Properties of Riemann

we have

(I) $R_{[abc]d} = 0$ (II) $R_{[ab]cd} = 0$ (III) $R_{ab[cd]} = 0$

want to show

$$R_{abcd} = R_{cdab}$$

first,

(I) $\Rightarrow R_{abcd} - R_{acbd} - R_{bacd} + R_{bcad} + R_{cabd} - R_{cbad} = 0$

by (II) and (III)

$\rightarrow 2R_{abcd} + 2R_{cabd} + 2R_{bcad} = 0$

$\rightarrow R_{abcd} = -R_{cabd} - R_{bcad}$

$= R_{cadb} + R_{bcda}$

now, using $R_{[cad]b} = 0$ and $R_{[bcd]a} = 0$

$\rightarrow R_{abcd} = -R_{dcab} - R_{adcb} - R_{dbca} - R_{cdba}$

$$= R_{cdab} - \underbrace{R_{adcb} - R_{dbca}}_{} + R_{cdab}$$

$= +R_{adbc} + R_{dbac}$

$= -R_{dabc} - R_{bdac}$

$= +R_{abdc}$

$= -R_{abcd}$

$\rightarrow R_{abcd} = 2R_{cdab} - R_{abcd} \Rightarrow 2R_{abcd} = 2R_{cdab}$

$\Rightarrow \boxed{R_{abcd} = R_{cdab}}$

consequences of metric compatibility

(i) verify $\nabla_\sigma g^{\mu\nu} = 0$

First,

$$0 = \nabla_\sigma \delta^\mu{}_\alpha = \nabla_\sigma g^{\mu\nu} g_{\nu\alpha} = g_{\nu\alpha} \nabla_\sigma g^{\mu\nu}$$

$$\rightarrow \quad 0 = g^{\alpha\beta} g_{\nu\alpha} \nabla_\sigma g^{\mu\nu} = \nabla_\sigma g^{\mu\beta} = 0 \quad \checkmark$$

(ii) verify $\nabla_\lambda \epsilon_{\mu\nu\rho\sigma} = 0$

First check $\nabla_\lambda (g \tilde{\epsilon}_{\mu\nu\rho\sigma})$

↑ Levi-civita set of 0, 1

$$\nabla_\lambda (g \tilde{\epsilon}_{\mu\nu\rho\sigma}) = \nabla_\lambda (\tilde{\epsilon}^{\mu'\nu'\rho'\sigma'} g_{\mu'\mu} g_{\nu'\nu} g_{\rho'\rho} g_{\sigma'\sigma})$$

$$= g_{\mu'\mu} g_{\nu'\nu} g_{\rho'\rho} g_{\sigma'\sigma} \nabla_\lambda (\tilde{\epsilon}^{\mu'\nu'\rho'\sigma'}) = 0$$

$$\nabla_\lambda (g \tilde{\epsilon}_{\mu\nu\rho\sigma}) = 0 \Rightarrow \quad \nabla_\lambda g = 0$$

$$\Rightarrow \nabla_\lambda \epsilon_{\mu\nu\rho\sigma} = \nabla_\lambda \left(\sqrt{|g|} \, \tilde{\epsilon}_{\mu\nu\rho\sigma} \right) = \sqrt{|g|} \, \nabla_\lambda \tilde{\epsilon}_{\mu\nu\rho\sigma} = 0 \quad \checkmark$$

christoffel symbols w/ diagonal metric:

in general

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} (\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho})$$

Now, in the following let $\mu \neq \nu \neq \rho$ and repeated indices are <u>not</u> summed over

~~say~~ since $g_{\mu\nu} = 0$ for $\mu \neq \nu$ and $g^{\mu\mu} = \frac{1}{g_{\mu\mu}}$

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\mu} (\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda}) = 0$$

$$\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2} g^{\lambda\lambda} (\cancel{\partial_\mu g_{\lambda\mu}}^{0} + \cancel{\partial_\mu g_{\mu\lambda}}^{0} - \partial_\lambda g_{\mu\mu}) = \frac{-1}{2} g^{\lambda\lambda} \partial_\lambda g_{\mu\mu}$$

$$= \frac{-1}{2 g_{\lambda\lambda}} \partial_\lambda g_{\mu\mu}$$

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2} g^{\lambda\lambda} (\partial_\mu g_{\lambda\lambda} + \cancel{\partial_\lambda g_{\lambda\mu}}^{0} - \cancel{\partial_\lambda g_{\mu\lambda}}^{0}) = \frac{1}{2 g_{\lambda\lambda}} \partial_\mu g_{\lambda\lambda}$$

$$= \partial_\mu \ln \sqrt{|g_{\lambda\lambda}|}$$

$$\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2} g^{\lambda\lambda} (\partial_\lambda g_{\lambda\lambda} + \partial_\lambda g_{\lambda\lambda} - \partial_\lambda g_{\lambda\lambda}) = \frac{1}{2 g_{\lambda\lambda}} \partial_\lambda g_{\lambda\lambda}$$

$$= \partial_\lambda \ln \sqrt{|g_{\lambda\lambda}|}$$

2-sphere

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

a) use our diagonal metric exp

$\Gamma^\theta_{\theta\theta} = 0$ $\quad \Gamma^\theta_{\theta\varphi} = \cancel{\partial_\theta \ln\sqrt{\sin^2\theta} = \frac{\cos\theta}{\sin\theta}} = 0$

$$\Gamma^\theta_{\varphi\varphi} = -\frac{1}{2}\partial_\theta g_{\varphi\varphi} = -\frac{1}{2} 2\sin\theta\cos\theta = -\sin\theta\cos\theta$$

$\Gamma^\varphi_{\varphi\varphi} = 0$ $\quad \Gamma^\varphi_{\varphi\theta} = \partial_\theta(\ln\sqrt{\sin^2\theta} = \frac{\cos\theta}{\sin\theta}$

$$\Gamma^\varphi_{\theta\theta} = 0$$

Geodesic Equations

$$\frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\varphi\varphi}\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \qquad \frac{d^2\varphi}{d\lambda^2} + \Gamma^\varphi_{\varphi\theta}\left(\frac{d\varphi}{d\lambda}\right)\left(\frac{d\theta}{d\lambda}\right) = 0$$

$\rightarrow \frac{d^2\theta}{d\lambda^2} - \sin\theta\cos\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$ and $\frac{d^2\varphi}{d\lambda^2} + \cot\theta \frac{d\varphi}{d\lambda}\frac{d\theta}{d\lambda} = 0$

if $\frac{d\varphi}{d\lambda} = 0$ (constant longitude)

$\rightarrow \frac{d^2\theta}{d\lambda^2} = 0 \Rightarrow x^\mu(\lambda) = (a\lambda + b, c)$ is a geodesic suppose now $\theta = \text{const}$ $\rightarrow \frac{d\theta}{d\lambda} = 0$, $\frac{d\varphi}{d\lambda} \neq 0$

geodesic eqns give

$$-\sin\theta\cos\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \quad \text{and} \quad \frac{d^2\varphi}{d\lambda^2} = 0$$

need $\sin\theta\cos\theta = 0$ only non-trivial ex is $\theta = \frac{\pi}{2}$
(i.e. $\theta = 0$ just gives a point
π not in the chart)

so, only geodesic w/ $\theta = \text{const}$ is $\theta = \frac{\pi}{2}$ ✓

b) parallel transport a vector $V^\mu = (1, 0)$
around a curve of constant lattitude ($\theta = \text{const}$)

tangents to curve w/ constant lattitude

$$t^\mu = \left(0, \frac{d\varphi}{d\lambda}\right)$$

can choose $\varphi = \lambda$ as parameterization of curve

$$\rightarrow t^\mu = (0, 1)$$

now, equation of parallel transport

$$\rightarrow t^\mu \nabla_\mu V^\nu = 0$$

$$\rightarrow \partial_\varphi V^\nu + \Gamma^\nu_{\varphi\alpha} V^\alpha = 0$$

$$\rightarrow \frac{\partial V^\theta}{\partial\varphi} + \underbrace{-\sin\theta\cos\theta}_{\Gamma^\theta_{\varphi\varphi}} V^\varphi = 0 \quad \text{and} \quad \frac{\partial V^\varphi}{\partial\varphi} + \underbrace{\cot\theta}_{\Gamma^\varphi_{\varphi\theta}} V^\theta = 0$$

So we need to solve coupled equations

$$(\mathrm{I}) \quad \frac{\partial V^\theta(\varphi)}{\partial \varphi} - \sin\theta\cos\theta\, V^\varphi(\varphi) = 0$$

$$(\mathrm{II}) \quad \frac{\partial V^\varphi(\varphi)}{\partial \varphi} + \cot\theta\, V^\theta(\varphi) = 0$$

w/ initial conditions $V^\theta(0) = 1 \quad V^\varphi(0) = 0$

and assuming $\theta = \text{const}$

$$\to \quad \partial_\varphi(\mathrm{I}) = \frac{\partial^2 V^\theta}{\partial \varphi^2} - \sin\theta\cos\theta\frac{\partial V^\varphi}{\partial \varphi} = 0$$

$$= \frac{\partial^2 V^\theta}{\partial \varphi^2} - \sin\theta\cos\theta\left[-\cot\theta\, V^\theta(\varphi)\right] = 0$$

$$\frac{\partial^2 V^\theta}{\partial \varphi^2} + \cos^2\theta\, V^\theta = 0$$

$$\to \quad V^\theta(\varphi) = A\cos(\cos\theta\,\varphi) + B\sin(\cos\theta\,\varphi)$$

at $\varphi = 0$

$$V^\theta(\varphi) = A \cancel{\cos(\ldots)} \quad \to \quad A = 1$$

and

$$\frac{\partial V^\varphi}{\partial \varphi} + \frac{\cos\theta}{\sin\theta}\left(\cos(\cos\theta\,\varphi) + B\sin(\cos\theta\,\varphi)\right) = 0$$

$$\to \quad V^\varphi = \frac{-\cos\theta}{\sin\theta}\left[\frac{\sin(\cos\theta\,\varphi)}{\cos\theta} - \frac{B}{\cos\theta}\cos(\cos\theta\,\varphi)\right] + C$$

at $\varphi = 0$

$$V^\varphi = 0 \quad \Rightarrow \quad B = 0 \quad \text{and} \quad C = 0$$

so solution ~~stuff~~ is

$$V^\theta(\varphi) = \cos(\cos\theta\,\varphi)$$

$$V^\varphi(\varphi) = \frac{-\sin(\cos\theta\,\varphi)}{\sin\theta}$$

after $\varphi = 2\pi$

$$\Rightarrow \quad V^\theta(2\pi) = \cos(2\pi\cos\theta)$$

$$V^\varphi(2\pi) = -\frac{\sin(2\pi\cos\theta)}{\sin\theta}$$

so vector returns to itself only for $\theta = \frac{\pi}{2}$

(i.e. it's a geodesic)...

Metric of Earth

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\Phi = \frac{-GM}{r}$$

a) Clock at surface measures time intervals

$$d\tau_{surface} = \left(1 - \frac{2GM}{R_{surface}}\right)^{1/2} dt$$

clock on building measures

$$d\tau_{building} = \left(1 - \frac{2GM}{R_{building}}\right)^{1/2} dt$$

elapsed proper time at position r

$$\Delta\tau = \int dt \sqrt{1 - \frac{2GM}{r}} = \sqrt{1 - \frac{2GM}{r}}\, \Delta t$$

$$\approx \left(1 - \frac{GM}{r}\right) \Delta t$$

for fixed interval Δt, clocks w larger r measure larger $\Delta\tau$ i.e. they "tick" more rapidly

b) Geodesic equation for circular orbit around equator $x^\mu(\lambda) = (t(\lambda), R, \theta = \frac{\pi}{2}, \phi(\lambda))$

radius of orbit

at equator

know $\frac{dx^\mu}{d\lambda} = 0$ for $x^\mu = R, \theta$ but $\frac{dt}{d\lambda} \neq 0$ $\frac{d\phi}{d\lambda} \neq 0$

won't need all Christoffel symbols

have Christoffel symbols

$$\Gamma^t_{tt} = 0 \qquad \Gamma^t_{t\varphi} = 0 \qquad \Gamma^t_{\varphi\varphi} = 0$$

$$\Gamma^r_{tt} \doteq -\frac{1}{2}(1+2\Phi)\,\partial_r\left[(1+2\Phi)\right] = \partial_r\Phi + \mathcal{O}(\Phi^2) \qquad \Gamma^r_{rt} = 0 = \Gamma^r_{r\varphi}$$

$$\Gamma^r_{\varphi\varphi} \doteq -\frac{1}{2}\overset{(1+2\Phi)}{\cancel{(1+2\Phi)}}\cdot 2r\sin^2\theta = -(1+2\Phi)\,r\sin^2\theta$$

$$\Gamma^\theta_{tt} = 0 \qquad \Gamma^\theta_{t\varphi} = 0 \qquad \Gamma^\theta_{\varphi\varphi} = \frac{-1}{2}\frac{1}{r^2}r^2\,2\sin\theta\cos\theta$$

$$= -\sin\theta\cos\theta$$

$$\Gamma^\varphi_{tt} = 0 \qquad \Gamma^\varphi_{\varphi t} = 0 \qquad \Gamma^\varphi_{\varphi\varphi} = 0$$

$$\frac{d^2t}{d\lambda^2} + \Gamma^t_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \Rightarrow \boxed{\frac{d^2t}{d\lambda^2} = 0} \Rightarrow t = a\lambda + b$$

can choose $b = 0$
$a = 1$

$$\frac{d^2r}{d\lambda^2} + \Gamma^r_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \Rightarrow \frac{d^2r}{d\lambda^2} + \partial_r\Phi\left(\frac{dt}{d\lambda}\right)^2 - (1+2\Phi)\,r\sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

$$\frac{dr}{d\lambda} = 0$$

$$\Rightarrow \left(\frac{d\varphi}{d\lambda}\right)^2 = \frac{\partial_r\Phi}{r\sin^2\theta}\left(\frac{dt}{d\lambda}\right)^2 + \mathcal{O}(\Phi^2)$$

$$\rightarrow \boxed{\frac{d\varphi}{dt} = \pm\sqrt{\frac{\partial_r\Phi}{r\sin^2\theta}}}$$

$$\frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \Rightarrow -\sin\theta\cos\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

but $\theta = \frac{\pi}{2}$
so
true

c.) One orbit $\rightarrow \Delta\phi = 2\pi$

$$\Delta t = \int_0^{2\pi} d\phi \frac{dt}{d\phi} = \pm\sqrt{\frac{r\sin^2\theta}{\frac{GM}{r^2}}} \int_0^{2\pi} d\phi = \pm 2\pi r \sin\theta \left(\frac{r}{GM}\right)^{1/2}$$

now, for sattelite $\theta = \frac{\pi}{2}$ $\frac{dr}{dt} = \frac{d\theta}{dt} = 0$

$$\Delta\tau = \int\sqrt{-g_{tt} + g_{\phi\phi}\left(\frac{d\phi}{dt}\right)^2}\, dt$$

$$= \int\sqrt{+\left(1+\frac{2GM}{r}\right) + r^2\frac{GM}{r^3}}\, dt$$

$$\approx \left(1+\frac{3}{2}\frac{GM}{r}\right)\Delta t$$

for Earth $\frac{GM}{R_E} = \frac{6.674\cdot 10^{-11}\frac{m^3}{kg\, s^2}\cdot 5.972\cdot 10^{24} kg}{6.371\cdot 10^6 m}$

$$= 6.256\cdot 10^7 \frac{m^2}{s^2}$$

$c=1 \Rightarrow 2.9979\cdot 10^8 \frac{m}{s} = 1$

$\rightarrow \frac{GM}{R_E} = 6.961\cdot 10^{-10}$ so working to linear order in Φ is super accurate!

$\rightarrow \Delta\tau_{\text{sattellite}} = 2\pi\cdot R_E\cdot\left(\frac{R_E}{GM}\right)^{1/2}\times\left(1+\frac{3}{2}\frac{GM}{R_E}\right)$

$= 5060.98$ seconds

compare to clock at rest on surface

$$\Delta\tau = \int\sqrt{-g_{tt}}\,dt = \left(1+\frac{2GM}{R_E}\right)\Delta t$$

so

$$\frac{\Delta\tau_{sat} - \Delta\tau_{earth}}{\Delta\tau_{eart}} \approx -\frac{1}{2}\frac{GM}{R_E} \approx -3.48\cdot 10^{-10}$$

or a total of $-1.76\cdot 10^{-6}$ s

↑ microsecond diff!!