Name:______________________________ 39

Please write your answers to each part of the problem clearly and show the logic you have used to arrive at your solutions.

There are three problems and you may not have time to solve all three. I suggest skimming the exam and deciding which problems you want to solve first before you begin.

The following expressions may be useful:

- Christoffel Symbol in coordinate basis:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{1}$$

- The Christoffel Symbol in a coordinate basis for a diagonal metric (below assume $\mu \neq \nu \neq \lambda$ and repeated indices *are not* summed over):

$$\Gamma^{\lambda}_{\mu\nu} = 0 \quad \Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_\lambda g_{\mu\mu}, \quad \Gamma^{\lambda}_{\mu\lambda} = \partial_\mu \left(\ln\sqrt{g_{\lambda\lambda}}\right), \quad \Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda \left(\ln\sqrt{g_{\lambda\lambda}}\right) \tag{2}$$

- Riemann tensor in coordinate basis:

$$R^{\rho}{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu} \tag{3}$$

- Symmetries of the Riemann tensor:

$$R_{\rho\mu\sigma\nu} = -R_{\mu\rho\sigma\nu} = -R_{\rho\mu\nu\sigma}, \quad R_{\rho\mu\sigma\nu} = R_{\sigma\nu\rho\mu}, \quad R_{\rho\mu\sigma\nu} + R_{\rho\sigma\nu\mu} + R_{\rho\nu\mu\sigma} = 0 \tag{4}$$

- Ricci tensor in coordinate basis: $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$
- Einstein's Equations: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$

Questions

12✓ 1. **Rindler Space**

Consider an observer in 2-D Minkowski space with a world line given by the parametric equations

$$t(\tau) = a\sinh(\tau/a) \quad x(\tau) = a\cosh(\tau/a) \tag{5}$$

where τ is the proper time for the accelerated observer and a is a constant.

(a) Compute the components and the magnitude of the observer's 2-velocity and acceleration.

what does a correspond to?

2-velocity $\left(\frac{dt}{d\tau}, \frac{dx}{d\tau}\right) = \left(\cosh(\tau/a), \sinh(\tau/a)\right)$

magnitude $= -\cosh^2(\tau/a) + \sinh^2(\tau/a) = -1$

2-acceleration $= \left(\frac{d^2t}{d\tau^2}, \frac{d^2x}{d\tau^2}\right) = \left(\frac{1}{a}\sinh\left(\frac{\tau}{a}\right), \frac{1}{a}\cosh(\tau/a)\right)$

magnitude $= -\frac{1}{a^2}\sinh^2\left(\frac{\tau}{a}\right) + \frac{1}{a^2}\cosh^2\left(\frac{\tau}{a}\right) = \frac{1}{a^2}$

(b) Let $\lambda = \tau/a$. Show that the space-like line defined by setting $\lambda = const.$ and varying a is orthogonal to the world line of the accelerated observer where the lines intersect.

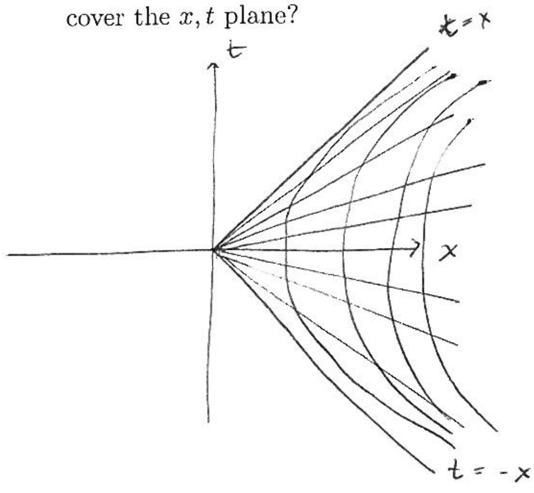
define a line parametrized by $(a\sinh\lambda, a\cosh\lambda)$

at value of $\lambda = \tau/a$, a lines intersect line segments in each direction have tangents $(\sinh\lambda, \cosh\lambda)$ and $(\cosh\frac{\tau}{a}, \sinh\frac{\tau}{a})$

at $\lambda = \tau/a$ so

inner product is $-\sinh\lambda\cosh\lambda + \cosh\lambda\sinh\lambda = 0$

(c) For each λ, the curves of constant λ and varying a define a family of space-like lines. This map gives a new coordinate system (λ, a) from (t, x). Sketch the new coordinates on the (t, x) plane. Do these coordinates cover the x, t plane?



$(t = a\sinh\lambda, x = a\cosh\lambda)$

$t = \tanh\lambda\, x$ ← lines of fixed λ

$t = \pm a\sqrt{\cosh^2\lambda - 1} = \pm\sqrt{x^2 - a^2}$ ← lines of fixed a

only cover 1/4 of the plane

(d) Calculate the components of the metric in the (λ, a) coordinate system.

$$g_{\lambda\lambda} = \left(\frac{\partial x}{\partial \lambda}\right)^2 g_{x\lambda} + \left(\frac{\partial t}{\partial \lambda}\right)^2 g_{tt}$$

$$= (a \sinh^2 \lambda)^2 - (a \cosh \lambda)^2$$

$$= -a^2$$

$$g_{\lambda a} = \left(\frac{\partial x}{\partial \lambda}\right)\left(\frac{\partial x}{\partial a}\right) g_{xx} + \left(\frac{\partial t}{\partial \lambda}\right)\left(\frac{\partial t}{\partial a}\right) g_{tt}$$

$$= (a \sinh \lambda) \cosh \lambda - a \cosh \lambda \sinh \lambda = 0$$

$$g_{aa} = \left(\frac{\partial x}{\partial a}\right)^2 g_{xx} + \left(\frac{\partial t}{\partial a}\right)^2 g_{tt}$$

$$\cosh^2 \lambda - \sinh^2 x$$

$$= 1$$

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2. **Vector calculus in spherical coordinates**
In this problem you will work out some familiar facts using the mathematical structure introduced in this course. In $\mathbb{R}^3$, we can use Cartesian coordinates (x^1, x^2, x^3). The components of the (Riemannian) metric tensor in the Cartesian coordinate basis are just δ_{ij}. Alternatively, we can use spherical coordinates (r, θ, ϕ) defined through

$$x^1 = r\sin\theta\sin\phi\,, \qquad x^2 = r\sin\theta\cos\phi\,, \quad x^3 = r\cos\theta \tag{6}$$

(a) Express the components of the coordinate basis vectors $\{\partial_r, \partial_\theta, \partial_\phi\}$ in terms of the Cartesian coordinate basis vectors $\{\partial_1, \partial_2, \partial_3\}$.

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \sin\theta\sin\varphi\,\partial_x + \sin\theta\cos\varphi\,\partial_y + \cos\theta\,\partial_z$$

$$\partial_\theta = r\cos\theta\sin\varphi\,\partial_x + r\cos\theta\cos\varphi\,\partial_y - r\sin\theta\,\partial_z$$

$$\partial_\varphi = r\sin\theta\cos\varphi\,\partial_x - r\sin\theta\sin\varphi\,\partial_y$$

(b) Express the components of the metric tensor in the spherical coordinate basis AND use this to show that the spherical coordinate basis vectors $\{\partial_r, \partial_\theta, \partial_\phi\}$ are orthogonal, but not orthonormal.

$$g_{rr} = \left(\frac{\partial x}{\partial r}\right)^2 g_{xx} + \left(\frac{\partial y}{\partial r}\right)^2 g_{yy} + \left(\frac{\partial z}{\partial r}\right) g_{zz} = \sin^2\theta\sin^2\varphi + \sin^2\theta\cos^2\varphi + \cos^2\theta = 1$$

$$g_{\theta\theta} = r^2\cos^2\theta\sin^2\varphi + r^2\cos^2\theta\cos^2\varphi + r^2\sin^2\theta = r^2$$

$$g_{\varphi\varphi} = r^2\sin^2\theta$$

$$\rightarrow \quad g(\partial_r, \partial_r) = 1 \quad g(\partial_r, \partial_\theta) = 0\,, \quad g(\partial_r, \partial_\varphi) = 0 \quad \Big| \quad g(\partial_\varphi, \partial_\varphi) = r^2\sin^2\theta$$

$$g(\partial_\theta, \partial_\theta) = r^2 \quad g(\partial_\theta, \partial_\varphi) = 0$$

orthogonal, not orthonormal

(c) Compute the non-zero Christoffel symbols in the (r, θ, ϕ) coordinates

$$\Gamma^r_{rr} = 0 \quad \Gamma^\theta_{\theta\theta} = 0 \quad \Gamma^\varphi_{\varphi\varphi} = 0 \qquad \Gamma^r_{r\theta} = \Gamma^r_{\theta\varphi} = \Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = 0$$

$$\Gamma^r_{\theta\theta} = \frac{-1}{2}\frac{\partial}{\partial r}(r^2) = -r \qquad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta$$

$$\Gamma^\theta_{\varphi\varphi} = \frac{-1}{2r^2}\frac{\partial}{\partial\theta}(r^2\sin^2\theta) = -\sin\theta\cos\theta$$

$$\Gamma^\theta_{r\theta} = \frac{\partial}{\partial r}\ln\left(\sqrt{r^2}\right) = \frac{1}{r} = \Gamma^\theta_{\theta r}\,; \qquad \Gamma^\varphi_{r\varphi} = \frac{\partial}{\partial r}\left[\ln\sqrt{r^2\sin^2\theta}\right] = \frac{1}{r}$$

$$\Gamma^\varphi_{\theta\varphi} = \frac{\partial}{\partial\theta}\ln\left(\sqrt{r^2\sin^2\theta}\right) = \frac{\cos\theta}{\sin\theta} = \Gamma^\varphi_{\varphi\theta}$$

(d) Given a vector $\mathbf{v}$ with coordinate basis components (v^r, v^θ, v^ϕ), write an expression for the divergence $\nabla \cdot \mathbf{v}$ (using Christoffel symbols from (c))

$$V = V^r \partial_r + V^\theta \partial_\theta + V^\phi \partial_\phi$$

$$\nabla \cdot V = \nabla_r V^r + \nabla_\theta V^\theta + \nabla_\phi V^\phi$$

$$= \partial_r V^r + \Gamma^r_{r\mu} V^\mu + \partial_\theta V^\theta + \Gamma^\theta_{\theta\mu} V^\mu + \partial_\phi V^\phi + \Gamma^\phi_{\phi\mu} V^\mu$$

$$= \partial_r V^r + \partial_\theta V^\theta + \frac{1}{r} V^r + \partial_\phi V^\phi + \frac{1}{r} V^r + \frac{\cos\theta}{\sin\theta} V^\theta$$

$$= \partial_r V^r + \frac{2}{r} V^r + \partial_\theta V^\theta + \frac{\cos\theta}{\sin\theta} V^\theta + \partial_\phi V^\phi$$

(e) An orthonormal *non-coordinate* basis $\{\hat{\mathbf{e}}_{(r)}, \hat{\mathbf{e}}_{(\theta)}, \hat{\mathbf{e}}_{(\phi)}\}$ can be constructed by normalizing the coordinate basis vectors $\{\partial_r, \partial_\theta, \partial_\phi\}$. Write an expression for $\nabla \cdot \mathbf{v}$ in terms of the components of $\mathbf{v}$ in this basis $(v^{\hat{r}}, v^{\hat{\theta}}, v^{\hat{\phi}})$

First, work out $(V^{\hat{r}}, V^{\hat{\theta}}, V^{\hat{\phi}})$ in terms of (V^r, V^θ, V^ϕ)

$$V = V^{\hat{r}} \hat{e}_{(r)} + V^{\hat{\theta}} \hat{e}_{(\theta)} + V^{\hat{\phi}} \hat{e}_{(\hat{\phi})} = V^r \partial_r + V^\theta \partial_\theta + V^\phi \partial_\phi$$

since $\{\partial_r, \partial_\theta, \partial_\phi\}$ already orthogonal can just read off components $\hat{e}_{(r)} = \partial_r$, $\hat{e}_{(\theta)} = \frac{1}{r}\partial_\theta$, $\hat{e}_{(\phi)} = \frac{1}{r\sin\theta}\partial_\phi$

$$V^{\hat{r}} = V^r$$

$$V^{\hat{\theta}} \frac{1}{r} = V^\theta$$

$$V^{\hat{\phi}} \frac{1}{r\sin\theta} = V^\phi$$

so, we can use result from (d) to see

$$\nabla \cdot V = \partial_r V^{\hat{r}} + \frac{2}{r} V^{\hat{r}} + \frac{1}{r} \partial_\theta V^{\hat{\theta}} + \frac{1}{r} \frac{\cos\theta}{\sin\theta} V^\theta + \frac{1}{r\sin\theta} \partial_\phi V^\phi$$

+12

3. de Sitter Space

Four-dimensional spacetime with positive constant spacetime curvature $\kappa = R/12$ is called de Sitter space. (If $\kappa < 0$ the spacetime is called anti de Sitter space). We say this spacetime is *maximally symmetric.* The Riemann tensor for a (4D) maximally symmetric space takes the simple form

$$R_{\rho\sigma\mu\nu} = \frac{R}{12}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \tag{7}$$

where R is the (constant) scalar curvature.

(a) What $T_{\mu\nu}$ gives rise to de Sitter space? (you do not need an explicit expression for $g_{\mu\nu}$ to answer this)

$$G_{\sigma\nu} = \left(\frac{R}{12} g^{\rho\mu}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) - \frac{1}{2}R\, g_{\sigma\nu}\right)$$

but find an expression in terms of K

$$= \frac{R}{12}\left(4 g_{\sigma\nu} - \delta^{\mu}{}_{\nu} g_{\sigma\mu}\right) - \frac{1}{2}R\, g_{\sigma\nu}$$

$$= R\left(\frac{4}{12} - \frac{1}{12} - \frac{1}{2}\right) g_{\sigma\nu}$$

$$= R\left(\cancel{\ldots}\right) g_{\sigma\nu} = -\frac{R}{4} g_{\sigma\nu}$$

$$\Rightarrow \quad -\frac{R}{4} g_{\sigma\nu} = 8\pi G\, T_{\mu\nu}$$

$$\rightarrow T_{\mu\nu} = \frac{-R}{32\pi G} g_{\sigma\nu} = \frac{-3K}{8\pi G} g_{\mu\nu}$$

↑ constant!

(b) Instead of having a source $T_{\mu\nu}$, can you change the left-hand side of Einstein's equation (e.g. the part that only depends on spacetime geometry) so that de Sitter space is a solution when $T_{\mu\nu} = 0$?

usual

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{-3K}{\cancel{8\pi G}} g_{\mu\nu} \times 8\pi G$$

i.e. redefine

$$\rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \frac{3K}{\cancel{8\pi G}} g_{\mu\nu} = 0$$

(c) One way to describe de Sitter space is as hyperboloid embedded in 5D Minkowski spacetime

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \ell^2 \tag{8}$$

where the 5D Minkowski metric is, of course, $ds_5^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2$ and $X^i \in (-\infty, \infty)$. Introduce coordinates on the hyperboloid $(\tau, \chi, \theta, \phi)$

$$X^0 = \ell \sinh(\tau/\ell), \quad X^1 = \ell\cosh(\tau/\ell)\cos(\chi), \quad X^2 = \ell\cosh(\tau/\ell)\sin\chi\cos\theta,$$
$$X^3 = \ell\cosh(\tau/\ell)\sin\chi\sin\theta\sin\phi, \quad X^4 = \ell\cosh(\tau/\ell)\sin\chi\sin\theta\cos\phi \tag{9}$$

Find the metric on the hyperboloid in terms of the $(\tau, \chi, \theta, \phi)$ coordinates.

$$g_{\tau\tau} = -\left(\frac{\partial X^0}{\partial\tau}\right)^2 + \left(\frac{\partial X^1}{\partial\tau}\right)^2 + \left(\frac{\partial X^2}{\partial\tau}\right)^2 + \left(\frac{\partial X^3}{\partial\tau}\right)^2 + \left(\frac{\partial X^4}{\partial\tau}\right)^2$$

$$= -\cosh^2 \tau/\ell + \sinh^2 \tau/\ell = -1$$

$$g_{\chi\chi} = \ell^2\cosh^2\frac{\tau}{\ell}\sin^2\chi + \ell^2\cosh^2\tau/\ell\,\cos^2\chi = \ell^2\cosh^2\tau/\ell$$

$$g_{\theta\theta} = \ell^2\cosh^2\frac{\tau}{\ell}\sin^2\chi$$

$$g_{\phi\phi} = \ell^2\cosh^2\frac{\tau}{\ell}\sin^2\theta\sin^2\chi$$

$\tau \in (-\infty, \infty)$

$\chi \in [0, \pi]$

$\theta \in [0, \pi]$

$\phi \in [0, 2\pi)$

$\cosh = \frac{1}{2}(e^{x} + e^{-x})$

(c) In a sentence and a sketch of some of the coordinates, describe de Sitter space in these coordinates. What should the ranges of $(\tau, \chi, \theta, \phi)$ be?

the metric is $ds^2 = -dz^2 + \ell^2\cosh^2\left(\frac{z}{\ell}\right)\underbrace{\left[d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi\sin^2\theta\, d\phi^2\right]}_{\text{metric of 3-sphere}}$

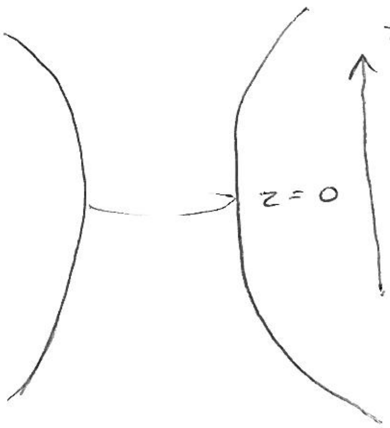
radius $\ell^{\cancel{2}}\cosh\frac{z}{\ell}$ at $z \to -\infty$ radius $\to \infty$

$z \to +\infty$ radiu $\to \infty$

at $z \to 0$ radius $\to \ell$

3-sphere that shrinks to size ℓ

expands again



(e) Now consider the coordinates (t, x^1, x^2, x^3) defined by

$$X^0 = \ell \sinh t/\ell - \frac{1}{2\ell}x_i x^i e^{-t/\ell}, \quad X^i = x^i e^{-t/\ell} \quad \text{for} \quad i = 1, 2, 3, \quad X^4 = \ell \cosh t/\ell - \frac{1}{2\ell}x_i x^i e^{-t/\ell} \tag{10}$$

Find the metric in terms of the (t, x^1, x^2, x^3) coordinates. Do these coordinates cover all of de Sitter space?