Killing Vectors and conserved quantities

Suppose we have a Killing vector field ξ. Show there is a conserved quantity along geodesics.

If ξ^a is Killing vector field we have

$$\mathcal{L}_\xi g = 0 \quad \Rightarrow \quad \nabla_a \xi_b + \nabla_b \xi_a = 0$$

Now, consider geodesic w/ tangent v^a

$$\Rightarrow \quad v^a \nabla_a v^b = 0$$

Now consider the quantity $v^a \xi_a$, change in this along geodesic is

$$v^b \nabla_b (v^a \xi_a) = v^b v^a \nabla_b \xi_a = -v^b v^a \nabla_a \xi_b$$
$$= -v^a \nabla_a (v^b \xi_b)$$
$$= -v^b \nabla_b (v^a \xi_a)$$

so we must have

$$v^b \nabla_b (v^a \xi_a) = 0 \quad \Rightarrow \quad v^a \xi_a$$ is conserved along a geodesic.

Carroll 5.3

Consider a (massive!) particle that has fallen to $r < 2GM$, Show that the radial coordinate must decrease at a minimum rate given by $\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}$

OK, if particle has four-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ then we must have

$$g_{\mu\nu} U^\mu U^\nu = -1$$

$$\Rightarrow -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 = -1$$

$$\Rightarrow \left(\frac{dr}{d\tau}\right)^2 = -\left(1 - \frac{2GM}{r}\right) + \left(1 - \frac{2GM}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{r}\right)\left[r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]$$

$$= \left(\frac{2GM}{r} - 1\right) + \underbrace{\left(\frac{2GM}{r} - 1\right)^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{2GM}{r} - 1\right)\left[r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]}_{\text{these terms are all positive (or zero)}}$$

so minimum $\frac{dr}{d\tau}$ is when $\left(\frac{dt}{d\tau}\right), \left(\frac{d\theta}{d\tau}\right), \left(\frac{d\phi}{d\tau}\right) \to 0$

$$\Rightarrow \left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}$$

Now, the lifetime of a particle traveling from

$r = 2GM$ to $r = 0$ is

$$\Delta\tau = d\tau = \left| \int_0^{2GM} \frac{dr}{\sqrt{\frac{2GM}{r} - 1}} \right|$$

Let $\frac{2GM}{r} = \sec^2\theta \implies \cos^2\theta = \frac{r}{2GM}$

θ ranges from $0, \frac{\pi}{2}$

$$\underbrace{-\frac{2GM}{r^2}}_{\frac{-1}{2GM}\sec^4\theta} dr = 2\tan\theta\sec^2\theta \, dr$$

so

$$\Delta\tau = \left| \int_0^{\pi/2} \frac{-4GM \cdot \cancel{\tan\theta} \cdot \cos^2\theta \, d\theta}{\cancel{\sqrt{\sec^2\theta - 1}}} \right|$$

$$= \left| -4GM \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta \right) d\theta \right| = 2GM \left(\theta + \frac{1}{2}\sin 2\theta \right) \Big|_0^{\pi/2}$$

$$= \pi GM = \pi \cdot \frac{R_s}{2}$$

For a solar mass black hole

$$\Delta\tau = \frac{\pi \cdot R_s}{2} \approx \frac{\pi}{2} \cdot \frac{3\text{ km}}{3 \cdot 10^5 \text{ km/s}} \approx 1.57 \cdot 10^{-5} \text{ seconds}$$

We know geodesics extremize proper time.

on a geodesic

$$E = -\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = \text{const.}$$

so

$$\left(\frac{dr}{d\tau}\right)^2 = -\left(\frac{2GM}{r} - 1\right) + \left(\frac{2GM}{r} - 1\right)^2 \frac{E^2}{\left(\frac{2GM}{r} - 1\right)^2} + \left(\frac{2GM}{r} - 1\right)\left[r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]$$

$$= -\left(\frac{2GM}{r} - 1\right) + E^2 + \left(\frac{2GM}{r} - 1\right)\left[r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]$$

for a radial trajectory w/ $E \to 0$ will have

minimum $\frac{dr}{d\tau}$ $\to$ maximum $\Delta\tau$.

Carroll 5.5

Comoving observer at constant $(r, \theta, \phi) = (r_*, \theta_*, \phi_*)$ in Schwarzschild. Drops beacon that emits radiation at λ_{em} in beacon frame.

a) find coordinate speed of beacon

Killing vectors $\Rightarrow$ $E = -g_{\mu\nu} K^\mu \frac{dx^\nu}{d\tau} = \text{const}$

$$\rightarrow \quad E = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = \text{const.}$$

beacon is massive particle $\Rightarrow$ $-1 = g_{\mu\nu} u^\mu u^\nu$ where $u^\mu = \frac{dx^\mu}{d\tau}$

$$\Rightarrow \quad -1 = -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \quad (*)$$

when beacon is dropped $E = \left(1 - \frac{2GM}{r_*}\right)\frac{dt}{d\tau}\Big|_{r_*}$

$$-1 = -\left(1 - \frac{2GM}{r_*}\right)\left(\frac{dt}{d\tau}\right)^2\Big|_{r_*}$$

$$\rightarrow \quad \frac{dt}{d\tau}\Big|_{r_*} = \frac{1}{\sqrt{1 - \frac{2GM}{r_*}}} \quad \rightarrow \quad E = \sqrt{1 - \frac{2GM}{r_*}}$$

now

$$(*) \Rightarrow \quad -1 = -\frac{E^2}{1 - \frac{2GM}{r}} + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2$$

$$\frac{E^2}{\left(1 - \frac{2GM}{r}\right)^3}\left(\frac{dr}{dt}\right)^2 - \frac{E^2}{1 - \frac{2GM}{r}} = -1$$

$$\rightarrow \left(\frac{dr}{dt}\right)^2 = \frac{\left(1-\frac{2GM}{r}\right)^3}{E^2}\left(\frac{E^2}{1-\frac{2GM}{r}} - 1\right)$$

$$= \left(1-\frac{2GM}{r}\right)^2\left(1 - \frac{1-\frac{2GM}{r}}{1-\frac{2GM}{r_*}}\right)$$

$$= \left(1-\frac{2GM}{r}\right)^2\left(\frac{\frac{2GM}{r}-\frac{2GM}{r_*}}{1-\frac{2GM}{r_*}}\right)$$

$$\rightarrow \boxed{\frac{dr}{dt} = \underset{\uparrow}{-}\left(1-\frac{2GM}{r}\right)\sqrt{\frac{2GM(r_*-r)}{r(r_*-2GM)}}}$$

– because we say infally

b) calculate proper speed of beacon w.r.t. locally inertial observer

Locally inertial observer measures intervals as

$$dt_{proper} = \left(1-\frac{2GM}{r}\right)^{1/2} dt$$

$$dr_{proper} = \left(1-\frac{2GM}{r}\right)^{-1/2} dr$$

$$\rightarrow \frac{dr_{proper}}{dt_{proper}} = \frac{1}{\left(1-\frac{2GM}{r}\right)}\frac{dr}{dt} = \sqrt{\frac{2GM(r_*-r)}{r(r_*-2GM)}}$$

as $r \rightarrow 2GM$

$$\frac{dr_{proper}}{dt_{proper}} \longrightarrow \sqrt{\frac{2GM(r_*-2GM)}{2GM(r_*-2GM)}} = 1 \; !$$

c.) Calculate λ_{obs}, wavelength of light observed by comoving observer at r_*

First, note that emitted frequency $\sim$ energy of photon an observer in beacon frame at r_{em} will measure

$$(\text{I}) \quad \frac{h}{\lambda_{em}} = -g_{\mu\nu}(r_{em})\, U^{\mu}_{beacon}(r_{em})\, p^{\nu}_{\gamma} \quad \leftarrow \text{4-momentum of emitted photon}$$

$$U^{\mu}_{beacon} = \frac{dx^{\mu}_{beacon}}{d\tau} \quad \leftarrow \text{we know from part (a)}$$

and

$$(\text{II}) \quad 0 = -g_{\mu\nu}(r_{em}) \frac{dx^{\mu}_{\gamma}(r_{em})}{d\lambda} \frac{dx^{\nu}_{\gamma}(r_{em})}{d\lambda} \rightarrow \frac{dr_{\gamma}}{d\lambda} = \left(1 - \frac{2GM}{r_{em}}\right) \frac{dt_{\gamma}(r_{em})}{d\lambda}$$

so (I) + (II) gives $\frac{dt_{\gamma}(r_{em})}{d\lambda}$

$$\frac{dt_{\gamma}}{d\lambda}(r_{em}) = \frac{h/\lambda_{em}}{\left(1 - \frac{2GM}{r_{em}}\right)\frac{dt_{beacon}}{d\tau_{beacon}} - \frac{dr_{beacon}}{d\tau_{beacon}}}$$

from (a)

$$\frac{dt_{beacon}}{d\tau} = \frac{E_{beacon}}{\left(1 - \frac{2GM}{r_{em}}\right)} = \frac{\sqrt{1 - \frac{2GM}{r_*}}}{\left(1 - \frac{2GM}{r_{em}}\right)}$$

$$\frac{dt_{\gamma}}{d\lambda}(r_{em}) = \frac{h/\lambda_{em}\left(1 - \frac{2GM}{r_{em}}\right)/E_{beacon}}{1 - \frac{2GM}{r_{em}} - \underbrace{\frac{dr}{dt}(r_{em})}_{= -\left(1 - \frac{2GM}{r_{em}}\right)\sqrt{\frac{\frac{2GM}{r} - \frac{2GM}{r_*}}{1 - \frac{2GM}{r_*}}}}}$$

$$= \frac{\frac{h}{\lambda_{em} E_{beacon}}}{1 + \sqrt{\frac{\frac{2GM}{r} - \frac{2GM}{r_*}}{1 - 2GM/r_*}}}$$

now, wavelength measured by comoving observer at radius r

is

$$\frac{h}{\lambda} = -g_{\mu\nu}(r)\, U^{\mu}_{\text{comov}} \frac{dx^{\nu}_{\gamma}}{d\lambda}(r)$$

where $U^{\mu}_{\text{comoving}} = \left(\frac{1}{\sqrt{1-\frac{2GM}{r}}}, 0, 0, 0\right)$

so

$$\lambda_{obs} = \frac{h}{\left(1-\frac{2GM}{r_*}\right)^{1/2} \frac{dt_\gamma}{d\lambda}(r_*)}$$

and, we can relate $\frac{dt_\gamma}{d\lambda}(r_*)$ to $\frac{dt_\gamma}{d\lambda}(r_{em})$ by

conserved energy

$$E_\gamma = -g_{\mu\nu} K^{\mu} \frac{dx^{\nu}}{d\lambda} = \text{const} \quad \text{for } K^{\mu} \sim \partial_t$$

$$\rightarrow \left(1-\frac{2GM}{r_{em}}\right)\frac{dt}{d\lambda}(r_{em}) = \left(1-\frac{2GM}{r_*}\right)\left(\frac{dt}{d\lambda}(r_*)\right)$$

$$\rightarrow \lambda_{obs} = \frac{h\left(1-\frac{2GM}{r_*}\right)}{\left(1-\frac{2GM}{r_*}\right)^{1/2}\left(1-\frac{2GM}{r_{em}}\right)\frac{dt}{d\lambda}(r_{em})}$$

$$= \frac{K\sqrt{1-\frac{2GM}{r_*}} \cdot E_{beacon} \cdot \lambda_{em}/K}{\left(1-\frac{2GM}{r_{em}}\right)\left[1+\sqrt{\frac{\frac{2GM}{r}-\frac{2GM}{r_*}}{1-2GM/r_*}}\right)^{-1}}$$

and,

$$E_{beacon} = \sqrt{1 - \frac{2GM}{r_*}}$$

$$\rightarrow \quad \lambda_{obs} = \left[\frac{\left(1 - \frac{2GM}{r_*}\right)}{\left(1 - \frac{2GM}{r_{em}}\right)} \frac{1}{\left(1 + \sqrt{\frac{\frac{2GM}{r_{em}} - \frac{2GM}{r_*}}{1 - \frac{2GM}{r_*}}}\right)} \right]^{-1} \cdot \lambda_{em}$$

ok, check if $r_{em} = r_*$

$\lambda_{obs} \rightarrow \lambda_{em}$ ✓

$r_{em} \rightarrow 2GM$

$\lambda_{obs} \rightarrow \infty$ ✓

d) calculate time at which beam emitted by beacon at $r = r_{em}$ will be observed at r_*

if $t=0$ is when beacon is dropped then have

$$\Delta t = \underbrace{\Delta t_1}_{\substack{\text{time for} \\ \text{beacon to} \\ \text{fall to } r_{em}}} + \underbrace{\Delta t_2}_{\substack{\text{time for photon} \\ \text{to go from} \\ r_{em} \text{ to } r_*}}$$

$$\Delta t_1 = \int_{t_*}^{t_{em}} dt = -\int_{r_*}^{r_{em}} \frac{dr}{\left(\frac{dr}{dt}_{beacon}\right)} = \int_{r_{em}}^{r_*} \frac{dr}{\left(1 - \frac{2GM}{r}\right)\sqrt{\frac{\frac{2GM}{r} - \frac{2GM}{r_*}}{1 - \frac{2GM}{r_*}}}}$$

$$x = \frac{r}{2GM}$$

then

$$\Delta t_1 = \frac{2GM}{\sqrt{1-\frac{1}{x_*}}} \int \frac{dx}{(1-\frac{1}{x})\sqrt{\frac{1}{x}-\frac{1}{x_*}}} = ?$$

time for photon to go to r_* from r_{em}

$$\Delta t_2 = \int_{t_{em}}^{t_*} dt \quad w/ \quad ds^2 = 0 \Rightarrow \frac{dr}{dt} = \left(1-\frac{2GM}{r}\right)$$

$$\Rightarrow \Delta t_2 = \int_{r_{em}}^{r_*} \frac{dr}{1-\frac{2GM}{r}} = r_* - r_{em} + 2GM \ln\left(\frac{r_* - 2GM}{r_{em} - 2GM}\right)$$

$$t_{obs} = \Delta t_1 + \Delta t_2$$

e.) as $t \to \infty$, $r_{em} \to 2GM$

$$\Delta t_1 \approx \int_{r_{em}}^{r_*} \frac{dr}{\underbrace{\left(1-\frac{2GM}{r}\right)}_{\text{diverging}}} \sqrt{\underbrace{\frac{\frac{2GM}{r}-\frac{2GM}{r_*}}{1-\frac{2GM}{r_*}}}_{\text{approaching } 1}}$$

$$\to \Delta t_1 \approx \int_{r_{em}}^{r_*} \frac{dr}{1-\frac{2GM}{r}} = r_* - r_{em} + 2GM \ln \frac{r_* - 2GM}{r_{em} - 2GM}$$

$$t_{obs} \approx 4GM \ln \frac{r_* - 2GM}{\underbrace{r_{em} \to 2GM}_{\text{diverging}}} \left(+ \underbrace{\sim (r_* - r_{em})}_{\text{finite}}\right)$$

similarly

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{r_{em}(r_* - 2GM)}{r_*(r_{em} - 2GM)}\left[\left(1 + \underbrace{\sqrt{\frac{\frac{2GM}{r_{em}} - \frac{2GM}{r_*}}{1 - \frac{2GM}{r_*}}}}_{\rightarrow 2}\right]\right.$$

as $t \to \infty$ $\quad r_{em} \to 2GM$

$$\frac{\lambda_{obs}}{\lambda_{em}} \longrightarrow 2\,\frac{r_{em}}{r_*}\,\frac{r_* - 2GM}{r_{em} - 2GM}$$

$$\simeq \frac{4GM}{r_*}\, e^{\frac{t_{obs}}{4GM}}$$

so timescale $T = 4GM$ $\quad$ ✓