Homework 8: Due at the beginning of class on Friday May 10th

You must write out clear, logical, and complete answers to the homework questions. You may work together on solving the homework problems, but you must write up the final answer on your own.

The scoring of each homework problem will be out of 3 points: 3 points will be given for a complete, 100% correct solution; 2 points will be given for a largely correct (and still clearly explained) solution, 1 point will be given for a poorly explained, incorrect solution. Excellent solutions by students may be photocopied and used as homework solutions available to everyone.

- *Cosmology*

  In this problem you will compute the evolution of the metric of the entire Universe! You will do this for various assumptions about the energy content of the Universe and will make use of both the usual form of Einstein's Equation, $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$, and the form of Einstein's Equation that includes a cosmological constant $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$.

  The metric for a spatially flat, homogeneous, isotropic universe can be written as

  $$ds^2 = -dt^2 + a(t)^2(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \tag{1}$$

  (a) The metric above is a solution to Einstein's equations with stress-energy tensor $T^{\mu\nu} = (\rho+P)U^\mu U^\nu + Pg^{\mu\nu}$ where ρ and P are functions of time only and U^μ is the four velocity which, due to the symmetries of the problem, can be chosen to be $U^\mu = (1,0,0,0)$. Write the continuity equations ($\nabla_\mu T^{\mu\nu} = 0$) in terms of ρ, P and a.

  (b) Suppose that $P = w\rho$ where w is a constant. Using your results from (a) to find an expression for ρ in terms of a. Check your results for $\rho(a)$ with the values $w = 0$, $w = \frac{1}{3}$, and $w = -1$. What kinds of matter do they describe?

  (c) Compute Einstein's Equations (without Λ) in terms of a, k, ρ and P and rearrange to get an equation with $\dot{a}/a$ only (the Friedmann Equation), and another equation with $\ddot{a}/a$ only (the acceleration equation).

  (d) Using your results from (b) and (c) find a power law solution for $a(t)$ in terms of w when $k = 0$. What is $a(t)$ for $w = -1$?

  (e) Find a parametric solution for a and t for a universe with a single fluid with $w = 0$, but $k \neq 0$. For $k > 0$, what is the maximum value of a? What is the lifetime of this universe?

  (f) Consider Einstein's equations with the cosmological constant Λ and matter with $w = 0$. Write down the Friedmann and acceleration equations and show that static solutions exist if and only if $k = +1$ and $\Lambda > 0$.

  (g) Continuing from (f), determine the relationship between a and ρ for the static universe and show that this universe (this static value of a) is unstable to small perturbations in $\delta\rho$.

(h) Matter with equation of state $w = -1$ and $\rho = const$ causes the exact same effect on $a(t)$ as the cosmological constant Λ. Let's call this matter ρ_{vac}. Observationally, we have determined that the sum of $\rho_{vac} + \Lambda/(8\pi G)$ has a total energy density of $\rho_{const,obs} \sim 3 \times 10^{-6}\text{GeV/cm}^3$. Suppose that ρ_{vac} is vacuum energy from (i) Planck-scale fluctuations with $M_{Planck} \sim 10^{19}$GeV (ii) SUSY breaking, say $M_{SUSY} \sim 10$ TeV (iii) electroweak symmetry breaking ~ 0.2TeV. For each case, compute what Λ has to be for the sum $\rho_{vac} + \Lambda/(8\pi G)$ to be equal to it's observed value (you may use units, eV/cm^3, or natural units eV^4, or whatever is convenient). How many decimal places do you need to keep to insure that $\rho_{const.,obs} \neq 0$?

- *Linearized Gravity and Gravitational Waves*
  In class, I introduced the perturbed form of the metric:

$$ds^2 = -(1+2\Phi)dt^2 - (\partial_i B - w_i)(dx^i dt + dt dx^i) + ((1-2\psi)\delta_{ij} \\ -2\partial_i\partial_j E - 2(\partial_i V_j + \partial_j V_i) + 2s_{ij})\, dx^i dx^j \tag{2}$$

(a) Show that a suitable coordinate transformation ($x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu$ where $\xi \sim \mathcal{O}(h_{\mu\nu})$) can eliminate the B, E, and V_i.

(b) Compute the Einstein tensor with this gauge choice. Show that Einstein's Equations are

$$\nabla^2\psi = 4\pi G T_{00} \tag{3}$$

$$\frac{1}{2}\nabla^2 w_i + 2\partial_i\partial_0\psi = 8\pi G T_{0i} \tag{4}$$

$$\left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)(\Phi - \psi) + \partial_0\partial_{(i}w_{j)} + 2\partial_0^2\psi\delta_{ij} - \Box s_{ij} = 8\pi G T_{ij} \tag{5}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ and $\Box = \partial_0^2 - \nabla^2$.

(c) Show that in the absence of any sources (e.g. $T_{\mu\nu} = 0$), all of the metric perturbations can be set to zero, up to s_{ij}.

(d) The full metric perturbation with all components except s_{ij} set to zero is often denoted $h^{TT}_{\mu\nu}$ for "transverse-traceless" gauge, e.g. $h^{TT}_{\mu\nu} = 2s_{ij}$ with the time components set to zero. Alternatively, the transverse-traceless gauge condition can be defined by requiring $h_{0\nu} = 0$, $\eta^{\mu\nu}h_{\mu\nu} = 0$, $\partial_\mu h^{\mu\nu} = 0$. In part (c) you found that $\Box h^{TT}_{\mu\nu} = 0$, or that $h^{TT}_{\mu\nu}$ satisfies a wave equation. Show that solutions of that equation are

$$h^{TT}_{\mu\nu} = C_{\mu\nu}e^{ik_\alpha x^\alpha} \tag{6}$$

where the components of $C_{\mu\nu}$ and k^α are constants. What is the condition on k^α for this to be true?

(e) Suppose that $h^{TT}_{\mu\nu}$ is a wave propagating in the z direction, what are the components of $C_{\mu\nu}$?