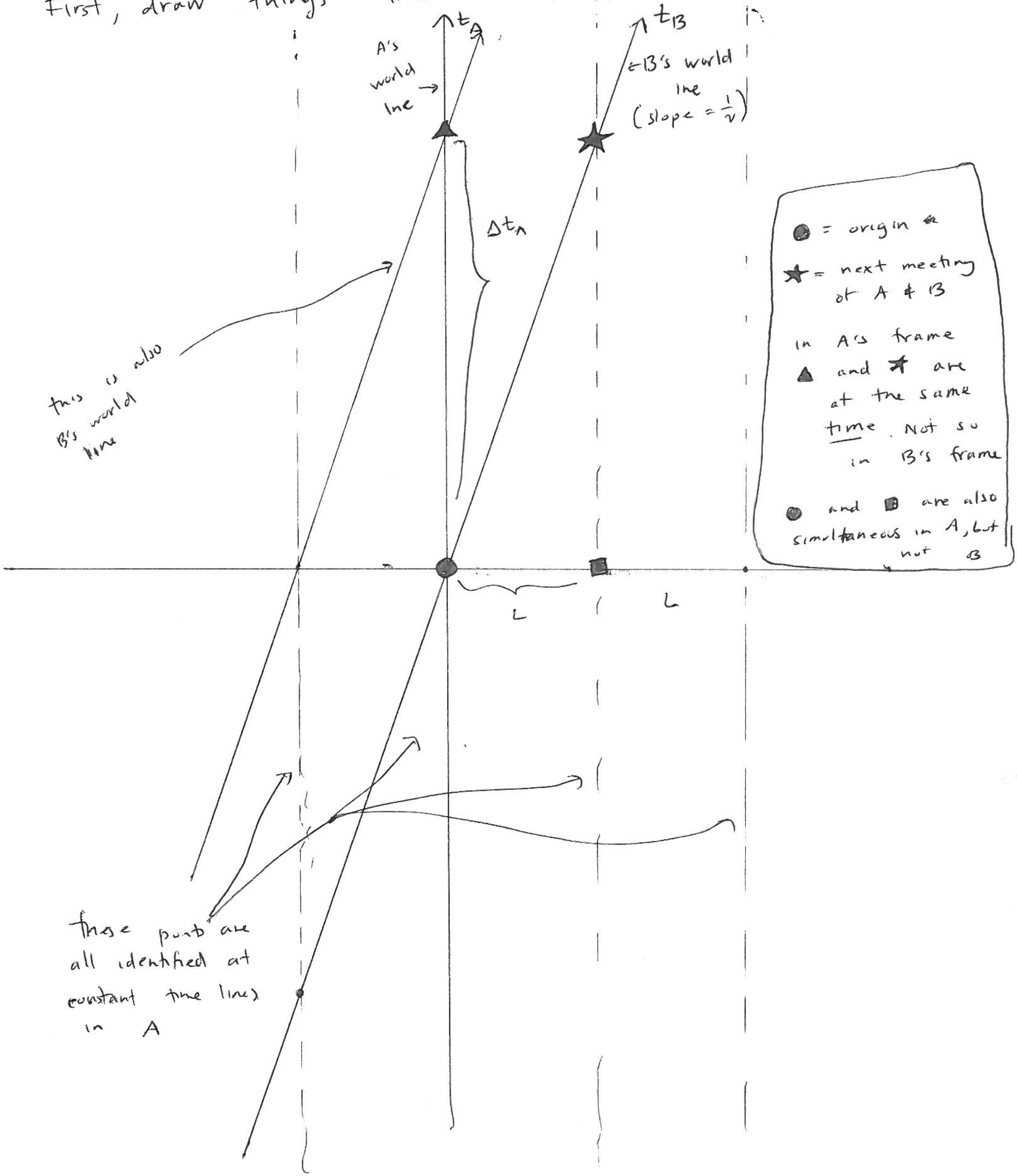
Carroll 1,2.

First, draw things from A's perspective



From A's perspective an amount of time Δt_A elapses between the ~~first~~ meeting of A and B at $(0,0)$ and the meeting again at $(\Delta t_A, 0)$ and $\Delta t_A = \frac{L}{v}$.

Now, the key is that in B's frame, the identification of (x) w/ $x+L$ is <u>not</u> simultaneous.

Let's map the points ●, ■, ★ and ▲ to B's frame.

$$● = (0,0) \text{ in } A \longrightarrow (0,0) \text{ in } B$$

$$■ = (0,L) \text{ in } A \longrightarrow \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ L \end{pmatrix} = \begin{pmatrix} -\gamma v L \\ \gamma L \end{pmatrix} \text{ in } B$$

$$★ = \left(\frac{L}{v}, L\right) \text{ in } A \rightarrow \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix}\begin{pmatrix} \frac{L}{v} \\ L \end{pmatrix} = \begin{pmatrix} \frac{\gamma L}{v} - v\gamma L \\ 0 \end{pmatrix} = \begin{pmatrix} L/v \, \frac{1}{\gamma} \\ 0 \end{pmatrix}$$

$$▲ = \left(\frac{L}{v}, 0\right) \longrightarrow \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix}\begin{pmatrix} L/v \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\gamma L}{v} \\ -\gamma L \end{pmatrix}$$

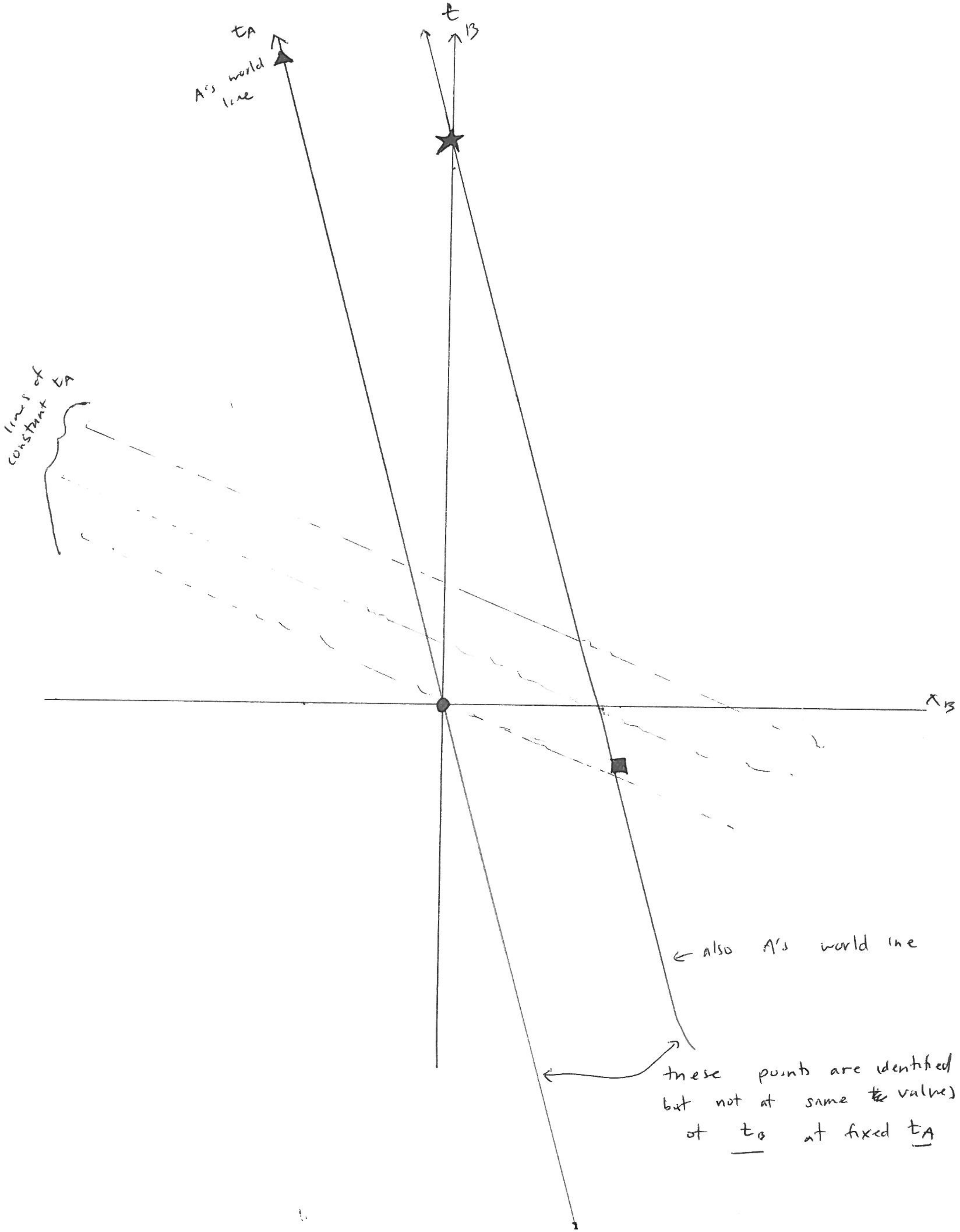
So in B ●, ■ and ▲, ★ are not simultaneous,

B will see the next meeting ~~at~~ with A at ★

so elapsed time in B is

$$\Delta t_B = \frac{L}{\gamma v}$$

in B's frame

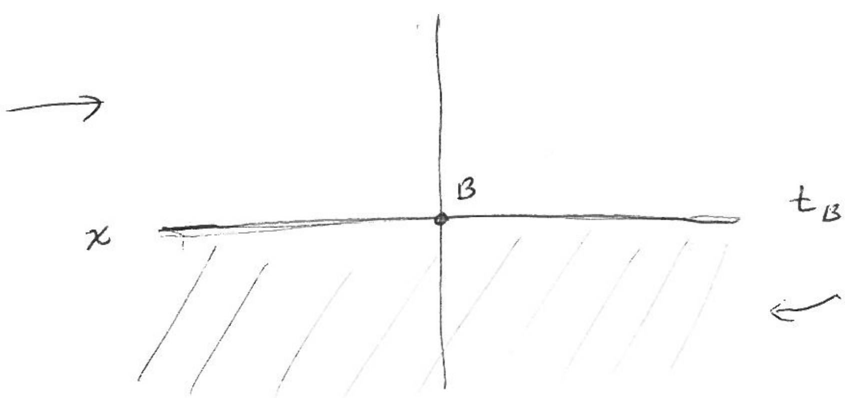

t
t_A
B
A's world line
lines of constant t_A
x_B
← also A's world line
these points are identified
but not at same values
of t_B at fixed t_A

Carroll problem 1.3

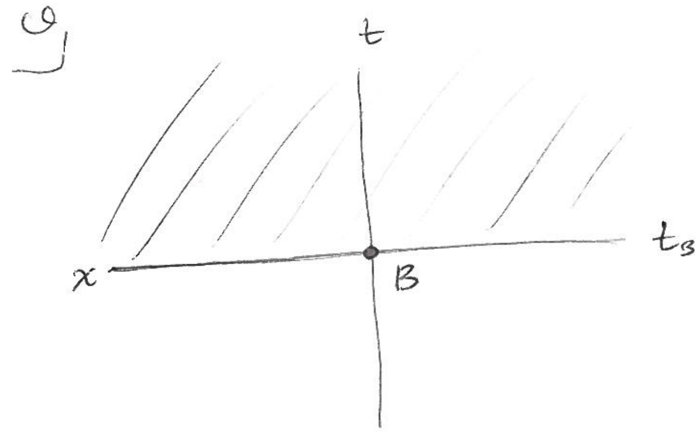
Three events are seen by observer $\mathcal{O}$ to occur ABC

in $\mathcal{O}$

lets put B at $t=0$, $x=0$



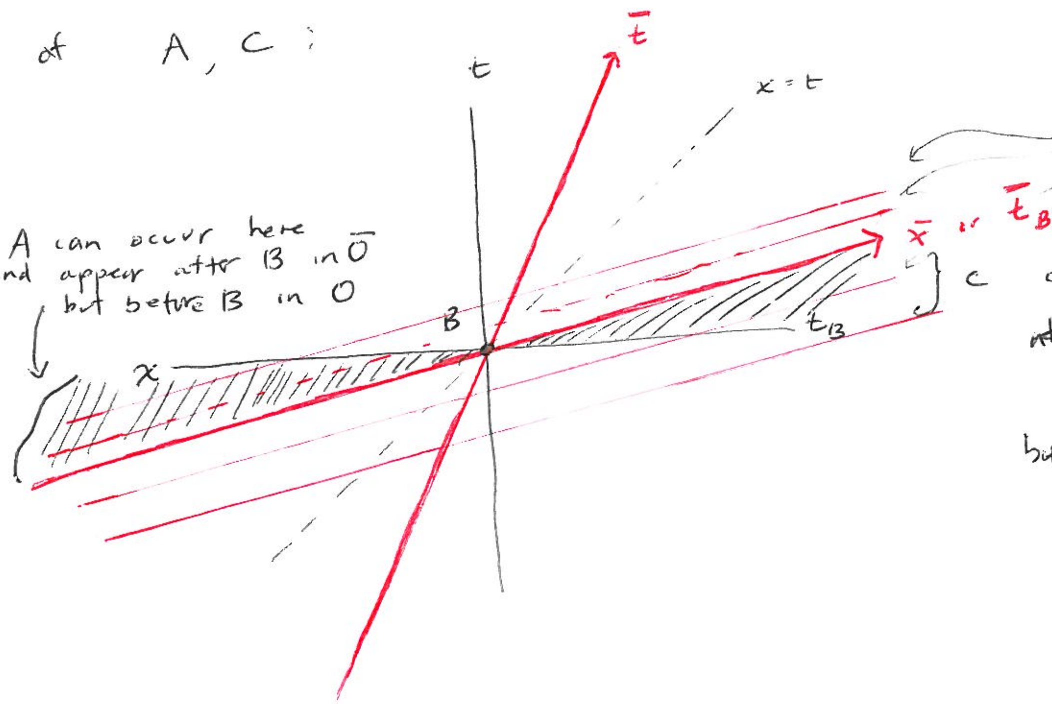
$\leftarrow$ A can occur anywhere here for ~~bec~~ $t_A < t_B$



$\leftarrow$ C can occur anywhere here for $t_c > t_B > t_A$

Observer $\bar{O}$ sees order CBA, this restricts locations of A, C:

lines of const $\bar{t}$ are given by $t = \frac{\bar{t}}{\gamma} + vx$ drawn here



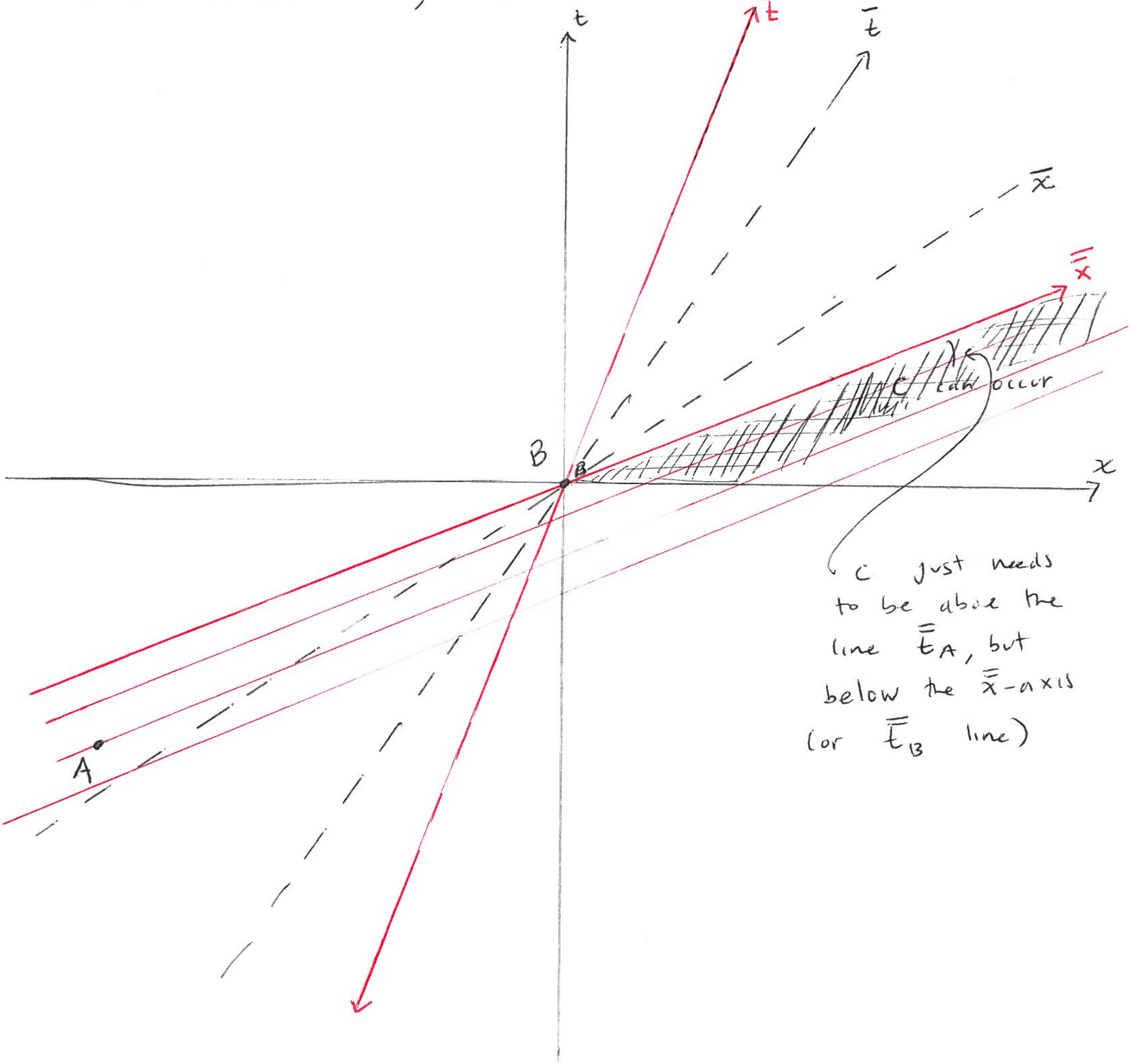
C can occur anywhere here ~~at the~~ ~~at~~ and have $t_c > t_B > t_A$

but $\bar{t}_c < \bar{t}_B < \bar{t}_A$

Now, is there a third observer who sees ACB?

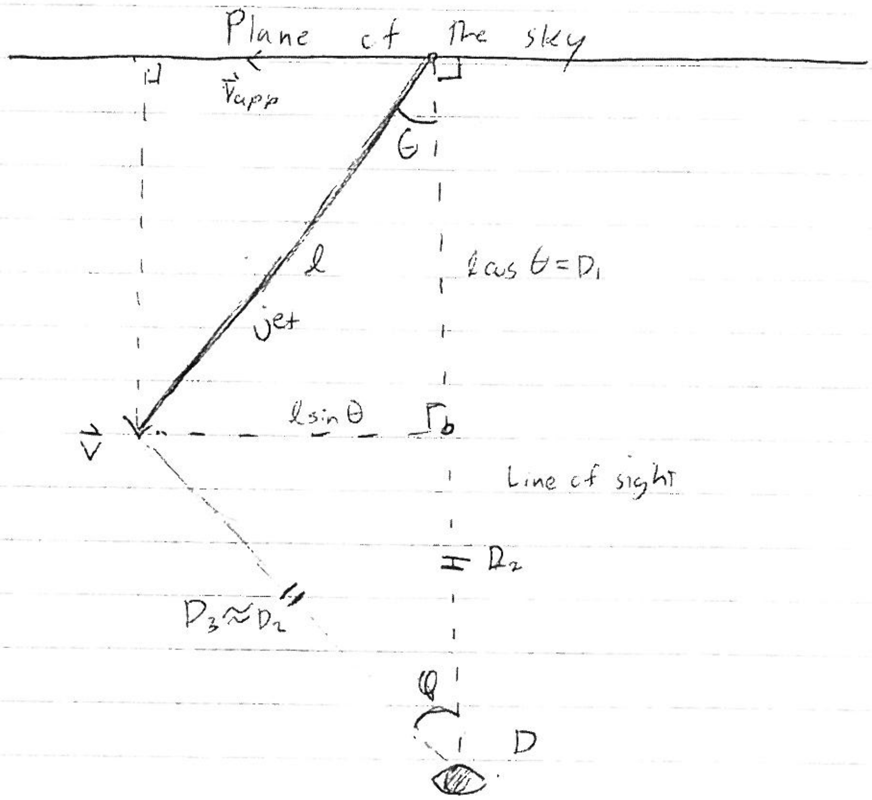
e.g an $\bar{\bar{O}}$ w $\bar{\bar{t}}_A < \bar{\bar{t}}_C < \bar{\bar{t}}_B$?

Claim: if $\bar{\bar{O}}$ is boosted w.r.t. O by a boost $\bar{\bar{v}}$ where $\bar{\bar{v}} < \bar{v}$ with $\bar{v}$ the velocity of $\bar{O}$ w.r.t. O, then this will work.



2) Carroll 1.4

+3



A portion of the jet at the origin(t_0) passes through length l along the jet stream to some point later (t_1), so $l = v\Delta t$, $\Delta t = t_1 - t_0$. If it emits light along its path, then an observer (the eye) recieves the light at times t_0' and t_1'

$$t_0' = t_0 + D_2 + l\cos\theta = t_0 + D_2 + v(t_1 - t_0)\cos\theta$$
$$t_1' = t_1 + D_2$$

where the distances $D_3 \approx D_2$ for very far objects like quasars.
The new time interval is

$$\begin{aligned}(\Delta t)' &= t_1' - t_0' \\ &= (t_1 + D_2) - (t_0 + D_2 + v(t_1 - t_0)\cos\theta)) \\ &= (t_1 - t_0) - v(t_1 - t_0)\cos(\theta) \\ &= (t_1 - t_0)(1 - v\cos(\theta)) = \Delta t(1 - v\cos(\theta))\end{aligned}$$

By skinny triangle approximation

$$D_3\sin\varphi \approx D_2\sin\varphi \approx l\sin(\theta) \approx D_2\,\delta\varphi \quad ; \; l = v\Delta t$$
$$\Rightarrow v\Delta t\sin(\theta) = D_2\,\varphi$$

since $V_{app} = D_L \, \Theta / \Delta t'$ in the observer's world view

$$\hookrightarrow V_{app} = \frac{D_L \Theta}{\Delta t'} = \frac{V \Delta t \sin(\theta)}{\Delta t'}$$

$$\boxed{V_{app} = \frac{V \sin(\theta)}{1 - V\cos(\theta)}}$$

(missing funtions ct c in the bottom if we want SI units)

If $0 \leq V \leq 1$ in our units, suppose we fixed $V \neq 0$.
Then if we vary θ

$$\partial V_{app}/\partial\theta = \frac{V\cos(\theta)}{1 - V\cos(\theta)} + \frac{V\sin(\theta)(-V\sin(\theta))}{(1 - V\cos(\theta))^2}$$

$$= \frac{V\cos(\theta)}{1 - V\cos(\theta)} - \frac{V^2\sin^2(\theta)}{(1 - V\cos(\theta))^2}$$

with maximum at

$$0 = \frac{V\cos(\theta)}{1 - V\cos(\theta)} - \frac{V^2\sin^2(\theta)}{(1 - V\cos(\theta))^2}$$

$$\cancel{(1 - V\cos(\theta))^2}\, 0 = V\cos(\theta)(1 - V\cos(\theta)) - V^2\sin^2(\theta)$$

$$V^2\sin^2(\theta) = V\cos(\theta)(1 - V\cos(\theta))$$

$$V^2(\sin^2(\theta) + \cos^2(\theta)) = V\cos(\theta)$$

$$V = \cos(\theta)$$

$$\Rightarrow \quad \theta = \arccos(V) \quad \text{and } \sin(\theta) = \sqrt{1 - V^2}$$

$$\Rightarrow \quad V_{app,max} = \frac{V\sqrt{1 - V^2}}{1 - V(V)}$$

$$V_{app,max} = \frac{V}{\sqrt{1 - V^2}}$$

and

$$\frac{V}{\sqrt{1 - V^2}} > 1 \quad \text{if } V^2 > 1 - V^2$$

ergo

$$V^2 > 1/2$$

which can occur for $V \in [0, 1]$ $\square$

Exercise 7

+3

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \quad V^{\mu} = (-1, 2, 0, -2)$$

a) $$X^{\mu}{}_{\nu} = g_{\nu\alpha} X^{\mu\alpha} = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

b) $$X_{\mu}{}^{\nu} = g_{\mu\alpha} X^{\alpha\nu} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$$

c) $$X^{(\mu\nu)} = \frac{1}{2}\left(X^{\mu\nu} + X^{\nu\mu}\right) = \begin{pmatrix} 2 & -1/2 & 0 & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{pmatrix}$$

d) $X_{[\mu\nu]}$, first $$X_{\mu\nu} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}$$

$$X_{[\mu\nu]} = \frac{1}{2}\left(X_{\mu\nu} - X_{\nu\mu}\right) = \begin{pmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & 1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{pmatrix}$$

e) $X^{\lambda}{}_{\lambda} = -4$ using a)

f) $V^{\mu} V_{\mu} = g_{\mu\nu} V^{\nu} V^{\mu} = -1 + 4 + 0 + 4 = 7$

g) $V_{\mu} X^{\mu\nu} = (4, -2, 5, 7)$, we use $V_{\mu} = (1, 2, 0, -2)$