Mostly supersymmetry

These are supplemental notes for PHY 621, Advanced Quantum Field Theory, Spring 2010. The plan is to cover supersymmetry from the beginning to advanced topics such as extended superspace (and maybe more if time allows), emphasizing quantum aspects. The approach requires us to start with a general treatment of coset spaces and conformal symmetry.

Citations of (sub)sections refer to *Fields*, hep-th/9912205 (see also my errata page), unless indicated to *Superspace*, hep-th/0108200.

Cosets & projective spaces

In this section we introduce useful methods to derive nonlinear representations of symmetry transformations on various spaces. We begin with the well-known bosonic example of the conformal group (and its subgroups) on Minkowski space, both off and on shell. Then we give a general construction that will be applied later to supersymmetry on superspace.

Introduction

Nonlinear σ-models & cosets (nonlinear realizations of groups): See IVA2-3.

Projective lightcone (more, from embedding in flat space)

- Conformal; RP(1) & CP(1): See IA6 (incl. exercise IA6.5), last page of XIB7.
- (Anti-)de Sitter & Poincaré: See IXC2.

Spinors

- Covering groups: See IB1,4-5, IC5.
- Spinor notation: See IA4, IIA5.
- Lightcone: See IIB1,3.
- Twistors & HP(1): See IIB6-7.

Group coordinates

Symmetry generators

A Lie group is a space, so we generally want to introduce some coordinates. Since it's a curved space, the choice of coordinates generally varies according to application. A simple choice is the exponential one,

$$g = e^{i\alpha^I G_I}$$

but it's usually not the most convenient one. For coset spaces, we often use

$$g = e^{i\alpha^i T_i} e^{i\alpha^\iota H_\iota}$$

since under the gauge group $g' = gh$, so h will transform only the α^ι, not the α^i. For various other purposes (see below), we may want to further factorize g. Group multiplication of such exponentials can be performed using the Baker-Campbell-Hausdorff theorem.

Generally, it's convenient to eliminate exponentials as much as possible, since it may be difficult to evaluate them explicitly in closed form. For example, we might use

$$g = e^{i\alpha^+ G_+} e^{i\alpha^0 G_0} e^{i\alpha^- G_-}$$

where the generators have been divided up into "raising operators" G_+, "lowering operators" G_-, and those of the "Cartan subalgebra" (a maximal Abelian subalgebra) G_0. (We here take $+, 0, -$ as multivalued indices.) Since G_0 is Abelian, its exponential is easily evaluated as phase factors. The expansions of the rest will terminate, leaving polynomials.

Exercise

Evaluate this group element in these coordinates using the definining representation of SU(2) for the generators. What are the reality properties of the coordinates?

Another possibility for classical groups is to work in the defining representation, and then solve the constraints on the group matrices in terms of some rational expression. We have already seen (incomplete) examples of this above for cosets represented as projective spaces.

Once a coordinate representation has been chosen, we also want such a representation for the action of the symmetry group on this space, i.e., a translation into coordinate language of $g' = g_0 g$. For many purposes it will be sufficient to evaluate the infinitesimal transformation (using, e.g., the BCH theorem)

$$\delta g \equiv i\epsilon^I G_I g = (e^{i\epsilon^I G_I} - I)g(\alpha) = i\epsilon^I \widehat{G}_I g$$

where $\widehat{G}_I$ is a differential operator. Since it generates an infinitesimal coordinate transformation, we can write

$$i\widehat{G}_I = L_I{}^M(\alpha)\partial_M, \qquad \delta\alpha^M = \epsilon^I L_I{}^M$$

where $\partial_M \equiv \partial/\partial\alpha^M$. We thus have

$$G_I G_J g = G_I \widehat{G}_J g = \widehat{G}_J G_I g = \widehat{G}_J \widehat{G}_I g$$

$$[G_I, G_J] = -if_{IJ}{}^K G_K \quad \Rightarrow \quad [\widehat{G}_I, \widehat{G}_J] = +if_{IJ}{}^K \widehat{G}_K$$

so technically it's $-\widehat{G}_I$ that's a coordinate representation of G_I. (Cf. subsection IC1, where we saw coordinate representations of the generators on spaces other than the group space.)

Equivalently, we can solve the "dual" equation, in terms of differential forms instead of derivatives,

$$(dg)g^{-1} \equiv [g(\alpha + d\alpha) - g(\alpha)]g^{-1}(\alpha) = i\, d\alpha^M L_M{}^I G_I$$

where $L_M{}^I$ is the matrix inverse of $L_I{}^M$. (As usual, when expressing transformations in terms of coordinates it's often convenient to eliminate all i's in the above equations by absorbing them into the G's and working with antihermitian operators.)

Covariant derivatives

If symmetry (known by mathematicians as "isometry") generators are defined by the left action of group generators on a group element, then generators of the gauge (known by mathematicians as "isotropy") group are defined by right action. The latter are known as "covariant derivatives" because they commute with the symmetry generators. (Commutativity of left and right multiplication is equivalent to associativity of multiplication.) From the same arguments as above, we have

$$gG_I = D_I g, \qquad iD_I = R_I{}^M(\alpha)\partial_M$$

$$g^{-1}dg = i\, d\alpha^M R_M{}^I G_I$$

$$[\widehat{G}_I, D_J] = 0$$

We now have

$$gG_IG_J = D_IgG_J = D_ID_Jg \quad \Rightarrow \quad [D_I, D_J] = -if_{IJ}{}^K D_K$$

There is a very simple relation between the symmetry generators and covariant derivatives. Consider the coordinate transformation that switches each group element with its inverse; then

$$g' = g_0gh \quad \Rightarrow \quad (g^{-1})' = h^{-1}g^{-1}g_0^{-1}$$

$$g \leftrightarrow g^{-1} \quad \Rightarrow \quad g_0 \leftrightarrow h^{-1} \quad \Rightarrow \quad G_I \leftrightarrow -D_I$$

(For the sake of this argument we need not distinguish between global and local groups, and h can be taken as in the full group.) This relation can also be seen from the explicit expressions for L and R as $(dg)g^{-1} \leftrightarrow -g^{-1}dg$. Thus, in the exponential coordinate system, we have simply $L(\alpha) = R(-\alpha)$ (with the extra "$-$" canceling the sign change of $\partial/\partial\alpha$).

We can "integrate" the (symmetry) invariant differentials $d\alpha^M R_M{}^I$ to get finite differences. But the result can be guessed directly:

$$g(\alpha_{12}) \equiv g^{-1}(\alpha_2)g(\alpha_1) = g^{-1}(\alpha_{21})$$

Thus the group element $g(\alpha_{12})$, and hence α_{12} itself, is symmetry invariant. α_{12} reduces to the above differential in the infinitesimal case. In coordinates where $g^{-1}(\alpha) = g(-\alpha)$ (for example, parametrization with a single exponential), we have also $\alpha_{21} = -\alpha_{12}$. The action of the covariant derivatives on the symmetry invariants is given by (using $d(g^{-1}) = -g^{-1}(dg)g^{-1}$)

$$D_I(\alpha_1)g(\alpha_{12}) = g(\alpha_{12})G_I, \qquad D_I(\alpha_2)g(\alpha_{12}) = -G_Ig(\alpha_{12})$$

The invariant differentials can also be used to define a group-invariant ("Haar") measure: The wedge product of all the differentials $d\alpha^M R_M{}^i$ (i ranges over the coset) is not only invariant under the symmetry group, but also under the gauge group, since the determinant of the gauge group element is 1 for the coset representation (even for GL(1), if we use the exponential parametrization).

Exercise

Evaluate all the above (L, R, α_{12}) for the coset U(1)/I.

Wave functions and spin

To define a Hilbert space for wave functions, we begin with a vacuum state defined to be invariant under the gauge group:

$$H_\iota|0\rangle = \langle 0|H_\iota = 0$$

(For some purposes, we can think of the gauge generators as "lowering operators". In general, we don't need a Hilbert space for this construction, but only a vector space; the bras then form the dual space to the kets, as described in subsection IB1.) A coordinate basis for the coset can then be defined as

$$|\alpha\rangle = g(\alpha)|0\rangle, \qquad \langle\alpha| = \langle 0|g^{-1}(\alpha)$$

(where $g(0) = I$) and thus invariant under a gauge transformation

$$g'|0\rangle \equiv gh|0\rangle = g|0\rangle$$

The wave function is then defined with respect to this basis as

$$\psi(\alpha) \equiv \langle\alpha|\psi\rangle = \langle 0|g^{-1}(\alpha)|\psi\rangle$$

from which it follows that its covariant derivative with respect to the gauge group vanishes:

$$-D_\iota\psi(\alpha) = \langle 0|H_\iota g^{-1}(\alpha)|\psi\rangle = 0$$

On the other (right) hand, the symmetry generators act in the expected way:

$$-\hat{G}_I\psi(\alpha) = \langle 0|g^{-1}(\alpha)G_I|\psi\rangle = (G_I\psi)(\alpha)$$

So far we have analyzed only coordinate representations. But usually in quantum mechanics we want to consider more general representations by adding "spin" to such "orbital" generators. This is accomplished by first introducing spin degrees of freedom, and then tying them to the group by modifying the gauge-group constraints. So we first introduce a basis $|^A\rangle$ (and its dual $\langle_A|$) for a matrix representation $\tilde{H}_\iota$ for the gauge group,

$$\langle_A|H_\iota = \tilde{H}_{\iota A}{}^B\langle_B|, \qquad H_\iota|^A\rangle = |^B\rangle\tilde{H}_{\iota B}{}^A$$

then define a basis for the Hilbert space by using this gauge group basis as our new (degenerate) vacuum,

$$|^A,\alpha\rangle \equiv g(\alpha)|^A\rangle$$

to get the generalizations of the previous

$$\psi_A(\alpha) \equiv \langle_A, \alpha|\psi\rangle \quad \Rightarrow \quad -D_\iota \psi_A(\alpha) = \tilde{H}_{\iota A}{}^B \psi_B(\alpha), \qquad -\hat{G}_I \psi_A(\alpha) = (G_I \psi)_A(\alpha)$$

The wavefunction now depends on the gauge-group coordinates, but this dependence is fixed independent of the state: For example, in the 2-exponential coordinate system

$$\psi_A(\alpha) = \langle_A| e^{-i\alpha^\iota H_\iota} e^{-i\alpha^i T_i}|\psi\rangle = (e^{-i\alpha^\iota \tilde{H}_\iota})_A{}^M \langle_M| e^{-i\alpha^i T_i}|\psi\rangle \equiv e_A{}^M(\alpha^\iota)\psi_M(\alpha^i)$$

where $e_A{}^M$ is a "vielbein" depending on only the gauge coordinates, and can be gauged to the identity, while ψ_M depends on only the coset coordinates. Since we know D in terms of derivatives, $D_\iota = -\tilde{H}_\iota$ can be solved to replace partial derivatives with respect to gauge-group coordinates with matrices, in both D_I and $\hat{G}_I$. We'll see applications of this to the conformal group (and thus also the Poincaré group) later.

The commutation relations of the surviving covariant derivatives

$$[D_i, D_j\} = f_{ij}{}^k D_k + f_{ij}{}^\kappa D_\kappa$$

then identify $f_{ij}{}^k$ as the "torsion", while $f_{ij}{}^\kappa$ is the "curvature".

Supersymmetry

We begin this section with a discussion of a few general properties of supersymmetry and superspace. We then focus on a particular treatment of superspace in D=4, based on the superconformal group, that has proven most useful in extending the success of superspace for N=1 supersymmetry to the case of N=2 (and shows some promise for N=4), at both the classical and quantum level.

Introduction

Super-symmetry, -space, -groups, and -twistors: See IIC.

Superspaces

Full

Cosets are easier with classical groups. The Poincaré group is a contraction of a classical group, so the conformal group is easier; it's also more useful, since nonconformal theories can be treated as broken conformal ones. In D=4 the superconformal group is (P)SU(N|2,2); to postpone considerations of reality properties, we'll Wick rotate to (P)SL(N|4) (corresponding to 2 space and 2 time dimensions). Furthermore we can treat not only "P" but also "S" as gauge invaiances rather than constraints; then an element of the group (or algebra) GL(N|4) is just an arbitrary real matrix (with appropriate grading). Thus, no consideration of exponentiation or constraints on the coordinates is necessary. The symmetry generators and covariant derivatives are then very simple:

$$G_{\mathcal{M}}{}^{\mathcal{N}} = g_{\mathcal{M}}{}^{\mathcal{A}} \partial_{\mathcal{A}}{}^{\mathcal{N}}, \qquad D_{\mathcal{A}}{}^{\mathcal{B}} = (\partial_{\mathcal{A}}{}^{\mathcal{M}}) g_{\mathcal{M}}{}^{\mathcal{B}}$$

(now dropping the $\widehat{\ }$ on G) where $\partial_{\mathcal{A}}{}^{\mathcal{M}} = \partial/\partial g_{\mathcal{M}}{}^{\mathcal{A}}$, and we ordered the derivatives to the left in D to keep grading signs trivial. (The derivatives are meant to act only to the right of the g. It's a kind of "normal ordering".)

The choice of gauge group is simply the choice of which constraints can be expressed linearly in covariant derivatives, instead of quadratically. In principle all constraints can be expressed quadratically, but this tends to be awkward in general. We saw this for the ordinary conformal group, with equations of motion quadratic in symmetry generators, which we implicitly translated into the same for covariant derivatives, i.e., momentum, spin, and conformal weight. They simplified because the covariant derivative for conformal boosts has already been set to vanish. Also, the Lorentz and scale coordinates had implicitly been replaced with spin and scale weight; i.e., they took fixed "values".

We now look at this construction in more detail, and generalize to supersymmetry. Because of the use of GL(N|4) for D=4, this construction is simpler than using (the defining representation of) SO(D,2) for arbitrary D (for N=0), or the superconformal groups OSp(N|4) for D=3 or OSp*(8|2N) for D=6, since the latter require a quadratic constraint on the matrices. The fact that the relevant cosets are projective spaces is a significant further simplification.

For a preliminary analysis, we divide up the graded matrices into their bosonic and fermionic parts:

$$g_{\mathcal{M}}{}^{\mathcal{A}} = \begin{array}{c c} & \begin{array}{cc} \bar{a} & \underline{a} \end{array} \\ \begin{array}{c} \overline{m} \\ \underline{m} \end{array} & \begin{pmatrix} g_{\overline{m}}{}^{\bar{a}} & g_{\overline{m}}{}^{\underline{a}} \\ g_{\underline{m}}{}^{\bar{a}} & g_{\underline{m}}{}^{\underline{a}} \end{pmatrix} \end{array}$$

where barred indices are bosonic internal GL(N) indices and underlined are fermionic spacetime GL(4) spinor indices, and then further divide the latter into 2 Lorentz GL(2) Weyl spinor indices, but reordered as determined by dimensional analysis (as is apparent when the individual coordinates/generators are identified):

$$g_{\mathcal{M}}{}^{\mathcal{A}} = \begin{array}{c c} & \begin{array}{ccc} \alpha & \bar{a} & \dot{\alpha} \end{array} \\ \begin{array}{c} \mu \\ \overline{m} \\ \dot{\mu} \end{array} & \begin{pmatrix} g_\mu{}^\alpha & g_\mu{}^{\bar{a}} & g_\mu{}^{\dot{\alpha}} \\ g_{\overline{m}}{}^\alpha & g_{\overline{m}}{}^{\bar{a}} & g_{\overline{m}}{}^{\dot{\alpha}} \\ g_{\dot{\mu}}{}^\alpha & g_{\dot{\mu}}{}^{\bar{a}} & g_{\dot{\mu}}{}^{\dot{\alpha}} \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} Lorentz + scale & supersymmetry & translation \\ S\text{-}supersymmetry & internal & supersymmetry \\ conformal\ boost & S\text{-}supersymmetry & Lorentz - scale \end{pmatrix}$$

Thus the scale weights (engineering dimensions) increase from lower-left to upper-right. The usual full superspace is then obtained by gauging away the diagonal blocks, as well as the lower-left triangle ("lowering operators"), leaving only the coordinates for translations and supersymmetry:

$$g_{\mathcal{M}}{}^{\mathcal{A}} \to \begin{pmatrix} I & \theta_\mu{}^{\bar{a}} & x_\mu{}^{\dot{\alpha}} \\ 0 & I & \bar{\theta}_{\overline{m}}{}^{\dot{\alpha}} \\ 0 & 0 & I \end{pmatrix}$$

More choices can be obtained by also subdividing the N-valued internal indices, perhaps not equally, into n and N−n:

$$g_{\mathcal{M}}{}^{\mathcal{A}} = \begin{array}{c c} & \begin{array}{cccc} \alpha & a & a' & \dot\alpha \end{array} \\ \begin{array}{c} \mu \\ m \\ m' \\ \dot\mu \end{array} & \begin{pmatrix} g_\mu{}^\alpha & g_\mu{}^a & g_\mu{}^{a'} & g_\mu{}^{\dot\alpha} \\ g_m{}^\alpha & g_m{}^a & g_m{}^{a'} & g_m{}^{\dot\alpha} \\ g_{m'}{}^\alpha & g_{m'}{}^a & g_{m'}{}^{a'} & g_{m'}{}^{\dot\alpha} \\ g_{\dot\mu}{}^\alpha & g_{\dot\mu}{}^a & g_{\dot\mu}{}^{a'} & g_{\dot\mu}{}^{\dot\alpha} \end{pmatrix} \end{array}$$

Again gauging away diagonal blocks and the lower-left triangle, we are left with an additional n(N−n) internal coordinates:

$$g_{\mathcal{M}}{}^{\mathcal{A}} \to \begin{pmatrix} I & \theta_\mu{}^a & \theta_\mu{}^{a'} & x_\mu{}^{\dot\alpha} \\ 0 & I & y_m{}^{a'} & \bar\theta_m{}^{\dot\alpha} \\ 0 & 0 & I & \bar\theta_{m'}{}^{\dot\alpha} \\ 0 & 0 & 0 & I \end{pmatrix}$$

For N=1 ("simple" superspace) this is identical to the previous case, but for N>1 it allows for generalizations that have proven necessary for most practical applications. However, so far only N=2 superspace ("hyperspace") has been developed to a point approaching the usefulness of N=1.

Projective as cosets

Projective spaces are obtained by gauging away parts of GL groups in the same manner as above (diagonal blocks + lower triangle), but dividing up the indices into only 2 parts. So we reassemble the previous 4 parts, but differently than the 2 original blocks (bosonic + fermionic) as indicated by our reordering. We then do a second reordering, as the standard bosonic + fermionic within each block:

$$g_{\mathcal{M}}{}^{\mathcal{A}} = \begin{array}{c c} & \begin{array}{cc} A & A' \end{array} \\ \begin{array}{c} M \\ M' \end{array} & \begin{pmatrix} g_M{}^A & g_M{}^{A'} \\ g_{M'}{}^A & g_{M'}{}^{A'} \end{pmatrix} \end{array} \to \begin{pmatrix} I & w_M{}^{A'} \\ 0 & I \end{pmatrix} = \begin{array}{c c} & \begin{array}{cccc} a & \alpha & a' & \dot\alpha \end{array} \\ \begin{array}{c} m \\ \mu \\ m' \\ \dot\mu \end{array} & \begin{pmatrix} I & 0 & y_m{}^{a'} & \bar\theta_m{}^{\dot\alpha} \\ 0 & I & \theta_\mu{}^{a'} & x_\mu{}^{\dot\alpha} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{array}$$

This case has the same bosonic coordinates but half the anticommuting coordinates of the previous. This is the smallest number of fermions we can get, since the gauge algebra must close, and we can't kill both a supersymmetry and its complex conjugate without killing translations. This is useful for constructing actions, since $\int d\theta = \partial/\partial\theta$ has positive mass dimension, so more θ's would require a Lagrangian lower in dimension. For N=1, this is either chiral superspace (n=0, no $\bar\theta$'s), in the chiral representation, or antichiral (n=1, no θ's), in the antichiral representation. For general N, n=0 is again chiral, and n=N antichiral, both with no y's. We also have a simple expression for the inverse matrix, in this gauge:

$$g_{\mathcal{A}}{}^{\mathcal{M}} = \begin{array}{c} \\ A \\ A' \end{array}\!\!\begin{array}{c} \begin{array}{cc} M & M' \end{array} \\ \begin{pmatrix} g_A{}^M & g_A{}^{M'} \\ g_{A'}{}^M & g_{A'}{}^{M'} \end{pmatrix} \end{array} \to \begin{pmatrix} I & -w_A{}^{M'} \\ 0 & I \end{pmatrix}$$

where this matrix w is the same as the previous. (Symmetry and gauge indices lose their distinction after gauge fixing.)

As we'll discuss in detail later, only the case n=N/2 (and thus even N) allows real superfields, since only it makes w a square matrix, with equal range for the A index and its "charge conjugate" A'. This is especially clear if we note that it's the only case where there are equal numbers of θ's and $\bar{\theta}$'s. (Of course, the full superspaces also allow real superfields.) Since this makes them the most useful, we'll often use the term "projective" to refer to them specifically.

We now derive the form of the symmetry generators and covariant derivatives before gauge fixing, in a convenient coordinate representation, using matrix methods. We write in matrix notation

$$g = \begin{pmatrix} I & w \\ 0 & I \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \bar{u}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -v & I \end{pmatrix} = \begin{pmatrix} u - w\bar{u}^{-1}v & w\bar{u}^{-1} \\ -\bar{u}^{-1}v & \bar{u}^{-1} \end{pmatrix}$$

which defines the coordinates $w_M{}^{M'}$, $u_M{}^A$, $\bar{u}_{A'}{}^{M'}$, and $v_A{}^{A'}$. Note that in this representation we have (using $sdet(XY) = sdet(X)sdet(Y)$)

$$sdet\ g = \frac{sdet\ u}{sdet\ \bar{u}}$$

Exercise

What is g^{-1} in terms of $w, u, \bar{u}, v$?

It's actually easier to derive the generators from the form of finite transformations, rather than using $G = g\partial_g$ and $D = (\partial_g)g$ and then using the above redefinitions of the elements of g in terms of $w, u, \bar{u}, v$. Using

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g_0^{-1} = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix}$$

in $g' = g_0 g$ and $g'^{-1} = g^{-1}g_0^{-1}$ (whichever is simpler), we find the finite superconformal transformations

$$u' = (w\tilde{c} + \tilde{d})^{-1}u, \qquad \bar{u}' = \bar{u}(cw + d)^{-1},$$

$$w' = (aw + b)(cw + d)^{-1} = (w\tilde{c} + \tilde{d})^{-1}(w\tilde{a} + \tilde{b}),$$

$$v' = v - \bar{u}(cw+d)^{-1}cu = v - \bar{u}\tilde{c}(w\tilde{c}+\tilde{d})^{-1}u$$

Exercise

Find the infinitesimal transformations from the finite ones, in terms of both the above parametrizations of g_0 and g_0^{-1}, and relate the two forms of the infinitesimal parameters.

Continuing to use matrix notation, we can write the infinitesimal transformations as

$$\delta = str[(\delta w)\partial_w + (\delta u)\partial_u + (\delta\bar{u})\partial_{\bar{u}} + (\delta v)\partial_v]$$

using $w, u, \bar{u}, v$ as indices labeling the blocks of the derivatives. This allows cycling all parameters inside the supertrace to the far left, again using the "normal-ordering" convention that derivatives are understood to act only to the right of everything in the generators. (We can do the same for D by right multiplication.) We thus have

$$G = g\partial_g = \begin{pmatrix} w\partial_w + u\partial_u & -w\partial_w w - u\partial_u w - w\partial_{\bar{u}}\bar{u} - u\partial_v\bar{u} \\ \partial_w & -\partial_w w - \partial_{\bar{u}}\bar{u} \end{pmatrix} \equiv \begin{pmatrix} G_u & -G_v \\ G_w & -G_{\bar{u}} \end{pmatrix}$$

$$D = \partial_g g = \begin{pmatrix} \partial_u u + \partial_v v & -\partial_v \\ \bar{u}\partial_w u + v\partial_u u + \bar{u}\partial_{\bar{u}} v + v\partial_v v & -\bar{u}\partial_{\bar{u}} - v\partial_v \end{pmatrix} \equiv \begin{pmatrix} D_u & -D_v \\ D_w & -D_{\bar{u}} \end{pmatrix}$$

(introducing some convenient signs by convention).

Projective by projection

The interesting properties of these cases follow from the fact that the coset coordinates fit into a rectangle. Furthermore, although the full, "left" index is required for manifest symmetry, the gauge group necessarily breaks the "right" index into 2 pieces. We can therefore begin with a rectangle that keeps the full left index, but only the part of the right index that contains the coset:

$$g_{\mathcal{M}}{}^{\mathcal{A}} \to \bar{z}_{\mathcal{M}}{}^{A'} = \begin{array}{c} \\ M \\ M' \end{array} \begin{array}{c} A' \\ \begin{pmatrix} z_M{}^{A'} \\ z_{M'}{}^{A'} \end{pmatrix} \end{array}$$

And we can do the analogous for the inverse group element:

$$g_{\mathcal{A}}{}^{\mathcal{M}} \to z_A{}^M = \begin{array}{cc} & \begin{array}{cc} M & M' \end{array} \\ A & \begin{pmatrix} z_A{}^M & z_A{}^{M'} \end{pmatrix} \end{array}$$

Then all that's left of the relation between the group element and its inverse is the orthogonality relation

$$z_A{}^{\mathcal{M}}\bar{z}_{\mathcal{M}}{}^{A'} = 0$$

Furthermore, all that's left of the original gauge invariance is the block diagonal pieces, one of which acts only on z (GL(n|2) for the superconformal group), and the other only on $\bar{z}$ (GL(N−n|2)). Note that neither z nor $\bar{z}$ contains the coordinates for conformal boosts.

As for other projective spaces (cf. RP(1) and CP(1), considered previously), the surviving coordinates w can be defined in a gauge-invariant way, which is a simpler way to see their symmetry transformations. An easy way to do this is by solving the orthogonality condition, as

$$\bar{z}_{\mathcal{M}}{}^{A'} = (w_M{}^{N'}, \delta^{N'}_{M'})\bar{u}_{N'}{}^{A'}, \qquad z_A{}^{\mathcal{M}} = u_A{}^N(\delta^M_N, -w_N{}^{M'})$$

which reproduces some of the coordinates of the coset formulation. Specifically, we can identify these if we write in matrix notation

$$\bar{z} = \begin{pmatrix} w \\ I \end{pmatrix} \bar{u}^{-1}, \qquad z = u^{-1}\,(\, I \quad -w\,)$$

Only u and $\bar{u}$ transform under their respective gauge transformations. This defines w as the "ratio" of the 2 blocks of either z or $\bar{z}$:

$$w_M{}^{M'} = \bar{z}_M{}^{A'}(\bar{z}_{M'}{}^{A'})^{-1} = -(z_A{}^M)^{-1} z_A{}^{M'}$$

where the inverses are matrix inverses of those blocks.

The symmetry transformation of w then follows as a "fractional linear" ("projective") transformation: As for the coset case,

$$\bar{z}' = g_0\bar{z}, \quad z' = z g_0^{-1} \quad \Rightarrow \quad w' = (aw+b)(cw+d)^{-1} = (w\tilde{c}+\tilde{d})^{-1}(w\tilde{a}+\tilde{b})$$

A special case is ordinary conformal symmetry (N=0), where all the above are 2×2 matrices: This takes a simpler form than in the usual vector notation, just as for the case of SO(3,1) on 2D Euclidean space. Here the simplification arises from using "quaternions" instead of 4D vectors, while in the 2D case it was complex numbers in place of 2-vectors.

Exercise

Rewrite these finite conformal transformations in vector notation for N=0. Compare with the results of subsection IA6.

From the same derivation we also have the transformations of the u's: Again as from the coset treatment,

$$\bar{u}' = \bar{u}(cw+d)^{-1}, \qquad u' = (w\tilde{c}+\tilde{d})^{-1}u$$

Note that in the gauge (or subject to the constraint) $sdet\ g = 1$, we have $sdet\ u = sdet\ \bar{u}$; thus

$$sdet(g_0) = 1 \quad \Rightarrow \quad sdet(cw + d) = sdet(w\tilde{c} + \tilde{d})$$

We can also construct symmety invariants in a similar way to (and implied by) the coset construction, as differentials or finite differences:

$$z_A{}^{\mathcal{M}} d\bar{z}_{\mathcal{M}}{}^{A'} = u_A{}^M (dw_M{}^{M'}) \bar{u}_{M'}{}^{A'}, \qquad z_{2A}{}^{\mathcal{M}} \bar{z}_{1\mathcal{M}}{}^{A'} = u_{2A}{}^M (w_1 - w_2)_M{}^{M'} \bar{u}_{1M'}{}^{A'}$$

The u's are pure gauge; symmetry- and gauge-invariant quantities depend only on differentials or differences of w, according to the translation ("b") part of the symmetry. These translations include the usual spacetime ones, some of the internal symmetry, and half the supersymmetries (as in the special case of chiral superspace).

The form of the symmetry generators in terms of w and u can again easily be derived from the finite forms of the transformations (taking the infinitesimal limit). We thus find the basis

$$G_w = \partial_w, \qquad G_u = w\partial_w + u\partial_u, \qquad G_{\bar{u}} = \partial_w w + \partial_{\bar{u}} \bar{u}, \qquad G_v = w\partial_w w + u\partial_u w + w\partial_{\bar{u}} \bar{u}$$

which are the coset-space generators less the ∂_v term in G_v. One can also check that these operators are permuted by the "inversion" (a particular case of the above finite transformations)

$$g_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}: \qquad w \to -w^{-1}, \qquad u \to w^{-1}u, \qquad \bar{u} \to \bar{u}w^{-1}$$

Although the covariant derivatives D_u and $D_{\bar{u}}$ for the gauge group are obvious from the way they act on the group indices,

$$D_u = \partial_u u, \qquad D_{\bar{u}} = \bar{u}\partial_{\bar{u}}$$

the remaining derivatives D_w can't be found that commute with the symmetry generators G_v. However, we can define

$$D_w = \bar{u}\partial_w u$$

that commute with all but G_v. This is the usual procedure for ordinary conformal symmetry, where coordinates are not introduced for conformal boosts, so ("covariant") translational derivatives don't commute with them.

Exercise

Derive G from $\bar{z}\bar{\partial} - \partial z$, subject to the constraint $z\bar{z} = 0$, in terms of w and the u's. Derive D_u from $-z\partial$ and $D_{\bar{u}}$ from $-\bar{\partial}\bar{z}$.

Off-shell superfields

Spin

As mentioned above, we can get more general representations of coset spaces by introducing spin through the covariant derivatives. This is effectively what is done in the usual analysis of the ordinary conformal group, or just the Poincaré group: For example, in Wigner's analysis of 4D Poincaré representations, the spin is defined essentially as the covariant derivative left over when orbital angular momentum is subtracted from the full Lorentz generators. The Pauli-Lubański equation (as well as the Klein-Gordon), expressed in terms of group generators, then directly reduces to covariant derivatives.

In our approach we keep $D_v = 0$ unmodified, since it's automatic in the projective description, but introduce spin to replace D_u and $D_{\bar{u}}$. (Of course, D_w is not in the gauge group.) Then

$$D_u \equiv \partial_u u = s_u \equiv u^{-1}\hat{s}_u u, \qquad D_{\bar{u}} \equiv \bar{u}\partial_{\bar{u}} = s_{\bar{u}} \equiv \bar{u}\hat{s}_{\bar{u}}\bar{u}^{-1} \quad \Rightarrow$$

$$G_w = \partial_w, \qquad G_u = w\partial_w + \hat{s}_u, \qquad G_{\bar{u}} = \partial_w w + \hat{s}_{\bar{u}}, \qquad G_v = w\partial_w w + \hat{s}_u w + w\hat{s}_{\bar{u}}$$

where the $\hat{s}$'s are defined to act on "curved" indices M, M' rather than "flat" indices A, A'.

Our flat/curved terminology is by analogy to general relativity, where "flat" indices carry the Lorentz gauge symmetry, and are how spin is introduced, while "curved" indices, and the coordinates that carry them, are acted on by any global symmetry of the space under consideration. In fact, in the bosonic case our gauge group GL(2)⊗GL(2) is just the Lorentz group, scale transformations (for which the "spin" part is the scale weight), and the purely gauge GL(1) that reduces GL(4) to the (Wick-rotated) conformal group SL(4).

Exercise

Translate the generators above for the bosonic case to vector notation, including spin (and scale weight).

Since our gauge group is GL(n|2)⊗GL(N-n|2), it's clear how this works: The gauge generators D_u and $D_{\bar{u}}$ carry flat indices; their irreducible matrix representations carry arbitrary mixtures of these defining indices, up and down, with arbitrary graded (anti)symmetrizations (but with arbitrary values of the Abelian GL(1) charges, and maybe some supertrace conditions). Thus our original fields $\mathring{\Phi}$ carry these flat indices, are scalars with respect to the symmetry group, and satisfy the constraints $D_u - s_u =$

$D_{\bar{u}} - s_{\bar{u}} = 0$. But we can explicitly solve these constraints in terms of fields Φ that carry only curved indices, by using u and $\bar{u}$ as "vielbeins" to convert flat indices into curved. The fields with curved indices then depend only on w, and are gauge invariant, but are no longer scalars: The $\hat{s}$'s in G act the same way on the curved indices as the s's acted on the flat (and themselves carry curved indices).

It's sufficient to consider an example with one of each type of index, primed and unprimed (up vs. down indices should be obvious):

$$s_A{}^C \mathring{\Phi}_{B'}{}^D = \delta_A^D \mathring{\Phi}_{B'}{}^C + r\delta_A^C \mathring{\Phi}_{B'}{}^D, \qquad s_{A'}{}^{C'} \mathring{\Phi}_{B'}{}^D = \delta_{B'}^{C'} \mathring{\Phi}_{A'}{}^D + \bar{r}\delta_{A'}{}^{C'} \mathring{\Phi}_{B'}{}^D$$

(with extra signs from index reordering implicit) where $r+\bar{r}$ is related to the superscale weight (see below) and $str\ s - str\ \bar{s}$ (the "$-$" comes from the definition of $D_{\bar{u}}$ and $G_{\bar{u}}$) is related to the super-(internal-)U(1) charge (or superhelicity; see the following section). The solution to the constraints is

$$\mathring{\Phi}_{A'}{}^A(w,u,\bar{u}) = (sdet\ \bar{u})^{r+\bar{r}} \bar{u}_{A'}{}^{M'} \Phi_{M'}{}^M(w) u_M{}^A, \qquad \mathring{\Phi}'_{A'}{}^A(w,u,\bar{u}) = \mathring{\Phi}_{A'}{}^A(w',u',\bar{u}')$$

$$\Rightarrow \quad \Phi'_{M'}{}^M(w) = [sdet(cw+d)]^{-r-\bar{r}} (cw+d)^{-1}{}_{M'}{}^{N'} \Phi_{N'}{}^N(w')(w\tilde{c}+\tilde{d})^{-1}{}_N{}^M$$

where r and $\bar{r}$ appear only in the combination $r+\bar{r}$ because we have used the "S" constraint on g, $sdet\, u = sdet\, \bar{u}$ (which implies the analogous on g_0, $sdet(w\tilde{c}+\tilde{d}) = sdet(cw+d)$).

Exercise

Relate r and $\bar{r}$ for arbitrary numbers of up/down, primed/unprimed indices.

Exercise

Show that in general $sdet(cw+d) = sdet(w\tilde{c}+\tilde{d})sdet(g_0)$.

A linear form of transformation on indices can be obtained by using z and $\bar{z}$ to convert flat indices into full GL(N|4) curved indices; e.g.,

$$\Phi_{\mathcal{M}}{}^{\mathcal{N}} \sim \bar{z}_{\mathcal{M}}{}^{A'} \Phi_{A'}{}^A z_A{}^{\mathcal{N}}$$

But such fields are constrained,

$$z_A{}^{\mathcal{M}} \Phi_{\mathcal{M}}{}^{\mathcal{N}} = \Phi_{\mathcal{M}}{}^{\mathcal{N}} \bar{z}_{\mathcal{N}}{}^{A'} = 0$$

Solving the constraints leads back to the above fields and yields their nonlinear transformations.

Note that the fermionic part of the spin is usually assumed to vanish, in agreement with known physical examples. This implies that their superpartners do also, so in those cases s vanishes except for the chiral case, where only s_u (consisting of just $s_\alpha{}^\beta$) is nonvanishing, or the antichiral case, where only $s_{\bar{u}}$ is.

Another way to account for the superscale weight is to define a field Φ to be a density by requiring that $dw\ \Phi^{1/\omega}$ transform as a scalar,

$$dw\ \Phi'^{1/\omega}(w) = dw'\ \Phi^{1/\omega}(w')$$

where "dw" is the naive integration measure over all the components of w, so Φ is a density of (superscale) weight "ω". (We may have switched active vs. passive transformations; see subsection IC2.) The transformation law for dw can be found from $d\bar{z}$ (or dz), which is invariant because $sdet(g) = 1$. (The relation of the superdeterminant to Jacobians, as the generalization of the bosonic case, follows from its definition in terms of a Gaussian integral.) This is true already for the part of the measure $d\bar{z}$ coming from any one particular value of A' in $\bar{z}_{\mathcal{M}}{}^{A'}$. We then separate out dw and $d\bar{u}$ in $\bar{z} = (w, I)\bar{u}^{-1}$:

$$d\bar{z}_{\mathcal{M}}{}^{A'} = (dw_M{}^{N'}, 0)\bar{u}_{N'}{}^{A'} + (w_M{}^{N'}, \delta_{M'}^{N'})d\bar{u}_{N'}{}^{A'}$$

$$\Rightarrow \quad d\bar{z} = dw\,(sdet\ \bar{u})^{-str\,I_u} \times d(\bar{u}^{-1}), \qquad str\,I_u = n - 2$$

where the exponent comes from multiplying the contributions from each particular value of M. The superconformal transformation of $d(\bar{u}^{-1})$ then follows from that of $\bar{u}^{-1}$ by a similar manipulation:

$$d(\bar{u}^{-1})' = d(\bar{u}^{-1})[sdet(cw + d)]^{str\,I_{\bar{u}}}, \qquad str\,I_{\bar{u}} = (N - n) - 2$$

$$\Rightarrow \quad dw' = dw\,[sdet(cw + d)]^{-str\,I}, \qquad str\,I = N - 4$$

(This derivation is thus singular for N=4, related to the additional "P" gauge invariance.) The superconformal transformation of Φ is then

$$\Phi'(w) = [sdet(cw + d)]^{-\omega\,str\,I}\Phi(w'), \qquad w' = (aw + b)(cw + d)^{-1}$$

We can identify $\omega\,str\,I$ with $r + \bar{r}$ of the previous derivation. ($r + \bar{r}$ needn't vanish for N=4, where dw is a scalar.)

Charge conjugation

As explained previously, only the cases N=2n, where w is square, allow the existence of real superfields. Because of the Wick rotation used to conveniently describe the superconformal group, fields will satisfy nontrivial reality conditions. We really don't need to Wick rotate: If you ignore reality, it doesn't make a difference; just treat any variable and its complex conjugate as algebraically independent. (However, there can be some topologcial complications, which we'll ignore, at least for now.) Reality for the superconformal group is expressed as a pseudo-unitarity condition (the "U" in (P)SU(N|2,2)),

$$g^\dagger \Upsilon g = g\Upsilon g^\dagger = \Upsilon, \qquad \Upsilon^2 = I, \qquad \Upsilon^\dagger = \Upsilon; \qquad \Upsilon^{\dot{\mathcal{M}}\mathcal{N}} = \begin{array}{c c} & \begin{array}{cccc} n & \nu & n' & \dot\nu \end{array} \\ \begin{array}{c} \dot m \\ \dot\mu \\ \dot m' \\ \mu \end{array} & \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & -iC \\ 0 & 0 & I & 0 \\ 0 & iC & 0 & 0 \end{pmatrix} \end{array}$$

in terms of the SL(2) and U(2) metrics, e.g.,

$$C^{\mu\nu} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad I^{\dot m n} = \delta_m^n$$

It isn't useful to solve for the reality conditions on the components of g because of the nonlinearity, and because some of the complex conjugates of components of w are in v. (So we have chosen a complex gauge by eliminating v.) Instead, we use this unitarity condition to define "charge conjugates" of elements of g that transform in the same way under the symmetry group, although differently under the gauge group. Specifically, we need this only for the coset:

$$\mathcal{C}(w') = (\mathcal{C}w)'$$

where $\mathcal{C}$ acts on w' as if it were w, and $'$ acts on $\mathcal{C}w$ as if it were w; thus superconformal transformations and charge conjugation commute. We therefore need to use only the fact that the symmetry transformation g_0 used in $g' = g_0 g$ satisfies the same unitarity condition as g above. This fact can then be applied as well to the transformations on the projective space, $\bar z' = g_0 \bar z$ and $z' = z g_0^{-1}$. The goal will be to define a charge conjugation $\mathcal{C}$ of fields that involves their complex (hermitian) conjugation, but still gives fields that depend on w (and not $w^\dagger$, whatever that is). Thus for flat superfields

$$(\mathcal{C}\mathring{\Phi})(w,u,\bar u) \equiv [\mathring{\Phi}(\mathcal{C}w, \mathcal{C}u, \mathcal{C}\bar u)]^\dagger$$

where "$\mathcal{C}w$" is some function of $w^\dagger$ (so $\Phi^\dagger$ gives back w) that transforms the same as w under superconformal transformations. The relation for curved superfields then follows. For real fields (when they can be defined), $\mathcal{C}\Phi$ is identified with Φ.

We thus define the action of charge conjugation $\mathcal{C}$ on the coordinates by

$$\mathcal{C}g \equiv g\Upsilon\hat{\Upsilon} = \Upsilon(g^{-1})^\dagger\hat{\Upsilon}, \qquad \hat{\Upsilon}^{\dot{\mathcal{A}}\mathcal{B}} = \begin{array}{c} \\ \dot{A} \\ \dot{A}' \end{array}\!\!\begin{array}{c} \begin{array}{cc} B & B' \end{array} \\ \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \end{array}$$

In the former form the symmetry transformation is obvious, while in the latter form Υ mixes only the symmetry indices, with $\widehat{\Upsilon}$ chosen to mix the gauge indices to relate the pieces appearing in the projective approach:

$$(\mathcal{C}\bar{z}_{\mathcal{M}}{}^{A'})^\dagger = -z_A{}^{\mathcal{N}}\Upsilon_{\mathcal{N}\dot{\mathcal{M}}}, \qquad (\mathcal{C}z_A{}^{\mathcal{M}})^\dagger = \Upsilon^{\dot{\mathcal{M}}\mathcal{N}}\bar{z}_{\mathcal{N}}{}^{A'}$$

relating z to the complex conjugate of $\bar{z}$. (The "$-$" sign, from $\hat{\Upsilon}$, preserves $sdet\, g = 1$.) The gauge indices don't match because charge conjugation switches primed and unprimed indices; but w is gauge invariant. We could match indices by putting back the identities in $\hat{\Upsilon}$; for the example of the previous subsection, the flat field would then satisfy

$$(\mathcal{C}\mathring{\Phi}_{A'}{}^{A})(w,u,\bar{u}) \equiv \hat{\Upsilon}_{A'\dot{B}}[\mathring{\Phi}(\mathcal{C}w,\mathcal{C}u,\mathcal{C}\bar{u})]^{\dagger\dot{B}}{}_{\dot{B}'}\hat{\Upsilon}^{\dot{B}'A}$$

Independent of coordinate choice, we find as a result

$$(\mathcal{C}G)^\dagger = -\Upsilon^{-1}G\Upsilon, \qquad (\mathcal{C}D)^\dagger = -\widehat{\Upsilon}^\dagger D\widehat{\Upsilon}^{\dagger-1}$$

(but we have chosen $\Upsilon^{-1} = \Upsilon$, $\widehat{\Upsilon}^{-1} = \widehat{\Upsilon}^\dagger = -\widehat{\Upsilon}$). More explicitly, and taking into account (i.e., undoing) that the above hermitian conjugation includes matrix transposition,

$$\mathcal{C}: \qquad D_w \to -D_w, \quad D_v \to -D_v, \quad D_u \leftrightarrow -D_{\bar{u}}$$

Exercise

Work out the explicit transformation on G (again undoing the matrix transposition in the above equation). Note that the effect of Υ is that fermionic (i.e., Lorentz) indices tend to make coordinates real, while bosonic (i.e., internal) indices make them complex. Thus, e.g., G_x is untransformed (except for a "$-$", because $G_x = \partial_x$ lacks an "i"), while G_y is switched with its analog in G_v.

We then find the conjugation of w, which we can write as

$$(\mathcal{C}w)^{\dagger\dot{M}'}{}_{\dot{N}} = \begin{array}{c} \\ \dot{m}' \\ \mu \end{array} \begin{array}{c} \begin{array}{cc} \dot{n} & \qquad\qquad \dot{\nu} \end{array} \\ \begin{pmatrix} -y^{-1}{}_{m'}{}^{n} & -iy^{-1}{}_{m'}{}^{n}\bar{\theta}_n{}^{\dot{\mu}}C_{\dot{\mu}\dot{\nu}} \\ -iC^{\mu\nu}\theta_\nu{}^{n'}y^{-1}{}_{n'}{}^{n} & -C^{\mu\nu}(x_\nu{}^{\dot{\mu}} - \theta_\nu{}^{n'}y^{-1}{}_{n'}{}^{n}\bar{\theta}_n{}^{\dot{\mu}})C_{\dot{\mu}\dot{\nu}} \end{pmatrix} \end{array}$$

(For N=0, $\mathcal{C}x$ is just x: The factors of C in its hermitian conjugation are because it's $x^{\mu\dot{\mu}}$ that's hermitian. Note that $C_{\mu\nu} = -C^{\mu\nu}$.) For deriving charge conjugation for spin, it's also useful to have

$$(\mathcal{C}u)^\dagger = \bar{u}\bar{\mathcal{A}}^{-1}(w), \qquad (\mathcal{C}\bar{u})^\dagger = \mathcal{A}^{-1}(w)u$$

$$\mathcal{A}_{M\dot{N}'} = \begin{array}{c} \\ m \\ \mu \end{array} \begin{array}{c} \begin{array}{cc} \dot{n}' & \quad \nu \end{array} \\ \begin{pmatrix} y_m{}^{n'} & 0 \\ \theta_\mu{}^{n'} & -iC_{\mu\nu} \end{pmatrix} \end{array}, \qquad \bar{\mathcal{A}}^{\dot{M}N'} = \begin{array}{c} \\ \dot{m} \\ \dot{\mu} \end{array} \begin{array}{c} \begin{array}{cc} n' & \quad \dot{\nu} \end{array} \\ \begin{pmatrix} y_m{}^{n'} & \bar{\theta}_m{}^{\dot{\nu}} \\ 0 & -iC^{\dot{\mu}\dot{\nu}} \end{pmatrix} \end{array}$$

$$\Rightarrow \quad sdet\, \mathcal{A} = sdet\, \bar{\mathcal{A}} = det\ y$$

For the same example, we then have

$$(\mathcal{C}\Phi_{M'}{}^{M})(w) = (det\ y)^{-r-\bar{r}}\bar{\mathcal{A}}^{-1}_{M'\dot{N}}[\Phi(\mathcal{C}w)]^{\dagger\dot{N}}{}_{\dot{P}'}\mathcal{A}^{-1\dot{P}'M}$$

Exercise

Derive $\mathcal{C}$ on $w, u, \bar{u}$ from the transformation of z and $\bar{z}$. (The derivation is similar to that for superconformal transformations, if one thinks of Υ as a particular superconformal transformation.)

Another way to generalize to nonvanishing superscale weight is by considering densities, as for superconformal transformations. We then need to relate $d(\mathcal{C}w)^\dagger$ to dw: With the help of the identity

$$sdet(e^X) = e^{str\, X} \quad \Rightarrow \quad sdet(X \otimes Y) = (sdet\, X)^{str\, I_Y}(sdet\, Y)^{str\, I_X}$$

to handle the two indices on $-d(y^{-1}) = y^{-1}(dy)y^{-1}$, we find

$$[d(\mathcal{C}w)]^\dagger = dw\,(det\ y)^{-str\, I}$$

Thus, requiring $dw\ \Phi^{1/\omega}$ act as a scalar under charge conjugation,

$$dw\,[(\mathcal{C}\Phi)(w)]^{1/\omega} \equiv \{d(\mathcal{C}w)[\Phi(\mathcal{C}w)]^{1/\omega}\}^\dagger$$

$$\Rightarrow \quad (\mathcal{C}\Phi)(w) = (det\ y)^{-\omega\, str\, I}[\Phi(\mathcal{C}w)]^\dagger$$

This will prove to be useful for the case N=2 when $\omega = \frac{1}{2}$ (so $-\omega\ str\, I = 1$), since then $S = \int dw\ \Phi\mathcal{C}\Phi$ is both superconformally invariant ($\Phi\mathcal{C}\Phi$ has $\omega = 1$) and real, following from charge conjugation invariance, and coordinate independence of S. (From the "S" constraint on r and $\bar{r}$, since also n=1, we also have $r = \bar{r} = -\frac{1}{2}$.)

Exercise

Show that $\mathcal{C}^2 = (-1)^{(N/2)(-\omega\, str\, I)}$. Thus the above N=2 example is complex.

On-shell superfields

Field equations

We have already seen that the group-space representation can be conveniently reduced by the use of constraints linear in the covariant derivatives, which define a coset space, of lower dimension. Another way to reduce a representation is by field equations. These are normally quadratic in the covariant derivatives, and thus can't be solved algebraically. (However, we'll find a lightcone/supertwistor solution below.) As in our similar treatment of the bosonic case in subsection IIB1, these equations apply only to field strengths.

These constraints, before introduction of the gauge group, carry the full range of indices, and thus can be written in terms of either the covariant derivatives or the symmetry generators, by virtue of the relation

$$G_{\mathcal{M}}{}^{\mathcal{N}} = g_{\mathcal{M}}{}^{\mathcal{A}} D_{\mathcal{A}}{}^{\mathcal{B}} g_{\mathcal{B}}{}^{\mathcal{N}}$$

We can then consider the possible reduction of DD or the equivalent GG constraints simply by graded (anti)symmetrization and supertracing of the 4 indices. The possible choices vary according to the number of spacetime dimensions in the final result: One choice gives the desired 4D Minkowski space; another gives 5D anti-de Sitter space (where (P)SU(N|2,2) is the super anti-de Sitter group); yet another (in the case N=4) gives the 10D space $\mathrm{AdS}_5 \times \mathrm{S}^5$, relevant for the AdS/CFT correspondence.

Restricting ourselves to the 4D case, the result can be obtained by noting that it is the tensor equation

$$G_{(\mathcal{M}}{}^{(\mathcal{P}} G_{\mathcal{N}]}{}^{\mathcal{Q}]} = 0 \ mod\ \delta\ terms$$

that includes the massless Klein-Gordon equation $p^2 = 0$. The equation is determined only up to Kronecker δ terms, which don't contribute to the Klein-Gordon equation, and has this ambiguity because of the gauge invariance

$$G_{\mathcal{M}}{}^{\mathcal{N}} \to G_{\mathcal{M}}{}^{\mathcal{N}} + \delta_{\mathcal{M}}^{\mathcal{N}} A$$

for arbitrary operator A. (Because it's Abelian, this is the same gauge symmetry as in the gauge group generated by D's; "Abelian" means it can be considered as either left or right.) The equation of motion also includes the general field equation we found for all spin in subsection IIB1 (from a similar analysis for just the ordinary conformal group, but in arbitrary dimensions), the general (massless) supersymmetry free field equation $\not{p}q = 0$, the Pauli-Lubański equation, several qq equations often seen in supersymmetry (usually as dd), equations involving the internal symmetry generators, and various redundant equations.

Exercise

List all the equations explicitly in terms of the superconformal generators p (translations), q (supersymmetry), J (Lorentz), $\mathit{\Delta}$ (scale), T (internal U(N)), s (S-supersymmetry), and k (conformal boosts), labeling them with SU(N) defining indices and Weyl spinor indices. (Drop the GL(1) generator of GL(N|4): Use SL(N|4).)

The number of field equations we wrote above in terms of the symmetry generators is reduced by the gauge constraints. In terms of these generators, many equations are redundant; alternatively, we can start with the equations written in terms of the covariant derivatives, where some drop out automatically. Either way, the net result is that the equations on projective space reduce to (all mod δ terms for the s's)

$$\partial_{(M'}{}^{(P}\partial_{N']}{}^{Q]} = s_M{}^{(P}\partial_{N'}{}^{Q]} = s_{(M'}{}^{P'}\partial_{N']}{}^{Q} = 0$$

$$s_M{}^P s_{N'}{}^{Q'} = s_{(M}{}^{(P}s_{N]}{}^{Q]} = s_{(M'}{}^{(P'}s_{N']}{}^{Q']} = 0$$

The first set of equations is for arbitrary massless representations of supersymmetry, the second set restricts the index structure for specialization to conformal supersymmetry. (We made a similar separation in our treatment of the bosonic case in subsection IIB1.) Specifically, the second set places the restriction that superconformal representations have only primed or only unprimed indices, and fixes the value of the superscale weight.

The list of the spin-free part of these reduced equations is:

$$\partial_x\partial_x = \partial_x\partial_\theta = \partial_\theta\partial_\theta = \partial_\theta\partial_{\bar\theta} + \partial_x\partial_y = \partial_y\partial_\theta = \partial_y\partial_y = 0$$

(and complex conjugates). Internal indices are symmetrized, while Weyl spinor indices are contracted (antisymmetrized). The ∂_y-free equations should be familiar from N=1 chiral scalars: They include the Klein-Gordon, Weyl spinor, and auxiliary field equations, respectively. The equation with all types of derivatives (and thus 2 different types of terms, each with only 1 of each kind of index, and thus no symmetrization possible) shows that any y-dependent term shows up without y at higher order in θ and $\bar\theta$ with x-derivatives, and that all terms with both θ and $\bar\theta$ are of this form.

Taylor expansion is sufficient for the y's, since setting both primed indices equal and both unprimed indices equal in the $\partial_y\partial_y$ equation says the field is linear in each y. (Of course we can always Taylor expand in the θ's.) Then the non-∂_x

equations say that all component fields in this Taylor expansion in y's and θ's are totally antisymmetric in unprimed internal indices and separately also in primed.

Exercise

Solve these equations for the case of N=2, n=1 by expanding $\Phi(w)$ in components over $\theta, \bar\theta, y$ (in terms of fields that are functions of just x).

Exercise

For N=2, n=1 there is just 1 y, so the $\partial_y \partial_y$ equation is simply $(\partial/\partial y)^2 = 0$. For N=4, n=2, $y_m{}^{m'}$ is an isospinor in each of the 2 internal SU(2)'s, and thus a 4-vector of SO(4). Show that this equation can be written in vector notation as

$$\partial_i \partial_j - \tfrac{1}{4}\delta_{ij}\partial^k \partial_k = 0$$

and solve it explicitly on an arbitrary function of y. (Hint: This is part of Einstein's equations for conformally flat spaces, as described in subsection IXC2.)

We now examine the component expansion for N=4, n=2. The result is straightforward:

$$\Phi = (\phi + y_m{}^{m'}\phi_{m'}{}^m + \tfrac{1}{2}y^2\bar\phi) + \theta_\mu{}^{m'}(\lambda_{m'}{}^\mu + y_{m'}{}^m \lambda_m{}^\mu) + \bar\theta_m{}^{\dot\mu}(\bar\lambda_{\dot\mu}{}^m + y_{m'}{}^m \bar\lambda_{\dot\mu}{}^{m'})$$

$$+(\theta^2_{\mu\nu} f^{\mu\nu} + \bar\theta^{2\dot\mu\dot\nu}\bar f_{\dot\mu\dot\nu}) - i\theta_\mu{}^{m'}\bar\theta_m{}^{\dot\mu}\partial_{\dot\mu}{}^\mu(\phi_{m'}{}^m + y_{m'}{}^m\bar\phi)$$

$$-i\theta^2_{\mu\nu}\bar\theta_m{}^{\dot\mu}\partial_{\dot\mu}{}^\mu\lambda^{m\nu} - i\bar\theta^{2\dot\mu\dot\nu}\theta_\mu{}^{m'}\partial_{\dot\mu}{}^\mu\bar\lambda_{\dot\nu m'} - \theta^2_{\mu\nu}\bar\theta^{2\dot\mu\dot\nu}\partial_{\dot\mu}{}^\mu\partial_{\dot\nu}{}^\nu\bar\phi$$

where we have used the internal SL(2)^2 metrics to raise, lower, and contract indices. Each component field, as a function of x, satisfies the Klein-Gordon equation, and each non-scalar satisifies a Weyl equation (which for f is the combination of the usual field equation and Bianchi identity for the Yang-Mills field strength). Note that all component fields appear at $y = 0$, but some only with x derivatives; as stated above, this is a general feature, following from the equation $\partial_\theta \partial_{\bar\theta} + \partial_x \partial_y = 0$; the same is not true off shell.

Supertwistors

We already saw in subsection IIC5 the supertwistor representation as a direct generalization of the bosonic case. Now we find it as a way to solve the above equations of motion: Direct substitution of

$$G_{\mathcal{M}}{}^{\mathcal{N}} = \tfrac{1}{2}[\bar\zeta_{\mathcal{M}}, \zeta^{\mathcal{N}}\} = \bar\zeta_{\mathcal{M}}\zeta^{\mathcal{N}} - \tfrac{1}{2}\delta^{\mathcal{N}}_{\mathcal{M}}$$

$$\{\bar{\zeta}_{\mathcal{M}}, \zeta^{\mathcal{N}}] = \delta_{\mathcal{M}}^{\mathcal{N}}, \quad \{\zeta,\zeta] = \{\bar{\zeta},\bar{\zeta}] = 0$$

verifies that it is a solution. (Just commute the ζ's together so the symmetrization gives commutators. Remember that twistors have statistics opposite to those suggested by the indices. We have used a symmetric ordering as the definition of normal ordering in this case, as in the analogous case of Dirac γ-matrices, for hermiticity. However, this is again ambiguous because of the Abelian gauge invariance.) Note that the supertwistor representation is also a projective space: Besides dividing up the N-valued $\bar{a}$ index as n + (N−n) for arbitrary n, we could have done the same for 4-valued index $\underline{a}$. (This would do the same kind of thing for the x coordinates as we have done for the y's.) The 0+4 case is trivial (it gives no spacetime coordinates), the 2+2 case gives normal 4D spacetime as discussed above, while the 1+3 case gives supertwistors. However, as mentioned above, this would give a complex space, so we need to include the complex-conjugate twistor to define real fields. Identifying the complex conjugate with the canonical conjugate (as for creation and annihilation operators) then prevents doubling the dimension of the space.

The supertrace piece $str\ G \equiv (-1)^{\mathcal{M}} G_{\mathcal{M}}{}^{\mathcal{M}}$ commutes with the superconformal generators, and should not be considered part of the superconformal group: It's the superhelicity. For the *off*-shell representation of the previous section,

$$superhelicity \equiv str\ G = str\ D = str\ s_u - str\ s_{\bar{u}}$$

The superhelicity is part of the "spin", and becomes nontrivial when thus relaxing the "S" constraint of the superconformal group, which we treated as a gauge condition. It's related to the Abelian gauge invariance $\delta G \sim I$ we considered, except in the case N=4, where $str\ I = 0$, and the latter gauge invariance is the definition of the "P" in "PSU(4|2,2)". In the twistor representation, it counts the number of $\bar{\zeta}$'s minus ζ's.

Exercise

Show that this choice of G satisfies exactly

$$G_{(\mathcal{M}}{}^{(\mathcal{P}} G_{\mathcal{N}]}{}^{\mathcal{Q}]} = \tfrac{1}{2}\delta_{(\mathcal{M}}^{\mathcal{P}} \delta_{\mathcal{N}]}^{\mathcal{Q}}$$

Show that with a particular choice of Abelian gauge parameter, the projective superspace representation of the previous section also solves it (without spin). Show that one can instead obtain either of

$$G_{(\mathcal{M}}{}^{(\mathcal{P}} G_{\mathcal{N}]}{}^{\mathcal{Q}]} = \pm\delta_{(\mathcal{M}}^{(\mathcal{P}} G_{\mathcal{N}]}{}^{\mathcal{Q}]}$$

for both cases. (Note however that such redefinitions change the relation of $str\ G$ to the superhelicity.)

The super Penrose transform then gives the solution to the equations of motion in projective superspace by identifying G_w in the two representations: For scalars,

$$-i\partial_{M'}{}^{M} = \pm\bar{\zeta}_{M'}\zeta^{M} \quad \Rightarrow \quad \Phi(w) = \sum_{\pm}\int d\zeta\, d\bar{\zeta}\; e^{\pm i\zeta w\bar{\zeta}}\chi_{\pm}(\zeta,\bar{\zeta})$$

(restoring the "i" for hermiticity), relating the projective superfield $\Phi(w_M{}^{M'})$ with the twistor superfields $\chi_{\pm}(\zeta^M,\bar{\zeta}_{M'})$ for positive and negative-energy solutions. The choice of n determines how the fermionic twistor coordinates are distributed between ζ and $\bar{\zeta}$. Note that, unlike $\zeta^{\mathcal{M}}$ or $\bar{\zeta}_{\mathcal{M}}$, these coordinates are not a representation (but only a nonlinear realization) of the superconformal group: For example, the conformal boosts are represented as quadratic in their "momenta".

The usual component (bosonic-)twistor fields are obtained by evaluating the expansion of χ over the fermionic ζ's. The expansion in y gives new component fields, but the expansion terminates because of the anticommutativity of the corresponding ζ's. The expansion in θ (and $\bar{\theta}$) also gives new component fields, but with spinor indices from bosonic ζ, which then satisfy the usual Weyl equation (as in the non-supersymmetric twistor formalism), and faster termination because there are fewer fermionic ζ's than θ's, and because y dependence may give extra fermionic ζ's. Also note that expansion in both θ and $\bar{\theta}$ will give both ζ^{μ} and $\bar{\zeta}_{\dot{\mu}}$, which is equivalent to an x derivative. We also see that all fields with y dependence also occur without y, but with x derivatives, because fermionic ζ's can come from either θ or y (but y's give only equal numbers of ζ^{μ} and $\bar{\zeta}_{\dot{\mu}}$). All of this agrees with our previous evaluation in terms of the field equations directly.

Exercise

For the case N=4, n=2, expand χ in the fermionic ζ's, identifying each component with one helicity in each field appearing in the expansion in the previous subsection.

As usual, the twistor superfields can be Fourier transformed to functions of just $\zeta^{\mathcal{M}}$ (or just $\bar{\zeta}_{\mathcal{M}}$, or something in-between): Integrating over just $\bar{\zeta}_{M'}$,

$$\Phi(w) = \sum_{\pm}\int d\zeta^{M}\; \tilde{\chi}_{\pm}(\zeta^{M}, -\zeta^{N}w_{N}{}^{M'}) = \sum_{\pm}\int d\zeta^{\mathcal{M}}\; \delta(\zeta^{M'} + \zeta^{N}w_{N}{}^{M'})\; \tilde{\chi}_{\pm}(\zeta^{\mathcal{M}})$$

In the last form, or the analog from integrating out ζ^{M} instead, the argument of the δ function can be replaced with

$$\zeta^{\mathcal{M}}\bar{z}_{\mathcal{M}}{}^{A'} \quad or \quad z_{A}{}^{\mathcal{M}}\bar{\zeta}_{\mathcal{M}}$$

since the u dependence factors out as a Jacobian *sdet*.

Introducing spin, we find that already the spin-dependent equations appearing in the first set of reduced equations of the previous subsection (i.e., those that also contain derivatives) restrict the supertwistor space solutions to the analog of those for the bosonic case:

$$\Phi_{M'\ldots N'}{}^{M\ldots N}(w) = \sum_{\pm} \int d\zeta\, d\bar{\zeta}\; e^{\pm i\zeta w\bar{\zeta}}\; \bar{\zeta}_{M'} \cdots \bar{\zeta}_{N'} \zeta^{M} \cdots \zeta^{N} \chi_{\pm}(\zeta, \bar{\zeta})$$

Since a ζ and $\bar{\zeta}$ are produced by a w derivative, this effectively reduces Φ to have only unprimed or only primed indices, graded antisymmetric in all of them, as implied by the second (spin-only) set of superconformal field equations. (However, fields that are total derivatives on shell need not be so off; but such field strengths are generally not conformal.) In the purely $\zeta^{\mathcal{M}}$ or $\bar{\zeta}_{\mathcal{M}}$ form, the full indices can be used, but because of the constraint enforced by the δ function, the fields will satisfy the analogous constraints on the indices, as described in the previous section. The superhelicity is now given by the number of unprimed minus primed indices.

Simple superspace

Our previous discussion was limited to what space(s) fields were defined on, and free field equations for field strengths. Interactions are harder, but gauge couplings allow us to find gauge fields. From there we can find actions and Feynman graphs.

N=1 supersymmetry and its formulation in superspace are thoroughly understood. All its aspects (both classical and quantum) are treated more easily with superspace than without. All the particulars of N=1 supersymmetry are pretty well covered in *Fields*; maybe we'll also work through some of the additional examples in *Superspace*.

Classical fields: See IVC.

Quantum fields

- Gauges: See VIB4-10.
- Graphs: See VIC1,3,5.
- Loops: See VIIIA5-6.
- More examples: See *Superspace*, 6.3.b,e, 6.4, 6.5.d, 6.7.

Hyperspace

The case of N=2 (“hyperspace”) is less well understood than N=1, but actions have been constructed, as well as some form of Feynman graph rules. The simplest proof of finiteness of N=4 Yang-Mills uses hyperspace, but was developed before any but the simplest Feynman graph calculations (including those needed for the proof) had actually been performed. N=4 Yang-Mills is the 4D theory for which the most amplitudes have been calculated (because it has the most symmetry, so the fewest independent things to calcuate), and is also the 4D theory easiest to relate to string theory, by the AdS/CFT correspondence. If hyperspace were better understood, calculations in N=4 Yang-Mills would be further simplified. (Of course, N=4 superspace would be even better, but it’s much less understood.)

Scalar hypermultiplet

Action

Much of projective hyperspace can be understood by analogy to full simple superspace, as a consequence of both having 2 θ’s and 2 $\bar{\theta}$’s. Then the problem is what to do with the single y coordinate. Since field strengths are Taylor expandable in y on shell, their charge conjugates must be Laurent expandable on shell: Charge conjugation introduces factors of powers of y, and in the field’s arguments y is replaced with $-1/y$. So it seems natural to use contour integration:

$$\oint \frac{dy}{2\pi i}\; y^m \frac{1}{y^n} = \delta_{m+1,n}$$

This makes the y space effectively compact, as expected for an internal symmetry; contour integration is always normalizable.

For the scalar hypermultiplet, the requirement of Laurent expandability turns out to be too weak off shell; we therefore require that it be Taylor expandable. This will be the analog of the “chirality” of simple supersymmetry: Since all our hyperfields are already projective, and there is no suitable hyperspace with fewer θ’s, we instead restrict y dependence. These “polar” superfields are called “arctic”; their charge conjugates, being regular instead at infinity (and singular at the origin), are called “antarctic”. Unlike the N=1 case, we now have an infinite number of auxiliary component fields, because of the infinite Taylor expansion in y.

A suitable free action is then, in analogy to N=1,

$$S = -\int dw\; \overline{\Upsilon}\Upsilon, \qquad \overline{\Upsilon} \equiv \mathcal{C}\Upsilon, \qquad \int dw = \int \frac{d^4x}{(2\pi)^2}\, d^2\theta\, d^2\bar{\theta} \oint \frac{dy}{2\pi i}$$

As mentioned earlier, for superconformal invariance and reality Υ must have "weight" $-r-\bar{r} = -\omega\, str\, I = 1$, so

$$\Upsilon'(w) = sdet(cw+d)\Upsilon(w'), \qquad \overline{\Upsilon}(w) = y[\Upsilon(\mathcal{C}w)]^\dagger$$

Due to the extra factor of y, $\overline{\Upsilon}$ has all powers of y less than or equal to 1. Note that the integral of Υ^2 or $\overline{\Upsilon}^2$ would give 0, just as for N=1, but now because of polarity rather than chirality. Also, since there is no analog to the chiral superpotential terms of N=1, there are no renormalizable self-interactions for this hypermultiplet; all its interactions will be through coupling to the vector hypermultiplet.

The contour integration is a little funny, because the SU(2) part of superconformal transformations could move the singularities of Υ or $\overline{\Upsilon}$ to cross the contour (or vice versa, depending on the active vs. passive point of view). One way to avoid this problem is to Wick rotate the SU(2) to SU(1,1): Then the contour should be considered as embedded in 2D anti-de Sitter space AdS_2 (or de Sitter dS_2), and not the sphere, and can be moved to one of the 2 boundaries (each of which is invariant under SU(1,1)). The limit is singular, but can be treated with some care. This gives this use of 1 y coordinate a "holographic" interpretation. In practice we'll ignore such subtleties.

Field equations

There isn't much to say about the off-shell components: Just Taylor expand in y and the θ's. So we examine the field equations to see how only a finite number of components survive on shell. The easiest way to derive the field equations is by varying Υ, since its only constraint is with respect to y. This gives the field equation for $\overline{\Upsilon}$; we can always get the one for Υ by charge conjugating back. Since Υ is constrained to have only nonnegative powers of y, the contour integral picks up only strictly negative powers of y in $\overline{\Upsilon}$, which has powers ≤ 1. Thus the field equation can be written as

$$\partial_y^2 \overline{\Upsilon} = 0$$

which kills all the negative powers, but not powers 1 and 0. This clearly imposes at least the superconformal equation $\partial_y^2 \Upsilon = 0$, because higher powers of θ/y in $\overline{\Upsilon}$ have extra $1/y$'s that bring those components sooner into the range of the field equation $\partial_y^2 \overline{\Upsilon} = 0$. By superconformal invariance, Υ therefore satisfies the rest of the superconformal equations.

Exercise

Taylor expand the field equation in y and the θ's. Check that the physical degrees of freedom agree with those described in subsection IIC5.

On shell, this multiplet has a hidden extra SU(2) symmetry (unrelated to supersymmetry), as we saw in the N=1 description in subsection IVC7. This is an analog of the on-shell U(1) (electric-magnetic) symmetry of free electromagnetism, described in subsection IIA7. We can manifest this symmetry in the field equations in hyperspace by introducing a "dual" scalar hypermultiplet (the analog of a "magnetic" potential in electromagnetism), and combining the 2 into a doublet of this SU(2): This is the usual trick of making a pseudoreal representation real by using the fact that an SU(2) isospinor is also pseudoreal, and combining a representation with its complex conjugate. (In this case we have $\mathcal{C}^2\Upsilon = -\Upsilon$. Note the "$i$" in the measure for the action to cancel this effect.) The field equations are then implied by

$$\Upsilon^i = C^{ij}\overline{\Upsilon}_j$$

Since $\overline{\Upsilon}$ terminates at order y, this implies Υ has only orders 1 and y, and thus the same for $\overline{\Upsilon}$, so we again get the field equation $\partial_y^2\overline{\Upsilon} = 0$ (and the 2 $\overline{\Upsilon}$'s are determined from one another by charge conjugation).

Thus the field equations are implied by the combination of Taylor expandability with this "reality" condition. Similar remarks apply to the field strength Φ for N=4 Yang-Mills considered previously: There $-r - \bar{r} = 1$ gives the charge conjugate a factor of $det\ y = y^2$ (4-vector square of the 4 y's), and the combination of Taylor expandability of Φ together with "ordinary" reality $\Phi = \mathcal{C}\Phi$ implies the field equations.

y nonlocality

A convenient way to write the constraint on Υ's y-dependence is using contour integration:

$$\Upsilon(y) = \oint_{0,y} \frac{dy'}{2\pi i}\,\frac{1}{y' - y}\,\Upsilon(y')$$

where the integration is over a contour enclosing both the origin and y. This means we can Taylor expand $1/(y' - y)$ in y/y' (but not y'/y):

$$\frac{1}{y' - y} = \sum_{n=0}^{\infty} \frac{y^n}{y'^{n+1}}$$

Then

$$\Upsilon(y) = \sum_{n=0}^{\infty} y^n \oint_0 \frac{dy'}{2\pi i}\,\frac{1}{y'^{n+1}}\Upsilon(y')$$

picks up only the nonnegative powers of y' in $\Upsilon(y')$ (with consistent coefficients), giving the same result as integrating about just y and not 0. The contour integral thus acts as an arctic projector.

As for Feynman diagrams in Minkowski space, it's often more convenient, when defining how to integrate around poles (especially when there's more than one integral to evaluate), to move the poles rather than the contour. In this interpretation, instead of having a bunch of integrals over various contours, with the poles for integration over each variable lying on the contour of another variable, we have all integrals over the same contour, with all poles in various different positions near that contour. For our case, the appropriate "ϵ prescription" is given by writing the arctic projection of Υ as

$$\Upsilon(y_2) = \oint \frac{dy_1}{2\pi i}\frac{1}{y_{12}}\Upsilon(y_1), \qquad \frac{1}{y_{12}} \equiv \frac{1}{y_1 - y_2 + \epsilon(y_1 + y_2)}$$

at least for the case of any convex contour (e.g., a circular one) about the origin. (Otherwise, we need to invent a fancier notation.) For y_1 near y_2, the direction of the ϵ contribution to the pole position $y_2 - \epsilon(y_1 + y_2)$ for y_1 integration is inward toward the origin (ϵ is positive; the second term just scales down the first), so the pole is inside the contour, as we found previously.

Other coordinates

Because of the nonlocality in y of arctic projection, the action for the vector hypermultiplet is also nonlocal in y, and as a result simpler in a "full" superspace. As defined previously, this involves doubling the number of fermionic coordinates (but not adding bosonic ones). We label the new coordinates "ϑ" (in u) for distinction from the old "θ" (in w).

Exercise

Derive the superconformal transformation of ϑ: Start with those already found for u (and $\bar{u}$); then apply the same type of coset/projective analysis, relating the way ϑ appears in u to the way w appears in g.

This complication suggests the use of a different coordinate system: One way to understand this is to note that the covariant derivatives D_w with respect to the projective coordinates w, in the "projective representation" we've been using, now depend on ϑ. We therefore change to a coordinate system where these covariant derivatives return to being just partial derivatives. A simple way to do this to switch them with symmetry generators by the coordinate transformation $g \to g^{-1}$, as discussed previously. To replace this with a transformation that can be obtained continuously from the identity, and to avoid a sign change in the commutation relations, we then change the sign of all coordinates. The combination of these two has the net result of switching the order of (the matrices containing) w and u (and $\bar{u}$) in our original definition of these coordinates. In this "reflective" representation, it is d_ϑ that is no longer just a partial derivative, and has picked up w dependence:

$$d_\vartheta = \partial_\vartheta + y\partial_\theta + \bar{\theta}\partial_x, \qquad \bar{d}_\vartheta = \bar{\partial}_\vartheta + y\bar{\partial}_\theta + \theta\partial_x$$

For many manipulations such details are unnecessary, and it's sufficient to know the commutation relations, and not the explicit representation. Because of the y dependence now in d_ϑ, the commutation relations of d_ϑ's at different values of y are nontrivial. Explicitly, we find

$$\{d_{1\vartheta}, \bar{d}_{2\vartheta}\} = y_{12}\partial_x \qquad (\{d_{1\theta}, \bar{d}_{2\vartheta}\} = \partial_x \quad \textit{and} \quad \{d_{1\theta}, d_{2\theta}\} = 0 \quad \textit{still})$$

$$\Rightarrow \quad d_{2\vartheta} d_{1\vartheta}^4 = y_{21} d_{1\theta} d_{1\vartheta}^4, \qquad \delta^8(\theta_{12}) d_{2\vartheta}^4 d_{1\vartheta}^4 \delta^8(\theta_{12}) = y_{21}^4 \delta^8(\theta_{12})$$

(and similarly for complex conjugates), where $y_{12} \equiv y_1 - y_2$, etc. This one modified, nonlocal (in y only) commutation relation is all we need; e.g., the latter relations appear in integration by parts in hypergraphs.

In these coordinates the scalar hypermultiplet field equations can be derived more directly, because antarctic projection can then be defined without reference to charge conjugation. Then not only can an arctic superfield be written in terms of an unconstrained (in both y and ϑ) superfield,

$$\Upsilon(2) = d_{2\vartheta}^4 \oint \frac{dy_1}{2\pi i} \frac{1}{y_{12}} \psi(1)$$

but its charge conjugate (for this weight) can be expressed similarly as

$$\overline{\Upsilon}(2) = d_{2\vartheta}^4 d_{2y}^2 \oint \frac{dy_1}{2\pi i} \frac{1}{y_{21}} \bar{\psi}(1)$$

The y derivatives appear because: (1) the antarctic projection makes the highest power $1/y$, (2) the y term in each d_ϑ increases this to y^3, and (3) the y derivatives decrease this to the correct power y (as we know from the analysis above in the projective representation). Unconstrained variation with respect to $\bar{\psi}$ (after using the d_ϑ^4 to make the $d^4\theta$ into $d^8\theta$) then gives the field equations $d_y^2 \Upsilon = 0$. (The arctic projection is redundant.) On the other hand, variation with respect to ψ gives just the arctic part of $\overline{\Upsilon}$ vanishing, which is the same as $d_y^2 \overline{\Upsilon} = 0$ by the same analysis as in the projective representation. (The asymmetry in Υ and $\overline{\Upsilon}$ is because d_ϑ is linear in y.)

Vector hypermultiplet

Coupling

Like the scalar hypermultiplet, we look for a description of the vector hypermultiplet in terms of a prepotential defined on projective hyperspace. Again in analogy to N=1, this should be a real prepotential, rather than a polar one. Because it lacks this polarity restriction, and is thus Laurent expandable in y, it's called "tropical". Like the scalar hypermultiplet, it will have only a few powers of y surviving on shell.

But the hyperfield strength lives in chiral superspace, and hence this multiplet has only a finite number of auxiliary fields. However, this property is less fundamental, as it depends on the dimension of spacetime: The scalar and vector hypermultiplets are almost independent of the number of spacetime dimensions for D$\leq$ 6, except that in D=6 the vector hypermultiplet has a vector component gauge field and no scalars; the scalars appear upon dimensonal reduction of this vector. (Of course the number of x's varies; but the number of θ's and y's stays the same. Also, free vector gauge fields are conformal only in D=4.) Thus the type of hyperfield strength varies in different dimensions (e.g., in D=6 it's a spinor, in the rest some scalars), but the type of prepotential is always the same.

Just as for both N=0 and N=1, gauge symmetry is understood as a generalization of global symmetry, so we derive its form by coupling to matter. The straightforward generalization of the N=1 coupling is then given by the action for the scalar hypermultiplet coupled to a vector hypermultiplet background:

$$S = -\int dw\ \overline{\varUpsilon} e^V \varUpsilon$$

This coupling fixes the weight of V to be 0 (as for N=1):

$$V'(w) = V(w'), \qquad \overline{V}(w) \equiv [V(\mathcal{C}w)]^\dagger = V(w)$$

The gauge transformations are then

$$\varUpsilon' = e^{i\varLambda}\varUpsilon, \qquad \overline{\varUpsilon}' = \overline{\varUpsilon} e^{-i\bar{\varLambda}}; \qquad e^{V'} = e^{i\bar{\varLambda}} e^V e^{-i\varLambda}$$

where $\varLambda$ is arctic like $\varUpsilon$, but has weight 0 like V. Thus, unlike $\overline{\varUpsilon}$, $\bar{\varLambda}$ has only nonpositive powers of y. Because of the $1/y$'s associated with $\mathcal{C}\theta$, and the $\theta\bar{\theta}/y$ term in $\mathcal{C}x$, this means $\varLambda = \bar{\varLambda}$ would set $\varLambda$ to a real constant (also like N=1), i.e., the global symmetry. (Thus reality is a stronger constraint on an arctic hyperfield of weight 0 than one of weight 1.)

With this gauge invariance we can examine the off-shell component fields of the vector hypermultiplet. Since $\mathit{\Lambda}$ contains all nonnegative powers of y, and $\bar{\mathit{\Lambda}}$ contains all nonpositive powers, it might seem that everything can be gauged away, but again the additional $1/y$'s associated with charge conjugation modify things: The $1/y$ in $\mathcal{C}\theta$ increases the number of non-gauge components of V with increasing θ, while the $\theta\bar{\theta}/y$ in $\mathcal{C}x$ leads to an x-derivative gauge transformation, again in analogy with the N=1 case. (We can also look at just what $\mathit{\Lambda}$ gauges away, and then apply "reality" to V.) The result is that, unlike the scalar hypermultiplet (but like the N=1 vector multiplet), this one has a finite number of auxiliary fields: In a Wess-Zumino gauge,

$$V = y^{-1}[(\theta\bar{\theta}A+\theta^2\phi+\bar{\theta}^2\bar{\phi})+\bar{\theta}^2\theta(\lambda+y^{-1}\tilde{\lambda})+\theta^2\bar{\theta}(\bar{\lambda}+y^{-1}\tilde{\bar{\lambda}})+\theta^2\bar{\theta}^2(\mathcal{D}+y^{-1}\mathcal{D}_0+y^{-2}\overline{\mathcal{D}})]$$

where the residual gauge invariance is the usual one for the vector A. We thus find, in addition to the expected physical 4-vector and complex scalar (or 6-vector in D=6), and SU(2) doublet of spinors, there is an SU(2) triplet of auxiliary scalars. This same set of fields is found if the vector hypermultiplet is reduced to N=1 supermultiplets, one vector supermultiplet plus one scalar supermultiplet.

Action

In the N=1 case, the construction of the vector multiplet action depended on the fact that a spinor derivative could kill the chiral gauge parameter. In the N=2 case, rather than chiral and antichiral gauge parameters, we have arctic and antarctic gauge parameters, and the only way to kill them is by antarctic or arctic projection. This leads to an action of the form

$$S = \frac{1}{g^2}\,tr\int dx\,d^8\theta\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\,\frac{dy_1}{2\pi i}\cdots\frac{dy_n}{2\pi i}\,\frac{(e^{V(1)}-1)\cdots(e^{V(n)}-1)}{y_{21}\cdots y_{1n}}$$

As we saw when considering the Gervais-Neveu gauge (subsection VIB5), the combination e^V-1 has the nonsuper-like gauge transformation

$$\delta(e^V-1) = (-i\bar{\mathit{\Lambda}}+i\mathit{\Lambda})+[-i\bar{\mathit{\Lambda}}(e^V-1)+(e^V-1)i\mathit{\Lambda}]$$

We then look at the inhomogeneous variation of the ith term for any n, and apply the identity (including ϵ's)

$$\frac{1}{y_{i,i-1}y_{i+1,i}} = \frac{1}{y_{i+1,i-1}}\left(\frac{1}{y_{i,i-1}}+\frac{1}{y_{i+1,i}}\right)$$

Then integrating over y_i, these arctic projectors then have the effect of "propagating" the $\mathit{\Lambda}$ in the inhomogeneous term to the left, and the $\bar{\mathit{\Lambda}}$ to the right (in terms of the labeling of y's, which we have associated with the ordering in the trace), where they cancel the corresponding homogeneous terms coming from the "$n-1$" term. (Variation kills the factor of $1/n$ itself.) This leaves only the inhomogeneous transformation of the $n=2$ term, which vanishes after $d^4\vartheta$ integration, since (after using the arctic projection) it's all at 1 value of y. (Alternatively, we could cancel it with the homogeneous transformation of the "$n=1$ term", which is itself 0 for the same reason.)

The $d^4\vartheta$ integration of this action can be explicitly performed, using the identities of the previous subsection, and gives a significant number of terms for each n. (Pick, e.g., $\int d^4\vartheta = d^4_{1\vartheta}$; then $d_{1\vartheta}V(2) = y_{12}d_{2\theta}V(2)$, etc.) After this integration, the V's can be taken to be in the projective representation, since the only difference in the 2 representations was in the ϑ dependence.

Exercise

For an analogous N=1 example, consider

$$S = \int dx\, d^4\theta\, f(\phi, d_\alpha\phi, d^2\phi)$$

Do the $d^2\bar{\theta}$ integration explicitly, and express the result in terms of ∂_x and ∂_θ on ϕ.

We now make a component analysis of the free action. The quadratic term is

$$S_2 = -\frac{1}{g^2}\, tr \int dx\, d^8\theta\, \frac{dy_1}{2\pi i}\, \frac{dy_2}{2\pi i}\; \tfrac{1}{2}\frac{1}{y_{12}^2} V(1)V(2)$$

Integration over ϑ (using, e.g., $d^4_{1\vartheta}$) eliminates the y_{12} poles:

$$S_2 \sim \int dx\, d^4\theta\, \frac{dy_1}{2\pi i}\, \frac{dy_2}{2\pi i}\; V(1)(\Box + y_{12}\bar{d}_\theta\partial_x d_\theta + y_{12}^2\bar{d}_\theta^2 d_\theta^2)V(2)$$

(All derivatives can now be taken as partial.) The y integration is now trivial: As expected above from the gauge transformation, the only terms that can contribute are

$$V = \frac{1}{y}\left(V_1 + \frac{1}{y}V_2 + \frac{1}{y^2}V_3\right)$$

$$\Rightarrow \quad S \sim \int dx\, d^4\theta\; (V_1\Box V_1 + V_1\bar{d}_\theta\partial_x d_\theta V_2 + V_1\bar{d}_\theta^2 d_\theta^2 V_3 + V_2\bar{d}_\theta^2 d_\theta^2 V_2)$$

Plugging in the Wess-Zumino gauge expression above for V,

$$V_1 = (\theta\bar{\theta}A + \theta^2\phi + \bar{\theta}^2\bar{\phi}) + (\bar{\theta}^2\theta\lambda + \theta^2\bar{\theta}\bar{\lambda}) + \theta^2\bar{\theta}^2\mathcal{D},$$

$$V_2 = (\bar{\theta}^2\theta\tilde{\lambda} + \theta^2\bar{\theta}\bar{\tilde{\lambda}}) + \theta^2\bar{\theta}^2\mathcal{D}_0, \qquad V_3 = \theta^2\bar{\theta}^2\overline{\mathcal{D}}$$

each term in the action gives

$$(A \cdot \Box A + \bar{\phi} \Box \phi) + (\bar{\lambda} \partial_x \tilde{\lambda} + \bar{\tilde{\lambda}} \partial_x \lambda + \mathcal{D}_0 \partial \cdot A) + \overline{\mathcal{D}} \mathcal{D} + \mathcal{D}_0^2$$

This gives the expected terms, after diagonalization of $\mathcal{D}_0$.

Superconformal invariance of the action might not be obvious, especially because of the nonlocality. The first thing to check is that $dx\, d^8\theta$ is superconformally invariant. (This would not be true for $dx\, d^4\theta$ or even $dx\, d^4\theta\, dy$.) Next is to use the results for the superconformal transformations of dw_i and w_{ij}, which are very similar, to find those for y:

$$dy'_i = (w_i \tilde{c} + \tilde{d})^{-1}{}_m{}^n (c w_i + d)^{-1}{}_{m'}{}^{n'} dy_i$$

$$y'_{ij} = (w_i \tilde{c} + \tilde{d})^{-1}{}_m{}^n (c w_j + d)^{-1}{}_{m'}{}^{n'} y_{ij}$$

where the indices m, n, m', n' each take just 1 value in this case, and we have used the facts that other dw's get killed by the $dx\, d^4\theta$, and the other w_{ij}'s vanish. These transformation factors then cancel.

Quantum

Scalar hypermultiplet

The propagator can be derived somewhat in analogy to the N=1 scalar multiplet: Introduce a “nondynamical” source term, and plug the solution to the (classical, free) field equation back into the action. (The source J will become $\delta/\delta\overline{\Upsilon}$ in the expression for the generting functional Z.) In this case, for (projective) source terms

$$-\int dw\ (\overline{\Upsilon}J + \bar{J}\Upsilon)$$

to not introduce dynamics requires that $y^2 J$ be Tayor expandable; or we can allow J to be Laurent expandable. We then use the expression for $\overline{\Upsilon}$ in terms of the general superfield superfield $\bar{\psi}$,

$$\overline{\Upsilon}(2) = d^4_{2\vartheta} d^2_{2y} \oint \frac{dy_1}{2\pi i} \frac{1}{y_{21}} \bar{\psi}(1)$$

and vary $\bar{\psi}$ in both the original action and the source term. The d^4_ϑ completes the integral to $d^8\theta$, so $\bar{\psi}$ can be easily varied. Using the fact that $\partial^2_y \Upsilon$ is still arctic (but the same doesn't hold for J), the field equations can then be written as

$$\partial^2_{1y}\Upsilon(1) = -\oint \frac{dy_2}{2\pi i} \frac{1}{y_{21}} \partial^2_{2y} J(2) = -\partial^2_{1y} \oint \frac{dy_2}{2\pi i} \frac{1}{y_{21}} J(2)$$

where we have used integration by parts to get the last form.

Using the identity

$$d^4_\vartheta d^2_y d^4_\vartheta = \Box d^4_\vartheta$$

and $\partial^2_{1y} 1/y_{21} = 2/y^3_{21}$, the solution to the field equations is

$$\Upsilon(1) = -\frac{d^4_{1\vartheta}}{\frac{1}{2}\Box} \oint \frac{dy_2}{2\pi i} \frac{1}{y^3_{21}} J(2)$$

As usual, instead of plugging into both the terms second-order and first-order in the fields, we can plug into just the first-order (source) terms and multiply by 1/2. In this case, it's sufficient to forget $\overline{\Upsilon}$ and plug in just for the one Υ term, and drop the 1/2.

The exponent $-S$ is thus replaced with

$$-\int dx\, d^8\theta \oint \frac{dy_1}{2\pi i} \frac{dy_2}{2\pi i}\ \bar{J}(1) \frac{1}{\frac{1}{2}\Box} \frac{1}{y^3_{21}}\ J(2)$$

This leads to the propagator

$$\langle \Upsilon(w_1)\overline{\Upsilon}(w_2)\rangle = d^4_{1\vartheta} d^4_{2\vartheta} \frac{1}{y^3_{21}} \frac{1}{-\frac{1}{2}\Box}\, \delta^8(\theta_{12})\delta(x_{12})$$

The derivatives can be evaluated explicitly: We again use

$$d^4_{1\vartheta} d^4_{2\vartheta} = y_{12}^2 (\Box + y_{12} \bar{d}_\theta \partial_x d_\theta + y_{12}^2 \bar{d}_\theta^2 d_\theta^2) d^4_{2\vartheta}$$

This gives the simple (but not necessarily useful) expression

$$\langle \Upsilon(w_1) \overline{\Upsilon}(w_2) \rangle = sdet(w_{12})$$

in agreement with superconformal covariance. To compare with components, use

$$sdet(w) = \frac{y - \theta x^{-1} \bar{\theta}}{x^2}$$

where the former term gives the scalar propagators, while the latter gives the spinors. Because of the subtlety of the $i\epsilon$ prescription, this form ignores the auxiliary fields, which have $\delta^4(x)$ propagators (arising from the $\Box$ term in the above expansion). These can be seen in the alternate form

$$sdet(w) = \frac{y}{(x - \theta y^{-1} \bar{\theta})^2}$$

where expansion of the denominator gives a term $\Box 1/x^2 \sim \delta(x)$.

Vector hypermultiplet

Gauge fixing looks similar to N=1, in the same sense that the scalar mutiplet action does: The main modifications are that now $d^4\theta$ is projective, there is also dy, the ghosts and Nakanishi-Lautrup fields are projective and arctic/antarctic instead of chiral/antichiral, etc. Thus the ghost action looks similar to that for N=1.

We start with gauge-fixing function

$$\left(y\tilde{C} + \frac{1}{y} \bar{\tilde{C}} \right) V$$

with extra factors of y whose utility will soon become apparent. Acting on this with BRST, and adding a "gauge-averaging" term (which can also be written as BRST on something), we get at quadratic order

$$\bar{B} \frac{1}{y \frac{1}{2} \Box} B + \left[\left(yB + \frac{1}{y} \bar{B} \right) V + \left(y\tilde{C} + \frac{1}{y} \bar{\tilde{C}} \right) (C + \bar{C}) \right]$$

(to be integrated dw.) We next redefine the charge conjugates of the ghost (superconformal weight 0), antighost (weight 2), and NL field (weight 2) as

$$\bar{C} \to \frac{1}{y} \bar{C}, \qquad \bar{\tilde{C}} \to y \bar{\tilde{C}}, \qquad \bar{B} \to y \bar{B}$$

so that they will have the same y dependence as if they had the same weight (1) as Υ (gauge fixing breaks conformal invariance anyway). Dropping crossterms that vanish after y integration (totally arctic or totally antarctic), this gives

$$\bar{B}\frac{1}{\frac{1}{2}\Box}B + (yB+\bar{B})V + (\tilde{C}\bar{C} + \tilde{\bar{C}}C)$$

The ghost kinetic terms thus have the same form as that for Υ.

On the other hand, integrating out the Nakanishi-Lautrup fields is similar to deriving the propagator for Υ (replacing J and $\bar{J}$ with V), except mainly for the extra $1/\frac{1}{2}\Box$ in the kinetic operator, which cancels the one that would have appeared in the propagator (again as for N=1). The result is to replace the bosonic terms with

$$\int dx\, d^8\theta \oint \frac{dy_1}{2\pi i}\frac{dy_2}{2\pi i}\; V(2)\,\frac{y_1}{y_{12}^3}\,V(1)$$

Note that at most only 1 of the $1/y_{12}$'s is really there, due to the identity

$$d^4_{1\vartheta}V(2) = y_{12}^2\tfrac{1}{2}\Box V(2) + \mathcal{O}(y_{12}^3)$$

which we'll use during ϑ integration, so we need apply the ϵ prescription to only one. Using symmetry in the V's to symmetrize in the y's, we then replace

$$\frac{y_1}{y_{12}} \to \tfrac{1}{2}\left(\frac{y_1}{y_{12}} + \frac{y_2}{y_{21}}\right)$$

We then combine this action term with (the quadratic part of) the gauge-invariant term, using the identity

$$\frac{1}{y_{12}} + \frac{1}{y_{21}} = 2\pi i\delta(y_{12}) \quad \Rightarrow \quad \frac{y_1}{y_{12}} + \frac{y_2}{y_{21}} - 1 = \frac{y_1+y_2}{2}\,2\pi i\delta(y_{12})$$

Note that this $y\delta(y)$ combination is an angular δ function $\delta(\phi)$ for circular contours $y = Re^{i\phi}$. Performing the ϑ integration, we obtain the gauge-fixed kinetic term

$$\int dw\; \tfrac{1}{2}Vy\tfrac{1}{2}\Box V$$

Hypergraphs

We can now collect the Feynman rules for both multiplets, in momentum space for x, and with all 8 θ's.

scalar multiplet propagator: $\dfrac{d^4_{1\vartheta}d^4_{2\vartheta}}{y_{21}^3}\dfrac{\delta^8(\theta_{12})}{p^2}$ $\quad(\langle\Upsilon(1)\overline{\Upsilon}(2)\rangle)$

vector multiplet propagator: $d^4_\vartheta\left[\dfrac{2}{y_1+y_2}\delta(y_{12})\right]\dfrac{\delta^8(\theta_{12})}{p^2}$ $\quad$(Fermi-Feynman gauge)

scalar multiplet vertex: $\displaystyle\int d^4\theta\, dy,\quad$ but use $\displaystyle\int d^4\theta\, d^4_\vartheta = \int d^8\theta$

vector multiplet (only) vertex: $\displaystyle(-1)^n\int d^8\theta\, dy_1...dy_n\frac{1}{y_{21}y_{32}\cdots y_{1n}}$

(However, for some purposes it might be more useful to stay completely in coordinate space, and use just the projective coordinates. Then the vector multiplet vertex is more messy. Here dy and $\delta(y)$ include the usual $2\pi i$'s.)

Note that, since there are no scalar self-interactions, all scalar lines are either closed loops or end on 2 external scalars. In the former case 1 d^4_ϑ can be taken off each scalar propagator to make each $\int d^4\theta$ into a $d^8\theta$, while in the latter case a propagator at one end will have no d^4_ϑ. Thus there is an $\int d^8\theta$ at every vertex of either kind, and almost all propagators, scalar and vector, have 1 d^4_ϑ. Then the main differences in the rules come from the y dependence.

As a simple example of the rules, consider the 1-loop correction to the scalar propagator. One is left with only a single d^4_ϑ in the loop, not enough to cancel the $\delta^8(\theta_{12})$, so it vanishes.

Exercise

Use the general result for N=1 (subsection VIIIA5) to confirm this. (The N=2 vector multiplet consists of N=1 vector and scalar multiplets, while the N=2 scalar multiplet is just N=1 scalar multiplets.)

Consider next the scalar contribution to the 1-loop vector propagator. The 2 remaining d^4_ϑ are just enough to make the loop nontrivial: Using an above identity, they produce a y^4_{12}. Together with 2 $1/y^3_{12}$'s from the propagators, they produce a term that looks like the (gauge-invariant) kinetic term, times a bosonic scalar propagator correction, as expected.

N=4

Nothing is really known about off-shell N=4 superspace.

Propagator

However, the on-shell properties are enough to write a propagator: Using the Penrose transform, written as

$$\Phi(w) = \int d\zeta^{\mathcal{M}}\ \delta(\zeta^{\mathcal{M}}\bar{z}_{\mathcal{M}}{}^{A'})\chi(\zeta^{\mathcal{M}})$$

we can write a propagator as a sum over physical states, Penrose transformed to the endpoints, as

$$\Delta = \int d\zeta^{\mathcal{M}}\ \delta(\zeta^{\mathcal{M}}\bar{z}_{1\mathcal{M}}{}^{A'})\delta(\zeta^{\mathcal{M}}\bar{z}_{2\mathcal{M}}{}^{A'})$$

Using the orthogonality of z and $\bar{z}$, we can solve either δ function as

$$\zeta^{\mathcal{M}} = \zeta^{A} z_{A}{}^{\mathcal{M}}$$

for some ζ^A, giving

$$\Delta = \int d\zeta^{A}\ \delta(\zeta^{A} z_{1A}{}^{\mathcal{M}}\bar{z}_{2\mathcal{M}}{}^{A'}) = sdet(z_{1A}{}^{\mathcal{M}}\bar{z}_{2\mathcal{M}}{}^{A'})$$

effectively from the Jacobian of the δ function.

A nontrivial example is the chiral-antichiral propagator for the N=1 scalar multiplet. Note that in this case we use 2 different projective superspaces: chiral for z_1, antichiral for $\bar{z}_2$. Using the usual (in the gauge $u = \bar{u} = 1$)

$$z_{1\alpha}{}^{\mathcal{M}} = (-\theta_{1\alpha}, \delta_{\alpha}^{\mu}, -x_{1\alpha}{}^{\dot{\mu}}), \qquad \bar{z}_{2\mathcal{M}}{}^{\dot{\alpha}} = (\bar{\theta}_2^{\dot{\alpha}}, x_{2\mu}{}^{\dot{\alpha}}, \delta_{\dot{\mu}}{}^{\dot{\alpha}})$$

we have

$$\Delta = \frac{1}{(x_{12} + \theta_1\bar{\theta}_2)^2}$$

which agrees with the previous result before detaching the $\bar{d}^2$ and d^2 at the ends,

$$\Delta = \bar{d}_1^2 d_2^2 \delta^4(\theta_{12})\frac{1}{x_{12}^2}$$

in the chiral representation for 1 and antichiral for 2.

For (real) projective superspace, the propagator is simply

$$\Delta = sdet(w_{12})$$

This includes N=0, where $\Delta = 1/x_{12}^2$. The N=2 result, describing the scalar hyermultiplet, agrees with that obtained above by other methods. The N=4 case gives the propagator for the field strength of the vector multiplet. It can also be analyzed by components: Expanding the explicit expression

$$sdet(w) = \frac{(y - \theta x^{-1}\bar{\theta})^2}{x^2}$$

in θ and y (corresponding to expansion of the associated field strengths) shows the usual propagators for the scalars ($1/x^2$) and spinors ($x/(x^2)^2$), and the field strengths for the vectors ($\langle f\bar{f}\rangle = xx/(x^2)^3$), and derivatives of these fields.

These propagators are a bit of a fudge: They are really "cut" propagators, homogeneous solutions to the wave equations obtained by summing over physical (positive energy), on-shell states. However, the Stückelberg-Feynman propagator can be obtained by taking this propagator for positive energy and using it for positive times (multiplying by $\Theta(x_{12}^0)$) and adding it to the negative-energy propagator for negative times. More simply, one can just fix the $i\epsilon$ prescription by hand: For example, for N=0 we can write the twistor integral as (in half-Fourier-transformed twistor variables)

$$\int d\zeta^\alpha\, d\bar{\zeta}^{\dot{\alpha}}\, e^{\pm i\zeta^\alpha \bar{\zeta}^{\dot{\alpha}} x_{\alpha\dot{\alpha}}} = \frac{1}{x^2}$$

To make this converge, we need

$$x_{\alpha\dot{\alpha}} \to x_{\alpha\dot{\alpha}} \pm i\epsilon\delta_{\alpha\dot{\alpha}}$$

Since the identity part of the matrix $x_{\alpha\dot{\alpha}}$ corresponds to the time component x_0, as in $e^{\pm i|p^0|x_0}$, this implies

$$\frac{1}{x^2} \to \frac{1}{x^2 \pm i\epsilon x^0}$$

with signs corresponding to those in the integral. By comparison, the Feynman propagator is $1/(x^2 + i\epsilon)$.

Tree

The 4-point amplitude has been derived by other means and translated into N=4 projective superspace. For example, we saw that MHV tree amplitudes could be written in chiral supertwistor space (subsection VIC3), and thus anti-MHV in antichiral. In particular, at 4 points each includes all amplitudes, and thus they are equivalent:

$$\mathcal{A}_{4\chi} = \frac{\delta^4(\sum p_{\alpha\dot{\alpha}})\delta^8(\sum \pi_{\bar{a}\alpha})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \quad \mathcal{A}_{4\bar{\chi}} = \frac{\delta^4(\sum p_{\alpha\dot{\alpha}})\delta^8(\sum \bar{\pi}^{\bar{a}}{}_{\dot{\alpha}})}{[12][23][34][41]}$$

They follow by supersymmetry from the 4-gluon amplitude we derived. We have rewritten the δ functions explicitly in terms of bosonic and fermionic momenta (i.e., translations and 1/2 the supersymmetries) as the first step of conversion from twistors.

The next step is to transform the chiral supertwistor into projective supertwistor space by Fourier transforming half the fermionic twistor coordinates. We'll find that the ubiquitous twistor denominator of MHV, and its complex conjugate of anti-MHV, are replaced in projective supertwistor space by their magnitude, which is directly expressible in terms of momenta.

We use the notation $ijkl$ to label the 4 distinct external lines. Then the only twistor identity we need is the equality of the MHV and anti-MHV expressions for the pure-gluon amplitude:

$$\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = \frac{[kl]^4}{[12][23][34][41]}$$

This allows us to evaluate the fermionic Fourier transform with respect to any *one* of the 4 twistor fermions (with respect to N=4, but all *four* of the external lines):

$$\int d^4\bar{\zeta}_i \; e^{i\bar{\zeta}_i\zeta_i} \; \delta^2(\lambda_{i\alpha}\bar{\zeta}_i) = \sum \langle ij\rangle\zeta_k\zeta_l = \delta^2(\bar{\lambda}_{i\dot{\alpha}}\zeta_i)\left(\frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}{[12][23][34][41]}\right)^{1/4}$$

(writing λ for the usual twistors and ζ for their fermionic superpartners), with Einstein summation understood on identical indices. Thus this Fourier transformation replaces the conservation δ-function for total $\pi_\alpha = \lambda_\alpha\bar{\zeta}$ with one for the corresponding $\bar{\pi}_{\dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}}\zeta$, and throws in a phase factor. In addition to reproducing the correct relation between the above forms of the amplitude in chiral and antichiral supertwistor space, it gives the result for projective supertwistor space intermediate between the two:

$$\mathcal{A}_{4\Pi} = \frac{\delta^4(\sum p_{\alpha\dot{\alpha}})\delta^4(\sum \pi_{\alpha a'})\delta^4(\sum \bar{\pi}_{a\dot{\alpha}})}{\frac{1}{4}st}$$

Note that this amplitude is missing an explicit δ-function for conservation of R-momentum (which would actually be a Kronecker δ, because of the compactness of the R-space): This conservation is implied by the other δ-functions (in twistor superspace, or on shell).

In this form, the amplitude is already expressed directly in momentum superspace; we need only attach external line factors, which are just the (linearized) projective superfield strengths Φ. We can then Fourier transform the projective amplitude back to coordinate superspace: The x dependence is as usual, the θ dependence is the local product, and the y dependence evaluates at $y = 0$:

$$\hat{\mathcal{A}}_{4\Pi} = \int d^{16}x_i \; d^8\theta \; \Phi(x_1,\theta,0)\Phi(x_2,\theta,0)\Phi(x_3,\theta,0)\Phi(x_4,\theta,0) \; \frac{\delta^4(x_1 - x_2 + x_3 - x_4)}{x_{12}^2 x_{23}^2}$$

4-point loop amplitudes differ only by the x-factor.

Loop

1-loop 4-point amplitude in components: See VIIIC4.

AdS/CFT

Limits

The Anti de Sitter/Conformal Field Theory correspondence proposes to relate 4D N=4 (super)conformal field theory to IIB superstring theory expanded about the 10D (bosonic) manifold of 5D anti de Sitter space $\times$ the 5-sphere. This incorporates the use of "holography": Instead of the usual procedure of solving the wave equation for time dependence, we solve it for spatial dependence, on the coordinate whose endpoint defines the boundary of AdS_5.

Here we'll apply a slightly different procedure: We saw previously that in lightcone quantization the wave equation is solved for dependence on a lightlike coordinate. Furthermore, for applying twistor techniques to Feynman diagrams we found it convenient to Wick rotate this idea to "spacecone" quantization, using a null, spatial coordinate. We'll find it convenient to use a similar procedure here, to find the correspondence between the superspaces of AdS and CFT.

To see why such a treatment naturally arises, we work in Poincaré coordinates for AdS_5:

$$ds^2 = \frac{dx^2 + dx_0^2}{x_0^2}$$

where x_0 is a spatial coordinate. (This is a particular choice of coordinates where the metric is flat times a scale: See subsection IXC2.) After an appropriate Wick rotation, we can do the same for S^5; combining the two spaces,

$$ds^2 = \frac{dx^2 + dx_0^2}{x_0^2} - \frac{dy^2 + dy_0^2}{y_0^2} = \frac{dx^2}{x_0^2} - \frac{dy^2}{y_0^2} + d\ ln(x_0 y_0) d\ ln(x_0/y_0)$$

We can then identify $x_0 y_0$ and x_0/y_0 as two null, spatial coordinates, to be used to define our spacecone quantization.

The usual boundary limit of AdS is $x_0 \to 0$; we modify this to $x_0 y_0 \to 0$ (x_0/y_0 fixed), in line with our interpretation of $x_0 y_0$ as the spacecone "time". This leaves us with 9 bosonic coordinates, 8 of which have translation invariance, and are to be identified with the 4 x's and 4 y's of 4D N=4 projective superspace. (There is a symmetry under translation of the 9th coordinate, but it requires also scaling of the other 8, as well as the fermions. It is associated with a combination of a dilatation with an R-symmetry U(1).)

The boundary limit can be related to the limit where the AdS_5 and S^5 (equal) radius R shrink to 0. This can be interpreted as the relation between active and passive approaches: Instead of (Muhammad) moving to the boundary, we shrink the distance scale, effectively moving the boundary closer. (It is a type of long-distance limit, in contrast to the short-distance limit $R \to \infty$ related to flat space.) In terms of the above metric, we first introduce the overall distance scale R^2 (for the 10D space, thus effectively treating all coordinates as "angles"), then rescale

$$x_0 \to Rx_0, \qquad y_0 \to Ry_0$$

$$R^2 ds^2 \to \frac{dx^2}{x_0^2} - \frac{dy^2}{y_0^2} + R^2 \; d\; ln(x_0 y_0) d\; ln(x_0/y_0)$$

(Alternatively, we can scale $x \to x/R, y \to y/R$ instead.) The limit $R \to 0$ pinches AdS into a lightcone, reducing the conformal analysis to that of the projective lightcone. (See subsection IA6.)

To generalize this limit to superspace, and see how it naturally arises in the projective approach, consider a general supergroup element of PSU(4|2,2), which is a symmetry on both the AdS and CFT sides (hence the correspondence). We want to define the boundary limit as one which picks out N=4 projective superspace, while preserving this symmetry (but perhaps not the gauge groups). Knowing how the projective space fits into the group element (and its inverse), this limit must be the $R \to 0$ limit after the rescaling

$$g_{\mathcal{M}}{}^{\mathcal{A}} \to \left(\sqrt{R} g_{\mathcal{M}}{}^{A}, \frac{1}{\sqrt{R}} \bar{z}_{\mathcal{M}}{}^{A'} \right), \qquad g_{\mathcal{A}}{}^{\mathcal{M}} \to \left(\frac{1}{\sqrt{R}} z_A{}^{\mathcal{M}}, \sqrt{R} g_{A'}{}^{\mathcal{M}} \right)$$

Note that the scaling by R is determined only by the $\mathcal{A}$ index, and is independent of the symmetry index $\mathcal{M}$.

This limit eliminates the v coordinates, which don't appear in the projective approach, and leaves w, but also some parts of u and $\bar{u}$, depending on the choice of gauge group. After eliminating v, and expressing z and $\bar{z}$ in terms of the rest, we see the R scaling is

$$w \to w, \qquad u \to \sqrt{R} u, \qquad \bar{u} \to \sqrt{R} \bar{u}$$

In particular, it's easy to pick out x_0 and y_0 as the pieces of u and $\bar{u}$ invariant under the manifest SO(3,1) Lorentz and SO(4) internal symmetries, after killing the "PS" pieces of PSU(4|2,2):

$$u = \begin{pmatrix} \sqrt{y_0} I & 0 \\ 0 & \sqrt{x_0} I \end{pmatrix} u_0, \qquad \bar{u} = \begin{pmatrix} \sqrt{y_0} I & 0 \\ 0 & \sqrt{x_0} I \end{pmatrix} \bar{u}_0; \qquad sdet\; u_0 = sdet\; \bar{u}_0 = 1$$

This can be seen, e.g., by considering the N=0 case, and noting that there $det(zd\bar{z}) = dx^2/x_0^2$ is the metric of the projective lightcone. We then have

$$sdet\ u = sdet\ \bar{u} = \frac{y_0}{x_0}$$

Then we see that the $R \to 0$ limit is the boundary limit, since now the scaling is on just $y_0 \to Ry_0$ and $x_0 \to Rx_0$:

$$w \to w, \qquad u_0 \to u_0, \qquad \bar{u}_0 \to \bar{u}_0, \qquad \frac{y_0}{x_0} \to \frac{y_0}{x_0}; \qquad x_0 y_0 \to R^2 x_0 y_0$$

10D field equations

The correspondence relates fundamental fields in the string theory to color-singlet composite fields in the conformal field theory. Of particular interest are fields that correspond to reducing the superstring to a superparticle: They describe 10D IIB supergravity (again perturbed about $AdS_5 \times S^5$).

Both 10D IIB supergravity and 4D N=4 super Yang-Mills are representations of the group PSU(4|2,2). But the physical interpretation is different: For example, they satisfy different field equations, even at the free level. We saw the free field equations for (the field strengths of) 4D super Yang-Mills, and applied them in projective superspace. On the other hand, 10D supergravity satisfies different, weaker equations (since more dimensions): Its free field equations are

$$G_{\mathcal{M}}{}^{\mathcal{P}} G_{\mathcal{P}}{}^{\mathcal{N}} = 0\ mod\ \delta\ terms$$

In the boundary limit these are not the 4D Yang-Mills equations, but the equations satisfied by certain color-singlet composites of the Yang-Mills fields.

We saw the stronger equations implied $p^2 = 0$ in D=4 by picking indices giving the highest (engineering) dimension; thus the rest of the equations followed by conformal supersymmetrization. That was easy, since all 4 indices were free in that case, whereas here some are contracted. Now we restrict to the bosonic sector of the weaker 10D equations, which is sufficient, as the supersymmetric generalization is unique. This means we truncate the symmetry group to SU(4)⊗SU(2,2), which is not the same as considering the N=0 case. The field equations are then of the form

$$G_{\bar{m}}{}^{\bar{p}} G_{\bar{p}}{}^{\bar{n}} = \delta_{\bar{m}}^{\bar{p}} \mathcal{O}, \qquad G_{\bar{\mu}}{}^{\bar{\rho}} G_{\bar{\rho}}{}^{\bar{\nu}} = \delta_{\bar{\mu}}^{\bar{\nu}} \mathcal{O}$$

for some operator $\mathcal{O}$. These can be translated into vector notation as

$$G_{[\underline{mn}} G_{\underline{pq}]} = G_{[\underline{\mu\nu}} G_{\underline{\rho\sigma}]} = G^{\underline{mn}} G_{\underline{mn}} - G^{\underline{\mu\nu}} G_{\underline{\mu\nu}} = 0$$

which generalize to arbitrary $\mathrm{AdS}_m\times \mathrm{S}^n$, where $\underline{m}$ and $\underline{\mu}$ are vector indices for SO(n+1) and SO(m-1,2). If we plug in the usual representations of these symmetry groups on these spaces, then the former 2 equations say that the corresponding spins vanish, while the last is the Klein-Gordon equation in m+n dimensions. If we had set $\mathcal{O}$ to vanish, decoupling the 2 spaces, we would instead have the m-dimensional Klein-Gordon equation on AdS, while on the sphere we would leave only a constant solution.

In fact, the supersymmetric 10D equations above are satisfied by *off-shell* 4D N=4 projective superpspace, simply as a consequence of reducing to just the w coordinates. This result can be generalized a bit: Converting to the DD form of the equations (by multiplying by g and g^{-1} appropriately to convert indices), still applying $D_v = 0$ (to allow the projective approach) and leaving D_w unconstrained, we can consider modifying the D_u and $D_{\bar{u}}$ constraints, as we did when considering arbitrary (super)spin. The solution is that the field is a scalar, which we already knew was true by construction as a superparticle, since no spin degrees of freedom were introduced. More specifically, we find

$$D_A{}^B = r\delta_A^B, \qquad D_{A'}{}^{B'} = \bar{r}\delta_{A'}^{B'}; \qquad r = \bar{r}$$

for some "central charge" r that commutes with D_w. From the previously given solution to this constraint for any eigenvalue of $r + \bar{r}$,

$$\overset{\circ}{\Psi}(w, u, \bar{u}) = (sdet\ u)^r (sdet\ \bar{u})^{\bar{r}} \Psi(w)$$

(we should solve before setting $sdet\ u = sdet\ \bar{u}$, etc.) and our above choice for defining y_0/x_0, we see that

$$r = \bar{r} = \frac{\partial}{\partial\ ln(y_0/x_0)}$$

and our general solution to the 10D field equations is in terms of a field that is an arbitrary function of w and y_0/x_0. (The same result is obtained if we calculate directly $D_u = \partial_u u$, etc., paying careful attention to signs from the grading. For example, $\partial_M{}^A u_B{}^N = \delta_A^B \delta_M^N$ has an implicit factor of $(-1)^A$ from the A being to the left of the B.) Thus, in the same way that 4D supertwistors solve the free 4D field equations, 4D N=4 projective superspace (plus the coordinate $ln(y_0/x_0)$) can be considered to be the supertwistor space of free 10D IIB supergravity on $\mathrm{AdS}_5\times \mathrm{S}^5$. It solves these 10D field equations in terms of "initial conditions" (in the spacecone sense) at the 9D boundary $x_0 y_0 = 0$.

Correspondence

We now investigate the significance of this 9th coordinate x_0/y_0 to the CFT. Consider expansion of the 10D theory over S^5 in terms of spherical harmonics. These can all be expressed in terms of those for the vector harmonic, which are given by a unit 6-vector; in the coordinates we've been using, these are (see again subsection IXC2)

$$Y = (Y^+, Y^i, Y^-) = \frac{(1, y^i, \frac{1}{2}(y^2 + y_0^2))}{y_0}$$

In the boundary limit, this becomes a null 6-vector,

$$Y \to \frac{(1, y^i, \frac{1}{2}y^2)}{y_0}$$

homogeneous in y_0. This y dependence can clearly be associated with that of the scalars of 4D N=4 Yang-Mills, i.e., the field strength $\mathit{\Phi}$ at $\theta = 0$.

A similar analysis can be made for the x_0 dependence of the scalars. In this case, it's easier to take the boundary limit first; then we can use the projective lightcone analysis directly. (Thus, $X^2 = 0$ is treated as the limit $R \to 0$ of $X^2 = -R^2$.) For general spin, this analysis was described for free theories in exercise IIB1.3. (In general, interactions modify this result; but for the fundamental fields of 4D N=4 Yang-Mills, and the composite operators considered here, ultraviolet finiteness preserves conformal weights.) Using an analysis of the type applied in exercise IA6.2 (paying careful attention to ordering), one finds in general D

$$\overset{\circ}{\phi}(x, x_0) = x_0^{(D-2)/2}\phi(x)$$

where in this case

$$D = 4 \quad \Rightarrow \quad \overset{\circ}{\phi}(x, x_0) = x_0 \phi(x)$$

This result also follows from dimensional analysis: Since the original field $\overset{\circ}{\phi}$ was a "scalar" under the conformal group (i.e., had vanishing scale weight), the engineering dimension of the usual field $\phi(x)$ must be canceled by an appropriate power of x_0. In the usual holographic analysis, this is identified with holographic-"time" dependence: If we write $x_0 = e^{-t}$, so the corresponding term in the metric is simply dt^2, then the dependence of a field in the limit $t \to \infty$ ($x_0 \to 0$) is $e^{-t\mathit{\Delta}}$, where $\mathit{\Delta}$ is the conformal weight. Amputation of this factor in AdS amplitudes is then equivalent to use of the interaction picture for this Euclidean time coordinate.

We can easily supersymmetrize this result to identify the other fields of the supermultiplet, and see how they appear in color singlets. Returning to our analysis of general spin, noting that Φ is a scalar with $r + \bar{r} = -1$, and again substituting for the *sdet*, we have

$$\mathring{\Phi}(w, u, \bar{u}) = \frac{x_0}{y_0}\,\Phi(w)$$

reproducing the x_0 and y_0 dependence found above for the scalars. (x_0 dependence is determined by the superscale weight of the multiplet, and y_0 by the super-U(1) weight. The corresponding symmetry generators also have $\theta\partial/\partial\theta$ terms, giving different component scale and U(1) weights to the higher spins.) It then follows that the supergravity superfield on the boundary must take the form

$$\mathring{\Psi}\left(w, \frac{x_0}{y_0}\right) = tr\left\{ f\left[\mathring{\Phi}\left(w, \frac{x_0}{y_0}\right)\right]\right\}$$

for some (Taylor expandable) function f, and thus contains terms of the form

$$tr\left\{\left[\frac{x_0}{y_0}\,\Phi(w)\right]^n\right\}$$

Thus, the 9th bosonic coordinate on the boundary just counts the number of supergluons. Note that, unlike the usual $x_0 \to 0$ limit, in this limit the supergravity fields are nonvanishing, having no dependence on $x_0 y_0$ (but string excitations will have positive powers of $x_0 y_0$, corresponding to anomalous dimensions in the 4D field theory). Also for these supergravity fields on the boundary, the "momentum" conjugate to the coordinate $ln(x_0/y_0)$ is quantized.

The 10D supergravity superfield is real. (Y is real: Because of Wick rotation, Y^i is real but $Y^{+*} = -Y^-$. This implies the usual charge conjugation for y on the boundary, while y_0 gives the density part of charge conjugation to Φ.) It's also nonsingular on S^5: Since it can be expanded in spherical harmonics, that means on the boundary only nonnegative powers of y will appear. Thus Φ is forced to satisfy its (*interacting*) field equations.

String cosets

So far we have avoided specifying the precise superpace used for describing the superstring. However, we have seen how 4D N=4 projective superspace can arise from taking the appropriate boundary limit, and how it appears upon solving the 10D equations of motion for the superparticle (supergravity). Certainly the string superspace must include at least these coordinates (and $x_0 y_0$). The action must also explicitly depend on R, as defined above in terms of the group elements. (For example, we saw for N=0 that R appears in the metric for AdS coset, but drops out in the flat-space coset.)

Superstrings, like superparticles, can be quantized in spacecone gauges. In such gauges only the physical fermions survive, 1/4 of the fermions of the full superspace. Also, all the physical bosons survive, corresponding to the dimension of spacetime. However, in the string case there are "oscillators", associated with excitations to massive levels of the string, only for the transverse dimensions. (See subsection XIB1.) Thus here we expect 8 fermions (and their canonical conjugates), with their associated oscillators, and 10 bosons, only 8 of which have oscillators. The 8 fermions + 8 bosons are associated directly with the 4D N=4 projective coordinates, while the 2 remaining "zero-modes" are associated with x_0 and y_0. Thus the AdS/CFT correspondence for the superspaces is clear in the spacecone gauge.

We also know that the 10D superspace in a general gauge must have more fermionic coordinates than the 8 fermions of the spacecone-gauge superspace: The full 32 fermions of the group PSU(4|2,2) correspond directly to those of the full superspace of 10D IIB supergravity. (An irreducible spinor of SO(9,1) has 16 components.) This superspace has an unambiguous definition: Since we keep all the fermions, we coset only the bosonic subgroup of the symmetry group PSU(4|2,2), namely SU(4)⊗SU(2,2). As described in subsections IVA2-3, S^5 is equivalent to the coset SO(6)/SO(5) = SU(4)/USp(4). (For covering spaces, see subsection IC5.) Then AdS_5 is a Wick rotation, SO(4,2)/SO(4,1) = SU(2,2)/USp(2,2). (See also subsection IXC2.) The full superspace is then the coset PSU(4|2,2)/USp(4)⊗USp(2,2).

It might also be useful to employ an intermediate superspace with 16 fermions, analogous to the projective and (anti)chiral superspaces of D=4. Chiral and antichiral 10D IIB superspaces for supergravity are straightforward: Under the gauge group USp(4)⊗USp(2,2), the 32 fermions divide up into a complex 4×4 and its complex conjugate. Chiral superspace uses just one of these, antichiral just the other. So either of these subspaces can be written as PSU(4|2,2)/I[USp(4)⊗USp(2,2)], where the "I" refers to inhomogeneous. This preserves the symmetry because the complex fermion (which was a 16-component 10D spinor in flat space) has a charge under a U(1) that isn't part of the symmetry algebra, and its complex conjugate the opposite charge, while the bosons are all neutral (as for 4D N=1). As a result, its covariant derivatives anticommute with themselves, and so can consistently vanish. Unfortunately, this U(1) symmetry of the superparticle action (describing supergravity) is not a symmetry of the superstring action (describing also massive 10D fields), so chiral superspace is not defined for the whole superstring.

A possible alternative is a projective-like superspace PSU(4|2,2)/OSp(4|4), which picks a real combination of the fermions, but leaves more bosons in the "internal" space, SU(4)/SO(4). These might be interpreted as the internal space accompanying the 10D spacetime, just as 4D projective superspace needs y coordinates in addition to the 4 spacetime coordinates x.