Homework 5: Due in class Friday April 5th (or beforehand in the box outside my office)

You must write out clear, logical, and complete answers to the homework questions. You may work together on solving the homework problems, but you must write up the final answer on your own.

The scoring of each homework problem will be out of 3 points: 3 points will be given for a complete, 100% correct solution; 2 points will be given for a largely correct (and still clearly explained) solution, 1 point will be given for a poorly explained, incorrect solution. Excellent solutions by students may be photocopied and used as homework solutions available to everyone.

1. **Rindler Space**
Consider an observer in 2-D Minkowski space with a world line given by the parametric equations

$$t(\tau) = a\sinh(\tau/a) \quad x(\tau) = a\cosh(\tau/a) \tag{1}$$

where τ is the proper time for the accelerated observer and a is a constant.

(a) Compute the components and the magnitude of the observer's 2-velocity and acceleration.

(b) Let $\lambda = \tau/a$. Show that the space-like line defined by setting $\lambda = const.$ and varying a is orthogonal to the world line of the accelerated observer where the lines intersect.

(c) For each λ, the curves of constant λ and varying a define a family of space-like lines. This map gives a new coordinate system (λ, a) from (t, x). Sketch the new coordinates on the (t, x) plane. Do these coordinates cover the x, t plane?

(d) Calculate the components of the metric in the (λ, a) coordinate system.

2. **Vector calculus in spherical coordinates**
In this problem you will work out some familiar facts using the mathematical structure introduced in this course. In $\mathbb{R}^3$, we can use Cartesian coordinates (x^1, x^2, x^3). The components of the (Riemannian) metric tensor in the Cartesian coordinate basis are just δ_{ij}. Alternatively, we can use spherical coordinates (r, θ, ϕ) defined through

$$x^1 = r\sin\theta\sin\phi\,, \qquad x^2 = r\sin\theta\cos\phi\,, \quad x^3 = r\cos\theta \tag{2}$$

(a) Express the components of the coordinate basis vectors $\{\partial_r, \partial_\theta, \partial_\phi\}$ in terms of the Cartesian coordinate basis vectors $\{\partial_1, \partial_2, \partial_3\}$.
(b) Express the components of the metric tensor in the spherical coordinate basis AND use this to show that the spherical coordinate basis vectors $\{\partial_r, \partial_\theta, \partial_\phi\}$ are orthogonal, but not orthonormal.
(c) Compute the non-zero Christoffel symbols in the (r, θ, ϕ) coordinates (d) Given a vector $\mathbf{v}$ with coordinate basis components (v^r, v^θ, v^ϕ), write an expression for the divergence $\nabla \cdot \mathbf{v}$ (e) An orthonormal *non-coordinate* basis $\left\{\hat{\mathbf{e}}_{(r)}, \hat{\mathbf{e}}_{(\theta)}, \hat{\mathbf{e}}_{(\phi)}\right\}$ can be constructed by normalizing the coordinate basis vectors $\{\partial_r, \partial_\theta, \partial_\phi\}$. Write an expression for $\nabla \cdot \mathbf{v}$ in terms of the components of $\mathbf{v}$ in this basis $(v^{\hat{r}}, v^{\hat{\theta}}, v^{\hat{\phi}})$

3. de Sitter Space

Four-dimensional spacetime with positive constant spacetime curvature $\kappa = R/12$ is called de Sitter space. (If $\kappa < 0$ the spacetime is called anti de Sitter space). We say this spacetime is *maximally symmetric*. The

Riemann tensor for a (4D) maximally symmetric space takes the simple form

$$R_{\rho\sigma\mu\nu} = \frac{R}{12}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \tag{3}$$

where R is the (constant) scalar curvature.
(a) What $T_{\mu\nu}$ gives rise to de Sitter space? (you do not need an explicit expression for $g_{\mu\nu}$ to answer this, but write your expression in terms of κ)
(b) Instead of having a source $T_{\mu\nu}$, can you change the left-hand side of Einstein's equation (e.g. the part that only depends on spacetime geometry) so that de Sitter space is a solution when $T_{\mu\nu} = 0$?
(c) One way to describe de Sitter space is as hyperboloid embedded in 5D Minkowski spacetime

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \ell^2 \tag{4}$$

where the 5D Minkowski metric is, of course, $ds_5^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2$ and $X^i \in (-\infty, \infty)$. Introduce coordinates on the hyperboloid $(\tau, \chi, \theta, \phi)$

$$X^0 = \ell\sinh(\tau/\ell), \quad X^1 = \ell\cosh(\tau/\ell)\cos(\chi), \quad X^2 = \ell\cosh(\tau/\ell)\sin\chi\cos\theta,$$
$$X^3 = \ell\cosh(\tau/\ell)\sin\chi\sin\theta\sin\phi, \quad X^4 = \ell\cosh(\tau/\ell)\sin\chi\sin\theta\cos\phi \tag{5}$$

Find the metric on the hyperboloid in terms of the $(\tau, \chi, \theta, \phi)$ coordinates.

(d) In a sentence and a sketch of some of the coordinates, describe de Sitter space in these coordinates. What should the ranges of $(\tau, \chi, \theta, \phi)$ be?

(e) Now consider the coordinates (t, x^1, x^2, x^3) defined by

$$X^0 = \ell\sinh t/\ell - \frac{1}{2\ell}x_i x^i e^{-t/\ell}, \quad X^i = x^i e^{-t/\ell} \quad \text{for} \quad i = 1,2,3, \quad X^4 = \ell\cosh t/\ell - \frac{1}{2\ell}x_i x^i e^{-t/\ell} \tag{6}$$

Find the metric in terms of the (t, x^1, x^2, x^3) coordinates. Do these coordinates cover all of de Sitter space?