Name:____

Guidelines

- The exam is due at 12PM (noon) on Tuesday May, 21st. You may hand it to me (or slide under my door if I'm out) anytime before then.
- You may consult the course textbook (Carroll) and any notes from class
- You may quote results from the book or class rather than recomputing things we/you have done previously (e.g. Christoffel symbols, Riemann and Ricci tensors, for particular solutions, mathematical tricks, etc)
- You may NOT discuss the content of this exam with anyone.
- You may NOT use the internet, nor look for solutions in other textbooks, journal articles, etc.
- You may use Mathematica or other software, but clearly explain all steps in your solution
- You may email me for clarifications or corrections if you find a typo — I will do my best to respond promptly.
- Some parts of each problem are related (require correct answers to previous parts) but not all parts of each problem require you to have completed the previous steps correctly.
- The solutions can be emailed or given to me personally (or put under my door, or in my mailbox in YITP). They may be handwritten or in Latex format, do not give me Mathematica. If you leave your solutions in my mailbox or put them under my door send me an email to confirm that I have received them.
- As always, clearly explained solutions that are legible and easy to follow are overwhelmingly preferred and earn higher marks. This take-home exam will be graded more finely than the homework. If your handwriting isn't legible, re-copy it before turning it in. Take pride in your work!

Questions

1. Properties of the FRW metric (20 points)

Consider the spatially flat FRW metric with line element $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$. You may quote the textbook or notes for the components of the Christoffel symbols, Riemann and Ricci tensor for this problem.

(a) Does this spacetime have a timelike Killing vector? Why or why not? What are the consequences of this?

(b) Prove that the energy of a massless particle (as measured by a comoving observer) evolves as $\propto 1/a$ and the 3-velocity of a non-relativistic massive particle evolves as $v \propto 1/a$.

(c) Prove that for universes dominated by non-relativistic matter or radiation, $t \to 0$ is a true curvature singularity (you may quote your homework, class notes, or the text for solutions to $a(t)$ for these cases). Does this seem reasonable given your results from (b)?

(d) Draw the conformal diagram for this space. For which observers is the past singularity at $t \to 0$ visible?

2. Modifications to General Relativity (40 points)

Suppose that instead of the usual Einstein-Hilbert action, $S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R$, we have the modified expression

$$S_{f(R)} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R + f(R)) \tag{1}$$

where $f(R)$ is a differentiable function. The total action for both gravity and matter is given by $S_{f(R)} + S_M$ where S_M is the action for whatever matter is present.

(a) Using the variational principle, determine the modified version of Einstein's equations. You may assume that the coupling of the matter fields to the metric is unchanged so that the usual expression for the energy-momentum tensor holds

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \qquad \text{where} \qquad S_M[g_{\mu\nu}, \Psi, \dots] \tag{2}$$

where $\Psi\dots$ indicates the matter fields (you may leave your answer in terms of $T_{\mu\nu}$, no need to express anything in terms of the $\Psi\dots$).

(b) Show that the equations of motion from (a) yield an equation for $f_R \equiv \frac{df}{dR}$

$$\Box f_R = \frac{dV_{eff}(f_R, R, T)}{df_R} \tag{3}$$

where $\Box = \nabla_\mu\nabla^\mu$, $T = g^{\mu\nu}T_{\mu\nu}$, and V_{eff} is an effective potential for f_R. Give an explicit expression for dV_{eff}/df_R. A typical requirement of a degree of freedom such as f_R is to have $m_{eff}(f_R) \equiv \frac{d^2V_{eff}}{df_R^2} > 0$. What condition does this impose of $f(R)$?

(c) The $f(R)$ theory in Eq. (1) is actually equivalent to a theory with an action given by

$$S_\Phi = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + f(\Phi) + \frac{df}{d\Phi}(R - \Phi)\right) + S_M[g_{\mu\nu}, \Psi, \dots] \tag{4}$$

where we have included the matter action.[1] Using the action above, show that the following redefinitions

$$\tilde{g}_{\mu\nu} = \left(1 + \frac{df}{d\Phi}\right) g_{\mu\nu} \qquad \text{and} \qquad \phi = -\sqrt{\frac{3}{2}}\sqrt{\frac{1}{8\pi G}} \ln\left(1 + \frac{df}{d\Phi}\right) \tag{5}$$

allow the action to be written as

$$S_\phi = \int d^4x \sqrt{-\tilde{g}} \left(\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - V(\phi) \right) + S_M[\omega^{-2}(\phi)\tilde{g}_{\mu\nu}, \Psi, \dots] \tag{6}$$

where $\tilde{g}$, $\tilde{R}$ are the determinant and Ricci scalar for the $\tilde{g}_{\mu\nu}$ metric and $\tilde{\nabla}_\mu$ is the covariant derivative with respect to $\tilde{g}_{\mu\nu}$. What are $V(\phi)$ and $\omega(\phi)$? (You may leave your answer in terms of an implicit solution for $\Phi(\phi)$)

(d) Do free particles in these theories travel on geodesics of $g_{\mu\nu}$ or $\tilde{g}_{\mu\nu}$? Is the weak equivalence principle satisfied in this theory?

2. Gravitational Waves (40 points)
In Minkowski space, far away from a region with non-zero energy-momentum tensor,

$$\bar{h}_{ij}(\mathbf{x}, t) = \frac{2G}{|\mathbf{x}|} \left.\frac{d^2 I_{ij}}{dt^2}\right|_{t_r = t - |\mathbf{x}|} \tag{7}$$

where $\mathbf{x}$ is a spatial 3-vector separation,

$$I_{ij}(t) = \int d^3y y^i y^j T^{00}(\mathbf{y}, t) \tag{8}$$

is the quadrupole moment tensor of the source and $h_{\mu\nu}$ is a small perturbation to the Minkowski metric and $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ with $h = \eta^{\mu\nu}h_{\mu\nu}$. We will consider a binary system of two objects each with mass M on a circular orbit with radius R, orbital frequency Ω, and the plane of the binary inclined at an angle i with respect to the line of sight (so that $\sin i = \pm 1$ is face on). You may assume the masses are moving non-relativistically to determine I.

(a) Determine the quadrupole moment tensor for the binary system in terms of M, R, i, Ω.

(b) Determine $\bar{h}_{ij}$ from I_{ij} and then impose the Lorentz gauge conditions ($\partial_\mu \bar{h}^{\mu\nu} = 0$) to determine the rest of the components of $\bar{h}^{\mu\nu}$ and then $h_{\mu\nu}$ from $\bar{h}_{\mu\nu}$. [HINT: we have already assumed that we are far from the binary to arrive at the quadrupole formula, you may assume $R/r \ll 1$, $\Omega \gg 1/r$ where $r = |\mathbf{x}|$.]

(c) Show that for $\sin i = \pm 1$ the components of $h_{\mu\nu}$ describe circularly polarized gravitational waves. What is the relationship between the observed frequency of the gravitational waves and the frequency of the orbit?

(d) We'd now like to show that for $\sin i = 0$ the gravitational waves are linearly polarized. This is not as obvious as in part (c) because the components of $h_{\mu\nu}$ that you have found should have residual scalar and vector degrees of freedom. By performing a suitable shift in coordinates, $x^\alpha \to x^\alpha - \xi^\alpha$, show that the metric can be put in the transverse-traceless gauge (i.e. $h = 0$, $\partial^\mu h_{\mu\nu} = 0$). [HINT: There are multiple ways of doing this, but I think it's easiest to first find a coordinate transform that eliminates the h_{00} component, then use the remaining gauge freedom to eliminate the h_{0i} components, keeping track of how the rest of the components transform as you do these gauge transforms.]

[1] This equivalence can be seen by varying the action and computing the equations of motion for Φ and $g_{\mu\nu}$.

(e) Show that your result from (d) corresponds to linear polarization. Is the amplitude the same as the amplitude from (c)?

(f) Assume that the binary orbit is large enough that it can be described by Kepler's law $R^3 = GM/(4\Omega^2)$ and that the binary system radiates gravitational waves with a power $P = -\frac{2}{5}\frac{G^4M^5}{R^5}$. Equating this radiated power with a change in the binding energy of $E_{bind} \sim -GM^2/R$ yields an expression for the rate of change of the orbital radius $\dot{R}$. Find an expression for $\dot{R}$ and then $\dot{\Omega}$.

(g) From parts (a) - (f) (or the book if you were unable to get parts (a)-(d)) write down three equations: (i) The overall amplitude of the gravitational waves (don't worry about factors of $\sin i$), (ii) The frequency of the waves ω in terms of Ω, and (iii) The rate of change of the frequency $\dot{\omega}$ all in terms of G, M, Ω, r.

(h) Attached is the PRL with the first LIGO detection of gravitational waves. By reading off of the plot in Figure 1 in the LIGO paper estimate (i) the distance to the binary in Mpc (1pc = $3.08610^{16}m$) (ii) The mass of the objects in the binary in M_{sun}. How do your estimates compare with the numbers reported by the LIGO team?