Surfaces of Revolution

2D surface embedded in $\mathbb{R}^3$, can be described in cylindrical coordinates (z, ρ, φ) by $z = f(\rho)$ and 2D surface can be written locally w/ coordinate (ρ, φ)

a) Show that the metric for the line element is given by $ds^2 = F(\rho)\,d\rho^2 + \rho^2 d\varphi^2$ and determine $F(\rho)$ in terms of $f(\rho)$

solution

in $\mathbb{R}^3$ metric is $ds^2 = dx^2 + dy^2 + dz^2$ in cartesian.

The surface is defined by $x = \rho\cos\varphi \quad y = \rho\sin\varphi \quad z = f(\rho)$

where $\rho^2 = x^2 + y^2$

so that in cylindrical coordinates

$$g_{\rho\rho} = \left(\frac{\partial x}{\partial \rho}\right)^2 g_{xx} + \left(\frac{\partial y}{\partial \rho}\right)^2 g_{yy} + \left(\frac{\partial z}{\partial \rho}\right) g_{zz}$$

$$= 1$$

$$g_{\varphi\varphi} = \left(\frac{\partial x}{\partial \varphi}\right)^2 g_{xx} + \left(\frac{\partial y}{\partial \varphi}\right) g_{yy} + \left(\frac{\partial z}{\partial \varphi}\right)^2 g_{zz}$$

$$= \rho^2$$

$$\vdots \text{ etc}$$

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

now, restricted to $z = f(\rho)$

$\rightarrow dz = f' d\rho$ where $f' = \frac{\partial f}{\partial \rho}$

$\rightarrow ds^2 = \underbrace{(1 + (f')^2)}_{\equiv F(\rho)} d\rho^2 + \rho^2 d\varphi^2$

b.) compute the non-zero christoffel symbols for surface

$$\Gamma^{\rho}_{\varphi\varphi} = \frac{-\rho}{1+(f')^2} \qquad \Gamma^{\rho}_{\rho\rho} = \frac{f'' f'}{1+(f')^2}$$

$$\Gamma^{\varphi}_{\varphi\rho} = \Gamma^{\varphi}_{\rho\varphi} = \frac{1}{\rho}$$

c.) compute the non-zero components of Riemann tensor

2D so

$$R^{\rho}{}_{\varphi\rho\varphi} = \partial_\rho \Gamma^{\rho}_{\varphi\varphi} - \partial_\varphi \Gamma^{\rho}_{\rho\varphi} + \Gamma^{\rho}_{\rho\alpha}\Gamma^{\alpha}_{\varphi\varphi} - \Gamma^{\rho}_{\varphi\alpha}\Gamma^{\alpha}_{\rho\varphi}$$

$$= \frac{1}{2}\rho \frac{F'}{F^2}$$

$$R^{\varphi}{}_{\rho\varphi\rho} = \frac{1}{2\rho}\frac{F'}{F}$$

d) compute the Ricci scalar

$$R_{\rho\rho} = \frac{1}{2\rho}\frac{F'}{F} \qquad R_{\rho\varphi} = R^{\varphi}{}_{\rho\varphi} = 0$$

$$R_{\varphi\varphi} = \frac{\rho}{2}\frac{F'^2}{F^2}$$

$$\rightarrow R = g^{\rho\rho}R_{\rho\rho} + g^{\varphi\varphi}R_{\varphi\varphi}$$

$$= \frac{1}{F}\frac{1}{2\rho}\frac{F'}{F} + \frac{1}{\rho^2}\frac{\rho}{2}\frac{F'^2}{F^2}$$

$$= \frac{1}{\rho}\frac{F'}{F^2}$$

e) prove that the 2D surface has zero Riemann tensor iff Ricci scalar vanished

if $R = 0 \Rightarrow F' = 0$ (for finite ρ, F)

$\Rightarrow$ all components of Riemann vanish

Riemann $= 0 \rightarrow F' = 0 \Rightarrow R = 0$

f) using your results from d) & e) to completely classify all surfaces of revolution w/ zero Riemann tensor

All functions f for which

$$\frac{\partial}{\partial \rho}\left[1+(f')^2\right]=0$$

$$\rightarrow \quad 2f''f'=0 \qquad f'' \text{ or } f'=0$$

$$\Rightarrow f(\rho)=A+B\rho \quad \text{for} \quad A, B \in \mathbb{R}$$

so they are <u>cones</u>