Homework 3: Due at the beginning of class on Friday Feb 22nd

You must write out clear, logical, and complete answers to the homework questions. You may work together on solving the homework problems, but you must write up the final answer on your own.

The scoring of each homework problem will be out of 3 points: 3 points will be given for a complete, 100% correct solution; 2 points will be given for a largely correct (and still clearly explained) solution, 1 point will be given for a poorly explained, incorrect solution. Excellent solutions by students may be photocopied and used as homework solutions available to everyone.

- Carroll, chapter 2, exercise 4
- Take the metric of flat Euclidean space in coordinates x, y, z,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1}$$

  Using what you've learned about the transformation properties of tensors, show that the metric in spherical coordinates is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{2}$$

- Carroll, chapter 2, exercise 6
- At the end of class I stated that differential forms and the Hodge star operator give a compact expression for Maxwell's equations. In this problem you will verify that. First, we need the following definition:

  The *exterior derivative* of a p-form $A_{a_1 \ldots a_p}$ is a p+1 form $dA_{ba_1 \ldots a_p}$ with components

$$dA_{\nu\mu_1 \ldots \mu_p} = (p+1)\partial_{[\nu} A_{\mu_1 \ldots \mu_p]} \tag{3}$$

  where [] denotes antisymmetrization over indices and ∂_ν is the ordinary partial derivative [1] acting on the components of $A_{a_1 \ldots a_p}$. In particular, the gradient of a scalar can be written $d\phi = \partial_\mu \phi dx^\mu$. In the following, I suppress notation showing the rank of tensors for brevity (e.g. the $_{a_1 \ldots a_p}$)

  a) Show that $d(dA) = 0$ for arbitrary p-form A.

  b) Show that for 1-form field $A = A_\mu dx^\mu$, the electromagnetic field strength tensor can be written as $F = dA$. As a consequence, $dF = d(d(A)) = 0$, which produces two of Maxwell's equations. The expression $F = dA$ also makes explicit that F will be unchanged if $A \to A + d\lambda$ for a scalar λ.

  c) Show that the expression $d(*F) = *J$, for a suitable definition of J gives the rest of Maxwell's equations.

  d) Enjoy looking at the simple expressions $dF = 0$ and $d(*F) = *J$ as opposed to the expressions in components on the last homework.

[1] We haven't introduced derivative operators, so for now we work with components and the partial derivative. We will in the next class or two and then it is easy to show that the definition of exterior derivative here is independent of the derivative operator so using the ordinary partial derivative is fine.