\end{align*}
\]

The free parameters

\[
B_{11}, B_{13}, B_{22}, B_{34}, b_{14}.
\]
which involve 21 real variables thus satisfy the following quaternion equation (provided \(b_{11} \neq 0\), which can be derived from (23)):

\[
B_{12}^* B_{12} B_{12}^* B_{12}^* (B_{12}^*)^{-1} - B_{12}^* B_{12} B_{12}^* (B_{12}^*)^{-1} - b_{14} B_{12}^* = \\
\frac{1}{b_{14}} B_{12} B_{12} B_{12} B_{12} (B_{12}^*)^{-1} (B_{12}^*)^2 + (B_{12}^*)^2 = 0.
\]

We can use this equation, which is a linear equation for the components of \(B_{11}\), to determine \(B_{11}^*\), where \(B_{11}^* = \sum_{14} + i \sum_{14} B_{11}^*\). It follows from our discussion that the number of free real parameters is at most 15. It can be verified that this assertion holds for all the degenerate cases.

We thus conclude that the solution of the duality equation at singular points of the general solution (26)–(28) depends on at most 15 free real parameters, i.e., the codimension is greater than or equal to four.

Appendix 3

The 't Hooft three-instanton solution depends on 13 parameters. We need to reduce our 15-dimensional manifold to a 15-dimensional manifold. The parameters \(B_{12}, B_{13}, B_{22}, B_{23}, B_{24}\) and \(b_{14}\) then clearly become functions of 15 independent parameters.

Let \(\lambda_a\) be quaternions and \(\lambda_a\) real numbers, \(a = 1, 2, 3\). We introduce numbers \(\rho_a\) and a matrix \(W\) by

\[
\rho_a = \det \xi_a, \quad W = \begin{bmatrix}
\rho_1 + \lambda_1^2, & \lambda_1 \lambda_2, & \lambda_1 \lambda_3 \\
\lambda_1 \lambda_2, & \rho_2 + \lambda_2^2, & \lambda_2 \lambda_3 \\
\lambda_1 \lambda_3, & \lambda_2 \lambda_3, & \rho_3 + \lambda_3^2
\end{bmatrix}.
\]

We define a matrix \(V\) as follows:

\[
VV^T = \text{diag}, \quad VV^T = I.
\]

Let us now parameterize \(B_{12}, B_{13}, B_{22}, B_{23}, B_{24}\) and \(b_{14}\) as follows:

\[
B_{12} = -\sum V_{12}\xi_{12} V_{12}, \quad B_{13} = -\sum V_{13}\xi_{13} V_{13}, \\
B_{22} = -\sum V_{22}\xi_{22} V_{22}, \quad B_{23} = -\sum V_{23}\xi_{23} V_{23}, \\
B_{24} = -\sum V_{24}\xi_{24} V_{24}, \quad b_{14} = -\sum V_{14}\lambda_{14}.
\]

Substituting (48) in the general solution (26)–(28), we obtain

\[
\begin{align*}
B_{11} = -\sum V_{11}\xi_{11} V_{11}, & \quad B_{21} = -\sum V_{21}\xi_{21} V_{21}, & \quad B_{23} = \sum V_{23}\lambda_{23}
\end{align*}
\]

After some trivial operations, we obtain the vector potential in the coordinate form

\[
A_2^+(x) = -\sum \eta_{2a} \rho_a \ln(1 + x^2_2 x^2_3) = -\sum \eta_{2a} \rho_a \ln\left(1 + \sum_{a=1}^7 \frac{x^2_a}{(x - x_a)^2}\right).
\]
where

\[ K^a_b = \frac{\lambda_a}{(x - x_a)^2} (x - x_a)_b \]

(there is no summation over repeated indices), i.e., in the parametrization (48), the general solution yields the 't Hooft solution.

Translated by D. NATHO