Reissner-Nordstrom Black Hole

$$ds^2 = -\underbrace{\left(1-\frac{2GM}{r}+\frac{GQ^2}{r^2}\right)}_{\equiv\,\Delta(r)}dt^2+\left(1-\frac{2GM}{r}+\frac{GQ^2}{r^2}\right)^{-1}dr^2+r^2d\Omega^2$$

a.) compute energy-momentum tensor

$$G_{tt} = \frac{GQ^2}{r^4}\left(1-\frac{2GM}{r}+\frac{GQ^2}{r^2}\right)$$

$$G_{rr} = -\frac{GQ^2}{r^4}\left(1-\frac{2GM}{r}+\frac{GQ^2}{r^2}\right)^{-1}$$

$$G_{\theta\theta} = \frac{GQ^2}{r^2} \qquad G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta}$$

$$\Rightarrow \quad 8\pi G\, T^0{}_0 = \frac{-GQ^2}{r^4}$$

recalling E&M $T^\mu{}_\nu$ from previous

$$\rightarrow \quad \vec{E} = \frac{Q}{r^2}\hat{r}$$

b.) Following the procedure from class.

radial null paths are $ds^2 = 0$ w/ $dt = \frac{dr}{\Delta(r)}$

coordinate
singularities at $r_\pm = GM \pm \sqrt{G^2M^2 + GQ^2}$
~~define~~

for $r > r_+$

$$-r_*(r) + r_{*\infty} = \int_r^\infty \frac{dr \cdot r^2}{(r-r_+)(r-r_-)} = \int \frac{r^2 dr}{r_+ - r_-}\left[\frac{1}{r-r_+} - \frac{1}{r-r_-}\right]$$

$$= \frac{1}{r_+ - r_-}\Big[\tfrac{1}{2}(\cancel{r^2} - 2rr_+ + r_+^2) + 2rr_+ - 2r_+^2 + r_+^2\ln(r-r_+)$$
$$-\tfrac{1}{2}(\cancel{r^2} - 2rr_- + r_-^2) - 2rr_- + 2r_-^2 - r_-^2\ln(r-r_-)\Big]_r^\infty$$

$$= \frac{1}{r_+ - r_-}\left[r(r_+ - r_-) + r_+^2\ln(r-r_+) - r_-^2\ln(r-r_-)\right]_r^\infty$$

$$\Rightarrow r_*(r) = r + \frac{r_+^2}{r_+ - r_-}\ln(r-r_+) - \frac{r_-^2}{r_+ - r_-}\ln(r-r_-)$$

for $r_- < r < r_+$

$$r_*(r) = r + \frac{r_+^2}{r_+ - r_-}\ln(r_+ - r) - \frac{r_-^2}{r_+ - r_-}\ln(r - r_-)$$

$0 < r < r_-$

$$r_*(r) = r + \frac{r_+^2}{r_+ - r_-}\ln(r_+ - r) - \frac{r_-^2}{r_+ - r_-}\ln(r_- - r)$$

ok so we can change coordinates to

t, r_* and have

$$ds^2 = -\Delta(r)\left[dt^2 - dr_*^2\right] + r^2 d\Omega^2$$

and

$$e^{r_*} = e^{r + \frac{r_+^2}{r_+ - r_-}\ln|r_* - r_+| - \frac{r_-^2}{r_+ - r_-}\ln|r - r_-|}$$

$$= e^{r}\,|r - r_+|^{\frac{r_+^2}{r_+ - r_-}}\,|r - r_-|^{\frac{-r_-^2}{r_+ - r_-}}$$

$$= e^{r}\,\underbrace{|(r - r_+)(r - r_-)|}_{\Delta(r) \times r^2}^{\frac{r_+^2}{r_+ - r_-}}\,|r - r_-|^{\frac{-r_+^2 - r_-^2}{r_+ - r_-}}$$

$$\rightarrow \left(e^{r_* - r}\,|r - r_-|^{\frac{r_+^2 + r_-^2}{r_+ - r_-}}\right)^{\frac{r_+ - r_-}{r_+^2}} = r^2|\Delta(r)|$$

$$\Delta(r) = \frac{1}{r^2}|r - r_-|^{\frac{r_+^2 + r_-^2}{r_+^2}}\,e^{\frac{r_+ - r_-}{r_+^2}(r_* - r)}$$

$$ds^2 = \underbrace{\frac{-1}{r^2}|r - r_-|^{\frac{r_+^2 + r_-^2}{r_+^2}}\,e^{\frac{r_+ - r_-}{r_+^2}(r_* - r)}}_{\text{not singular as } r \longrightarrow r_+}\left[dt^2 - dr_*^2\right]$$

✓

(still singular as $r \to r_-$

null radial paths given by $t \pm r_* = \text{const}$

change to coordinates $\xi_\pm = t \pm r_*$

then

$$ds^2 = \frac{-1}{r^2}(r-r_-)^{\frac{r_+^2+r_-^2}{r_+^2}} e^{\frac{r_+-r_-}{2r_+^2}(\xi_+-\xi_-)} e^{-\frac{r_+-r_-}{r_+^2}r} \left[d\xi_+ d\xi_-\right]$$

now, eliminate by changing to

$$U = -e^{-\frac{r_+-r_-}{2r_+^2}\xi_-} \qquad V = e^{\frac{r_+-r_-}{2r_+^2}\xi_+}$$

$$dU = \frac{+(r_+-r_-)}{2r_+^2} e^{-\frac{r_+-r_-}{2r_+^2}\xi_-} d\xi_- \qquad dV = \frac{(r_+-r_-)}{2r_+^2} e^{-\frac{r_+-r_-}{2r_+^2}\xi_+}$$

$$\to ds^2 = \frac{\cancel{+}1}{r^2}\left(\frac{2r_+^2}{r_+-r_-}\right)^2 (r-r_-)^{\frac{r_+^2+r_-^2}{r_+^2}} e^{-\frac{(r_+-r_-)r}{r_+^2}} dU\,dV$$

finally $T = \frac{U+V}{2}$ $\qquad R = \frac{V-U}{2}$

$r > r_+$

$$\rightsquigarrow ds^2 = -\frac{1}{r^2}\left(\frac{2r_+^2}{r_+-r_-}\right)^2 (r-r_-)^{\frac{r_+^2+r_-^2}{r_+^2}} e^{-\frac{(r_+-r_-)r}{r_+^2}} \left[-dT^2 + dR^2\right]$$

note for $r < r_+$ could use

$$ds^2 = \frac{-1}{r^2}\left(\frac{2r_-^2}{r_+-r_-}\right)^2 (r_+-r)^{\frac{r_+^2+r_-^2}{r_-^2}} e^{-\frac{(r_+-r_-)r}{r_-^2}} \left[-dT^2+dR^2\right]$$

$$\boxed{R^2-T^2 = -UV = e^{\frac{r_+-r_-}{2r_+^2}(\xi_+-\xi_-)} = e^{\frac{r_+-r_-}{2r_+^2} r_*}}$$

$$\boxed{= e^{\frac{r_+-r_-}{2r_+^2} r} \times (|r-r_+|)\,|r-r_-|^{-\frac{r_-^2}{r_+^2}}}$$

and

$$t = \tfrac{1}{2}(\xi_+ + \xi_-) = \frac{1}{2}\left(\frac{-2r_+^2}{r_+-r_-}\ln(|U|) + \frac{2r_+^2}{r_+-r_-}\ln V\right)$$

$$\boxed{= \frac{r_+^2}{r_+-r_-}\ln\left(\frac{V}{|U|}\right) = \frac{r_+^2}{r_+-r_-}\ln\left(\frac{R+T}{R-T}\right)}$$

can have $U,V \in (-\infty,\infty)$

how map to finite range to get penrose

e.g. $\tan\left(\frac{\tau \pm \rho}{2}\right) = \frac{T \pm R}{2r_+}$

spatial infinity $\quad r \to \infty, \ t$ - finite

$$\to \tan\frac{\tau \pm \rho}{2} \to \pm\infty \to \frac{\tau+\rho}{2} \to \frac{\pi}{2}$$

$$\frac{\tau - \rho}{2} \to -\frac{\pi}{2}$$

$$\tau \to 0 \quad \rho \to \pi$$

past null infinity $\quad r \to \infty, \ t \to -\infty \quad r + t = \text{const}$

$$R - T \to \infty \Rightarrow \frac{\tau - \rho}{2} \to \frac{\pi}{2} \to \rho = \tau - \pi$$

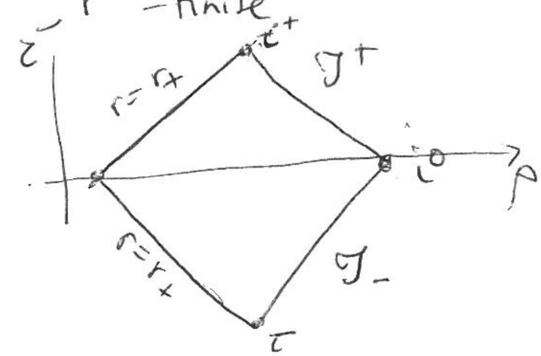
future null infinity $\quad r \to \infty \quad t \to +\infty \quad r - t = \text{const}$

$$R + T \to \infty \quad \Rightarrow \rho = \pi - \tau$$

past / future spatial infinity $\quad t \to \pm\infty, \ r$ - finite

$$\to \quad \tau = \pi/2, \ \rho = \pi/2$$

$$\tau = \pi/2 \quad \rho = \pi/2$$



$$r \to r_+ \quad \Rightarrow \quad R^2 - T^2 = 0 \quad \to \text{null}$$

NOW,

$r \to r_- \quad \Rightarrow \quad R^2 - T^2 = 0 \quad \to$ null $\quad$ but will have

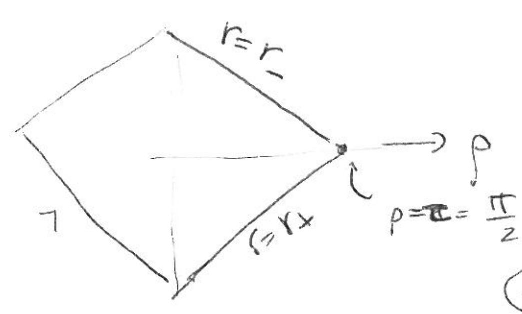
$U > 0$

$V > 0$

$$\Rightarrow 0 < \frac{\tau + \rho}{2} < \frac{\pi}{2} \quad \to \quad \begin{matrix} \tau > \rho \\ \tau < \pi - \rho \end{matrix}$$

horizon at $\quad r \to r_- \quad$ at $\quad \tau = \pi - \rho$

$\longrightarrow$



Finally, $0 < r < r_-$

$$r \to 0 \qquad R^2 - T^2 = -UV = -|r_+||r_-|^{-\frac{r_-^2}{r_+^2}} < 0$$

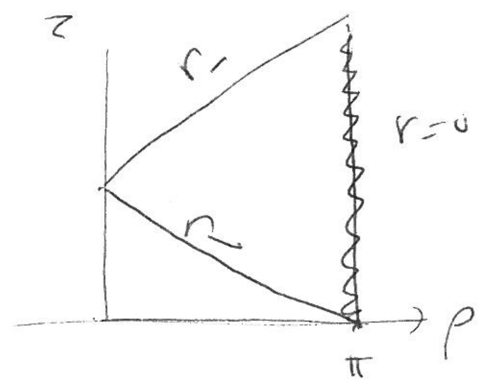
singularity timelike $\quad r_* \to \frac{r_+^2}{r_+ - r_-}\ln r_+ - \frac{r_-^2}{r_+ - r_-}\ln r_-$

$$U = r_+ e^{-\frac{r_+ - r_-}{2r_+^2}(t)} |r_-|^{\frac{r_-^2}{2r_+^2}} = T - R \qquad V = r_+ e^{\frac{r_+ - r_-}{2r_+^2}t} |r_+|^{-\frac{r_-^2}{2r_+^2}} = T + R$$

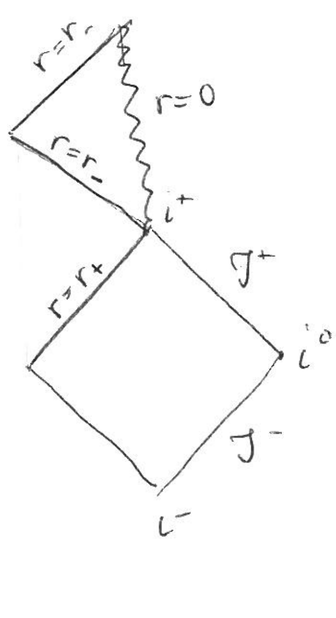
$$UV = r_+^2 \qquad T^2 - R^2 = r_+^2$$

$$\tan\left(\frac{\tau - \rho}{2}\right) = \frac{1}{\tan\left(\frac{\tau + \rho}{2}\right)} \xrightarrow{\text{solved by}} \rho = \frac{\pi}{2}$$

$U, V > 0$ as long as $\frac{3\pi}{2} < \tau < 2\pi$



together



d) Show that coordinates (T, χ, θ, ϕ) with

$\cosh \tau/\ell = \frac{1}{\cos T}$ make de Sitter metric conformal to

$$ds^2_{\text{Einstein-static}} = -dT^2 + d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

w/ $\cosh \tau/\ell = \frac{1}{\cos T}$ $\qquad \frac{1}{\ell}\sinh \tau/\ell\, d\tau = \frac{-\sin T}{\cos^2 T}\, dT$

$$\Rightarrow \underbrace{\frac{1}{\ell^2}\sinh^2 \tau/\ell\, d\tau}_{} = \frac{\sin^2 T}{\cos^4 T}\, dT^2$$

$$= \frac{1}{\ell^2}(\cosh^2 \tau/\ell - 1)\, d\tau^2 \qquad d\tau^2 = \cancel{\ell^2 \tan^2 T}\ \frac{\ell^2 \cancel{\cot^2 T \tan^2 T}}{\cos^2 T}\, dT^2$$

$$= \frac{1}{\ell^2}\Big(\underbrace{\frac{1}{\cos^2 T} - 1}_{\tan^2 T}\Big) \quad \Longrightarrow \quad = \frac{\ell^2}{\cos^2 T}\, dT^2$$

$\rightarrow$ our metric becomes

$$\boxed{ds_4^2 = -\frac{\ell^2}{\cos^2 T}\, dT^2 + \frac{\ell^2}{\cos^2 T}\left[d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi \sin^2\theta\, d\phi^2\right]}$$

$$= \frac{\ell^2}{\cos^2 T}\left[-dT^2 + d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

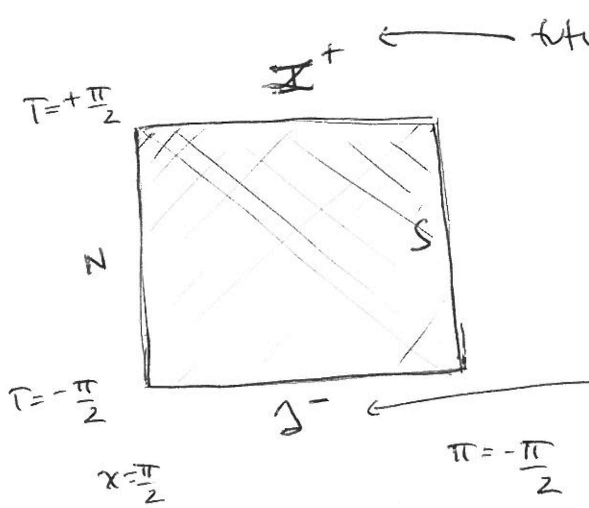
$\cosh \tau/\ell \in (1, \infty)$ $\qquad \tau \in (-\infty, \infty) \Rightarrow \boxed{T \in (-\pi/2, \pi/2)}$

$T \in (0, \pi/2)$

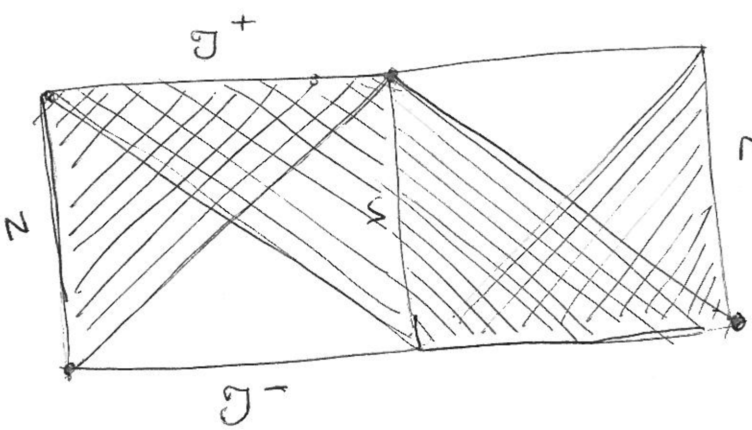
e.) use T, χ coordinates for conformal diagram.

$\chi = -\pi/2$ south pole null geodesics on $T \pm \chi = \text{const}$

$\chi = +\pi/2$ north pole



note, we can add a copy on the other side of south pole S to understand causal structure (points that are reached by rotations in θ, ϕ).



causal past of future null infinity is shaded region

does <u>not</u> cover all spacetime ⇒ <u>Horizon</u>

- light ray: at north or south pull at $\mathcal{I}^-$ gust makes it to opposite pole at $\mathcal{I}^+$.
- observer at either pole does <u>not</u> have access to entire spacetime. There