Part I: Suggested Reading

Langacker: parts of Chapters 2.10, 4.3, 7.1, 7.2, and 7.3.

Part II: Problems

1. When deriving the SM gauge boson mass eigenstates after electroweak symmetry breaking, i.e. the $W^{\pm}$-bosons, the $Z^0$-boson, and the photon $A^0$, we denote the fields as being either charged or neutral. But how do we see that the $W^{\pm}$-bosons have electric charge $\pm 1$ and the $Z$-boson and photon are neutral?

2. (Langacker, Problem 7.1) Consider a generalization of the SU(2)$_L \times$ U(1)$_Y$ model involving $k$ multiplets $\phi_i$ ($i = 1, \ldots, k$) of complex scalars. The dimension of the $i$-th multiplet is $2t_i + 1$, where $t_i$ can be $0, 1/2, 1, 3/2, \ldots$, and the elements have $T^3$ eigenvalues $t^3_i = -t_i, -t_i + 1, \ldots, t_i$. Also, the $i$-th multiplet has weak hypercharge $y_i$. Assume that each multiplet has one electrically neutral component $\phi^0_i$, i.e. with $q_i = t^3_i + y_i = 0$, and that this component acquires a vev, $\langle \phi^0_i \rangle = v_i / \sqrt{2}$.

(a) Show that the mass eigenstates $W^{\pm}$, $Z$, and $A$ are the same as in the SM.

(b) The $\rho$-parameter, $\rho \equiv \frac{m^2_W}{m^2_Z \cos^2 \theta_w}$ is predicted to be unity at tree level in the SM. Show that in the more general case,

$$\rho = \frac{m^2_W}{m^2_Z \cos^2 \theta_w} = \frac{\sum_{i=1}^{k} [t_i(t_i + 1) - (t^3_i)^2]|v_i|^2}{2 \sum_{i=1}^{k} (t^3_i)^2|v_i|^2}.$$  \hfill (1)

(c) Specialize to the case of one doublet and two triplets,

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi^{+} \\ \Phi^{0} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}.$$  \hfill (2)

where $v_\phi \equiv \sqrt{2}\langle \phi^0 \rangle \gg v_\Phi \equiv \sqrt{2}\langle \Phi^0 \rangle$ and $v_\Sigma \gg v_\Phi \equiv \sqrt{2}\langle \Sigma^0 \rangle$. Calculate $\rho$ to leading nontrivial order in $v_\Phi/v_\phi$ and $v_\Sigma/v_\phi$.

(d) Now specialize to the case where the Higgs sector involves only Higgs doublets. What is $\rho$ at tree-level? Argue that the couplings of the neutral physical Higgs bosons to fermions will no longer be flavor-diagonal. (Do not attempt to write the Higgs potential or find the exact Higgs mass eigenstates.)
3. This question will discuss custodial symmetry.

(a) Show that the SM Higgs potential,

\[ V = \lambda (\phi^\dagger \phi - \mu^2)^2, \tag{3} \]

where

\[ \phi = \frac{1}{\sqrt{2}} \left( \phi_1 + i\phi_2 \phi_3 + i\phi_4 \right), \tag{4} \]

is invariant under an SO(4) global symmetry acting on the four \( \phi_i \).

(b) The group SO(4) is isomorphic to the group SU(2)_L \otimes SU(2)_R. (Note that here the SU(2)_L global symmetry is just the global variant of the SM gauge symmetry.) Express \( \phi \) as a \( 2 \times 2 \) matrix written in terms of the \( \phi_i \), and show how \( U_L \in SU(2)_L \) and \( U_R \in SU(2)_R \) act on \( \phi \) to keep the potential invariant.

(c) Now assume \( \langle \phi_3 \rangle = v \neq 0, \) and write \( \phi_3 = v + H \). Show that the SO(4) symmetry is broken to SO(3). Show that one can identify this SO(3) symmetry with SU(2)_V, which is the diagonal part of SU(2)_L \otimes SU(2)_R. (The residual SU(2) symmetry left over after electroweak symmetry breaking is called custodial SU(2) or custodial isospin.)

(d) Show that (for \( g' = 0 \) and in the absence of Yukawa couplings) the custodial symmetry guarantees that the \( W \) and \( Z \) gauge bosons have equal masses and form a triplet of the SU(2)_V unbroken global symmetry.

(e) Now let \( g' \neq 0 \) (but still ignore the Yukawa interactions). Show that this breaks the SU(2)_V custodial symmetry and leads to different masses for the \( W \) and \( Z \) bosons, with

\[ \rho \equiv \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = 1 \tag{5} \]

at tree-level. (Note that since \( g' \neq 0 \) breaks the custodial symmetry, \( \rho \) will receive higher-order corrections proportional to \( g'^2 \). For \( g' = 0 \) (and still ignoring Yukawa terms), the custodial symmetry would guarantee that \( \rho = 1 \) even when radiative corrections are included.)

(f) Now consider the Yukawa Lagrangian for the first generation given by

\[ \mathcal{L}_Y = Y_u Q_L \tilde{\phi} u_R + Y_d Q_L \phi d_R + h.c., \tag{6} \]

where \( Q_L^\dagger = (u_L, d_L) \). Assume that the quarks have the same Yukawa couplings, \( Y_u = Y_d \). Write \( L_Y \) in a way that makes the global SU(2)_L \otimes SU(2)_R symmetry
(before electroweak symmetry breaking) obvious (note that this requires that the quark fields also transform under this symmetry; show how the $U_{L(R)} \in \text{SU}(2)_{(L,R)}$ act on the quarks).

(g) Consider now the same Yukawa Lagrangian after electroweak symmetry breaking, when the fermions are massive. Show that the resulting Lagrangian is invariant under $\text{SU}(2)_V$ if the quarks have the same mass. Of course, in the SM, the quark masses are not the same. Discuss which Yukawa term will give the largest correction to $\rho$.

4. Compute the partial decay widths of the $W$-bosons into pairs of quarks and leptons at tree-level. Determine the total width of the $W$-bosons and the fractions of the decays (i.e. branching ratios) that give hadrons and leptons. Compare your answers with those found in the summary tables of the Particle Data Group (PDG) [http://pdg.lbl.gov].

5. Compute the partial decay widths of the $Z$-boson into pairs of quarks and leptons at tree-level. Determine the total width of the $Z$-boson and the fractions of the decays (i.e. branching ratios) that give hadrons, charged leptons, and invisible modes $\nu\bar{\nu}$. Compare with the values in the PDG.

6. The four-component Dirac spinors $u(\vec{p},s)$ and $v(\vec{p},s)$ are solutions to the momentum space Dirac equation, i.e. $(\not\! p - m)u(\vec{p},s) = 0$ and $(\not\! p + m)v(\vec{p},s) = 0$. Consider the chiral projections of these spinors, i.e.

$$ u(\vec{p},s)_{L,R} \equiv P_{L,R} u(\vec{p},s), \quad v(\vec{p},s)_{L,R} \equiv P_{L,R} v(\vec{p},s). \quad (7) $$

Weak charged current interactions involve $u_L$ and $v_L$. Show that this means that in the relativistic limit weak charged current interactions mainly involve negative helicity particles or positive helicity antiparticles. Estimate the suppression factor for transitions involving the “wrong” helicity.