3. Cosmology

a) continuity equation

$$T_{00} = \rho$$

$$T_{11} = \frac{a^2 P}{1-kr^2} \qquad T_{22} = a^2 P r^2 \quad T_{33} = a^2 r^2 \sin^2\theta \, P$$

$$T^{00} = \rho \quad T^{11} = \frac{P}{a^2}(1-kr^2) \quad T^{22} = \frac{P}{a^2 r^2} \quad T^{33} = \frac{P}{a^2 r^2 \sin^2\theta}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$\nu = 0$ eqn $\Rightarrow$ $\frac{\partial \rho}{\partial t} + 3\frac{\dot a}{a}(\rho + P) = 0$

$\nu = i$ eqn $\Rightarrow$ $0 = 0$

b) Let $P = w\rho$ w/ $w$ = const.

$$\rightarrow \frac{\partial \rho}{\partial t} = -3\frac{\dot a}{a}(1+w)\rho \Rightarrow \frac{d\rho}{d\ln a} = -3(1+w)\rho$$

$$\Rightarrow \rho \propto a^{-3(1+w)} \qquad \text{w/ } \rho(a=1) = \rho_0$$

$$\Rightarrow \rho = \rho_0 a^{-3(1+w)}$$

$w = 0$ $\rho \propto 1/a^3$ dust

$w = 1/3$ $\rho \propto 1/a^4$ radiation

$w = -1$ $\rho = \underline{\text{const}}$ $\Lambda$

c.) $H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho$ $\frac{\ddot{a}}{a} = \frac{-4\pi G}{3} (\rho + 3P)$

where $H = \frac{\dot{a}}{a}$

d) $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_0 a^{-3(1+w)}$

w/ $a \sim t^\alpha$ $\rightarrow$ $\frac{\alpha^2}{t^2} = \frac{8\pi G}{3} \rho_0 t^{-3(1+w)\alpha}$

$\rightarrow \alpha^2 = \frac{8\pi G}{3} \rho_0 t^{-3(1+w)\alpha + 2}$

need $-3(1+w)\alpha = -2$ $\rightarrow$ $\alpha = \frac{2}{3(1+w)}$

for $w \neq -1$ $a(t) = (t/t_0)^{\frac{2}{3(1+w)}}$

$w = -1$ $\dot{a} = \pm\sqrt{\frac{8\pi G}{3} \rho_0}\, a \Rightarrow a(t) = e^{\pm\sqrt{\frac{8\pi G}{3}\rho_0}\, t}$

e.)

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_0 a^{-3}$$

$$\Rightarrow \frac{da}{dt} = \pm\sqrt{\frac{8\pi G}{3}\frac{\rho_0}{a} - k}$$

$$\rightarrow \frac{da}{\pm\sqrt{\frac{8\pi G}{3}\frac{\rho_0}{a} - k}} = dt$$

let $\frac{8\pi G\rho_0}{3ka} = \csc^2\theta$

$$-\frac{8\pi G\rho_0}{3k}\frac{da}{a^2} = -2\cot\theta\csc^2\theta\, d\theta$$

$$\rightarrow \frac{8\pi G\rho_0}{3k}\frac{2\cot\theta\csc^2\theta \frac{1}{\csc^2\theta}}{\sqrt{k}}\tan\theta\, d\theta$$

$$\rightarrow t(\theta) = \frac{16\pi G\rho_0}{3k^{3/2}}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)$$

$$a(\theta) = \frac{8\pi G\rho_0}{3k}\sin^2\theta$$

or $\omega = 2\theta$

$$\Rightarrow \boxed{\begin{aligned} t(\omega) &= \frac{4\pi G\rho_0}{3k^{3/2}}(\omega - \sin\omega) \\ a(\omega) &= \frac{4\pi G\rho_0}{3k}(1-\cos\omega) \end{aligned}}$$

maximum value of a at $\omega = \pi$ $\quad a_{max} = \frac{8\pi G\rho_0}{3k}$

lifetime → at $\omega = 0$, $a(\omega) = 0$ at $\omega = (2\pi)$ $a(\omega) \to 0$ again, then expands to max value of $a_{max}$, recollapses at $\omega = 2\pi$ $\quad t(2\pi) = \frac{8\pi^2 G\rho_0}{3k^{3/2}}$

f) Einstein's w/ $\Lambda$ & matter w/ $\rho \propto 1/a^3$

(I) $$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

(II) $$\frac{\ddot a}{a} = \frac{-4\pi G}{3}\rho + \frac{\Lambda}{3}$$

want $\frac{\ddot a}{a} = 0$, $\frac{\dot a}{a} = 0$ $\Rightarrow$ $\Lambda = 4\pi G\rho$

$$\rightarrow \frac{k}{a^2} = \frac{12\pi G}{3}\rho = 4\pi G\rho$$

$$\rightarrow a_{static} = \left(\frac{k}{4\pi G\rho}\right)^{1/2}$$

needed $\Lambda, k > 0$ $\rightarrow$ $k = +1$

g) Let $a = a_{static} + \delta a$

(I) $$\rightarrow \cancel{\frac{\delta\dot a\, \dot a_s}{a_s^2}}^{\,0} - \cancel{\frac{2\delta a\, \dot a_s^2}{a_s^3}}^{\,0} - \frac{\delta a\, k}{a_s^3} = \frac{8\pi G}{3}(\delta\rho)$$

$$\delta\rho = \frac{-3k}{8\pi G}\frac{\delta a}{a_s^3}$$

$\longrightarrow$

(II) $$\Rightarrow \frac{\delta\ddot a}{a_s} = \frac{-4\pi G\delta\rho}{3} = +\frac{4\pi G}{3}\left(\frac{-3k}{8\pi G}\frac{\delta a}{a_s^3}\right) = \frac{k\,\delta a}{2a_s^3}$$

$$\rightarrow \delta\ddot a - \underbrace{\frac{k}{2a_s^2}}_{>0}\delta a = 0$$

$$\Rightarrow \delta a(t) = c_1 e^{\sqrt{\frac{k}{2a_s^2}}t} + c_2 e^{-\sqrt{\frac{k}{2a_s^2}}t}$$

unstable!!

h.)

want

$$\rho_{const,obs} \sim 3\times10^{-6}\ \frac{GeV}{cm^3} \sim 3\times10^{-6}\ \frac{GeV}{cm^3} \times \left(197\times10^{-3}\ GeV \times 10^{-13}\ cm\right)^3$$

$$\simeq 2\times10^{-44}\ (GeV)^4$$

$$\Lambda \simeq 8\pi G\ (\rho_{obs} - \rho_{vac})$$

$$\simeq \frac{8\pi}{M_{pe}^2}(\rho_{obs} - \rho_{vac}) = 8\pi\frac{\rho_{vac}}{M_{pe}^2}\left(\frac{\rho_{obs}}{\rho_{vac}} - 1\right)$$

i) $\Lambda_1 = \frac{8\pi\rho_{vac}}{M_p^2}\left(\frac{\rho_{obs}}{\rho_{vac}} - 1\right)$ $\quad M_{pe} \simeq 1.29\times10^{19}\ GeV$

$$\simeq 8\pi\frac{M_{pe}^4}{M_{pe}^2}\left(10^{-44-4\times19} - 1\right)$$

$$\simeq 8\pi M_{pe}^2\left(10^{-120} - 1\right) \Rightarrow \Lambda \simeq -8\pi M_{pe}^2$$

need 120 decimal places to cancel!
but not 121 . . .

ii) $\Lambda = \frac{8\pi m_{susy}^4}{m_{pe}^2}\left(10^{-44-4\times4} - 1\right)$ $\quad$ use, $m_{susy} \simeq 10^4\ GeV$

$$\simeq 8\pi\frac{m_{susy}^4}{m_{pe}^2}\left(10^{-60} - 1\right) \rightarrow \Lambda \approx 8\pi\frac{m_{susy}^4}{m_{pe}^2}$$

need 60 decimal places!

iii) $\Lambda = 8\pi\frac{m_{EW}^4}{m_{pe}^2}\left(10^{-44-4\times2} - 1\right) \Rightarrow \Lambda \simeq 8\pi\frac{m_{EW}^4}{m_{pe}^2}$

need 52 decimal places