2.) <u>Spaces</u> <u>with</u> <u>Torsion</u>

In this problem we drop the torsion-free requirement on derivative operators so that we may have $\nabla_a \nabla_b f \neq \nabla_b \nabla_a f$ All other properties (linearity, Leibnitz, commutativity w/ contraction, consistency w tangents as directional derivatives when applied to scalars) still hold.

a) Let ∇_a be a derivative op w/ torsion while $\tilde{\nabla}_a$ is torsion-free, $\nabla_a \omega_b = \tilde{\nabla}_a \omega_b - C^c_{ab} \omega_c$ where C^a_{bc} is a $\binom{1}{2}$ tensor field assoc. w/ ∇_a, $\tilde{\nabla}_a$.

Define $\nabla_a \nabla_b f - \nabla_b \nabla_a f = -T^c_{ab} \nabla_c f$. Find T^c_{ab} in terms of C^c_{ab}

<u>solution</u>

let $\omega_b = \nabla_b f$, we know $\nabla f = \tilde{\nabla} f$

$$\rightarrow \nabla_a \omega_b - \nabla_b \omega_a = \tilde{\nabla}_a \omega_b - C^c_{ab} \omega_c - \tilde{\nabla}_b \omega_a + C^c_{ba} \omega_c$$

$$= (\underbrace{\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a}_{\text{vanishes by assumption}}) f - (C^c_{ab} - C^c_{ba}) \omega_c$$

$$\rightarrow T^c_{ab} = C^c_{ab} - C^c_{ba}$$

b) is T_{ab}^{c} a tensor field? why or why not;

T_{ab}^{c} is a tensor field of rank $\binom{1}{2}$ since

it's the sum of tensor field (or [ ] of

a tensor field)

c) Show that for two vector fields X^a, Y^a

we have that $X^a \nabla_a Y^c - Y^a \nabla_a X^c - [X,Y]^c = T_{ab}^{c} X^a Y^b$

solution

Apply LHS to a test function f

$$(X^a \nabla_a Y^c \nabla_c f - Y^a \nabla_a X^c \nabla_c f - X^a \nabla_a (Y^c \nabla_c f) + Y^a \nabla_a (X^c \nabla_c f)$$

$$= X^a \nabla_a Y^c \cancel{\nabla_c} f - Y^a \nabla_a \cancel{X^c \nabla_c} f - X^a \cancel{\nabla_a Y^c} \nabla_c f$$

$$- X^a Y^c \nabla_a \nabla_c f + Y^a \cancel{\nabla_a} X^c \nabla_c f + Y^a X^c \nabla_a \nabla_c f$$

$$= Y^a X^c \nabla_a \nabla_c f - X^a Y^c \nabla_a \nabla_c f$$

$$= - X^a Y^c (\nabla_a \nabla_c - \nabla_c \nabla_a) f = + X^a Y^c T_{ac}^{b} \nabla_b f$$

$$\rightarrow X^a \nabla_a - Y^a \nabla_a X^c - [X,Y]^c = T_{ab}^{c} X^a Y^b$$

d) Given a metric g_{ab} show that there is a unique derivative op ∇_a w/ torsion $T_{ab}^{\ \ c}$ that is metric compatible. ($\nabla_c g_{ab} = 0$). Show this by finding an expression in terms of ∂_a and $T_{ab}^{\ \ c}$ (hint: use $\tilde{\nabla} = \partial$ and set up equations to solve for C)

<u>want</u> $\nabla_c g_{ab} = 0$ know $\nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d_{ab} g_{cd} - C^d_{ac} g_{bd}$

using $\tilde{\nabla}$ = partial get 3 - equations

$$\partial_a g_{bc} = C_{cab} + C_{bac} \qquad (i)$$

$$\partial_b g_{ac} = C_{cba} + C_{abc} \qquad (ii)$$

$$\partial_c g_{ab} = C_{bca} + C_{abc} \qquad (iii)$$

add and solve (i) + (ii) − (iii)

$$\rightarrow C^d_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab})$$

$$- \frac{1}{2} g^{cd} (T_{abc} + T_{bac} - T_{cab})$$