Green-Schwarz superstrings

There are difficulties with covariant quantization of superstrings in a way that manifests supersymmetry, so previously we discussed only the lightcone gauge for Green-Schwarz superstrings, following from RNS by triality. However:

- Triality operates at the (nonperturbative) quantum mechanical level, so it's difficult to describe supersymmetry in the Ramond-Neveu-Schwarz formalism for cases like $\mathrm{AdS}_5/\mathrm{CFT}_4$.
- Lightcone approaches to strings have had little success beyond 1 loop.

So now we look at the covariant formulation (with respect to both Lorentz and supersymmetry).

Superparticle

We have already seen (e.g., in 1-loop 4-particle S-matrices) that superstrings can be understood by combining knowledge of bosonic strings with that of superparticles. So far we have discussed "superparticles" only in terms of supersymmetric field theory or the zero-modes of the lightcone superstring. Thus we now examine covariant approaches to classical mechanics of superparticles.

Action

Actually it's quite easy to write a manifestly supersymmetric action for a particle in superspace. Only the interpretation and application are difficult. To be general, we define supersymmetry q in arbitrary dimensions as (see subsection XC4)

$$\delta\theta = \epsilon \ , \quad \delta x = i\epsilon\gamma\theta = i(\delta\theta)\gamma\theta$$

where $\gamma^a_{\alpha\beta}$ are symmetric matrices so that

$$\{q_\alpha, q_\beta\} = 2\gamma^a_{\alpha\beta} p_a$$

for transformations generated by $\delta = [i\epsilon q, \ ]$ (the missing indices should be clear by context). Then the supersymmetry invariant currents (differentials) are

$$d\theta \ , \quad dx + i(d\theta)\gamma\theta$$

In terms of these we write the action for a massless, superspinless superparticle as (cf. the bosonic particle, section IIIB)

$$-\int d\tau \, \tfrac{1}{2} v^{-1} (\dot{x} + i\dot{\theta}\gamma\theta)^2$$

(The other invariant is forbidden by dimensional analysis, and sometimes even by Lorentz invariance.) One obvious complication is that the action is nonlinear even in the covariant gauge (worldline metric) $v = 1$.

A Hamiltonian analysis is rather straightforward, as we're already in "one dimension" for the worldline. The result can be written as

$$\int d\tau[-(\dot{x}p - i\dot{\theta}\pi) + (v\tfrac{1}{2}p^2 + \lambda d)] \ , \quad d = \pi + \not{p}\theta \quad [q = -i(\pi - \not{p}\theta)]$$

or manifestly supersymmetry covariantly as

$$\int d\tau\{-[(\dot{x} + i\dot{\theta}\gamma\theta)p - i\dot{\theta}d] + (v\tfrac{1}{2}p^2 + \lambda d)]$$

The Lagrangian form is directly obtained by eliminating the auxiliary field p and Lagrange multiplier λ. (Note the simplifications introduced by using the current $p \sim \dot{x} + i\dot{\theta}\gamma\theta$ in place of explicit $\dot{x}$'s in the constraints.)

κ symmetry

Quantization is a problem because unlike the "1st-class constraint" p^2 from varying the Lagrange multiplier v (which makes v a gauge field, see subsection IIIA5), the constraint d is a mixture of 1st and "2nd-class" (so λ is not pure gauge), as seen from the commutation relations:

$$\{d, d\} = 2\gamma p$$

but p is not a constraint. (Effectively $p = -i\partial_x$ and $\pi = \partial_\theta$.)

The problem is separating, and dealing with, both kinds of constraints. We won't provide a complete solution here, but note that that 1st-class ones can easily be separated: Introducing $\gamma^{a\alpha\beta}$ (in general up and down spinor indices may denote different, but dual, spinor representations; see subsections IA4, XC2, and XC4), we choose

$$\mathcal{B} = \not{p}d$$

since (normalizing $\{\gamma, \gamma\} = 2\eta$)

$$\{\mathcal{B}, \mathcal{B}\} = 2\not{p}p^2$$

shows that p^2 and $\mathcal{B}$ have an algebra that closes. (We could also choose $\not{p}\pi$, which commutes with itself, and differs only by a term proportional to p^2, but it isn't supersymmetry invariant like d and p.) In particular, the gauge transformation generated by $\mathcal{B}$ acts on θ and x as

$$\delta\theta = \not{p}\kappa \ , \quad \delta x = -i\kappa\not{p}\gamma\theta = -i(\delta\theta)\gamma\theta$$

modulo a term proportional to the constraint d (which can always be canceled by $\delta\lambda$; there are also the usual implied gauge transformations for v).

To see how much of the problem this solves, we examine the lightcone gauge. Since there p^+ is always invertible (by assumption), we concentrate on its terms: Besides the usual gauge $x^+ = \tau$ (for p^2) and ensuing manipulations, for $\mathcal{B}$

$$\delta\theta = -p^+\gamma^-\kappa + ... \quad \Rightarrow \quad \text{gauge } \gamma^+\theta = 0$$

(using the projection operator $\sim \gamma^+\gamma^-$). What's left is

$$\int d\tau [(\dot{x}^- + i\dot{\theta}\gamma^-\theta)p^+ - \tfrac{1}{2}\dot{x}^{i2}]$$

The remaining 2nd-class constraints thus state that what's left of θ is essentially canonically conjugate to itself. (There is also a factor of p^+, but it's a constant by the equations of motion, so it can be scaled away by a redefinition of θ, up to a sign.) But we should have expected this from triality in the string case, where the same is true for RNS (or spinning particle) fermions.

What's the spectrum of this superparticle? For the case of most interest, D=10 N=1 supersymmetry, it's easy to see using triality: x is described the same as for a scalar particle. So all spin comes from θ. But (after scaling out the p^+), it has the same action as for a spinning (Dirac) particle (in lightcone gauge), except that the SO(8) vector fermion has been replaced with an SO(8) Weyl-Majorana spinor fermion. It still has anticommutation relations of a Dirac/Clifford algebra. In the Dirac case, a vector operator led to states that were a Weyl spinor $\oplus$ the other chirality Weyl spinor. By triality, which simply permutes the 3 **8**-representations, a Weyl spinor operator must lead to states that are a vector $\oplus$ the other-chirality Weyl spinor. This is exactly the spectrum of 10D N=1 super Yang-Mills, the massless sector of the open superstring.

Chiral σ models

The main complication for the superstring (besides 1 more dimension for the worldsheet) is that we want to maintain separation into left and right-handed modes (with respect to worldsheet propagation) in a way that's clear even in a nonlinear action. Such a problem generally arises when defining 2D theories on group spaces. (Here the group is supersymmetry.) The solution is to include a 2-form ("Wess-Zumino") term in the action. This has a natural group-theoretic interpretation if we choose a 2-form whose 3-form field strength (which automatically appears upon varying that term in the action) is identified with the structure constants of the group (which appear upon reordering terms from varying the metric term in the action). In this subsection we analyze the case of simple groups at the classical level. (There are also interesting quantum effects, but they aren't relevant to the non-semisimple, supersymmetry case.)

Group

So our 2D fields g are elements of some group. For purposes of considering coset spaces (relevant for supersymmetry), we consider only global (on the worldsheet) symmetry transformations corresponding to left multiplication of this field by a constant group element g_0,

$$g' = g_0 g$$

(The actions we'll consider later for superstrings will break symmetry corresponding to right multiplication.) Then the symmetry invariant current ("field strength" for the scalars g), which is an element of the Lie algebra (and a worldsheet vector) is

$$iJ = g^{-1}dg$$

and the metric term in the action comes from squaring this, and taking the group-theory trace. (For cosets we take only some of these currents, whose square is invariant under a subgroup under right multiplication, as we already did for the superparticle.) We can also simplify this (for the full group) as (since $\partial(gg^{-1}) = 0$)

$$(g^{-1}\partial g)(g^{-1}\partial g) = -(\partial g^{-1})(\partial g)$$

We similarly note the "Bianchi identities" for the field strength ("Maurer-Cartan equations")

$$dJ + iJ \wedge J = 0$$

and defining a covariant variation (also in the Lie algebra, but a worldsheet scalar)

$$i\mathit{\Delta} \equiv g^{-1}\delta g$$

that (since δ commutes with d)

$$\delta J = d\mathit{\Delta} + i[J, \mathit{\Delta}]$$

Pulling out a factor of $\mathit{\Delta}$ (after integration by parts) from the trace and integral, the field equation from the metric term is then proportional to

$$-\partial \cdot J$$

In (2D) lightcone notation, if we normalize

$$A \cdot B = -A_+B_- - A_-B_+ \ , \quad A \wedge B = d^2\sigma\,(-A_+B_- + A_-B_+)$$

then we would find (using the field equations $\pm$ MC equations)

$$-2\partial_\pm J_\mp \mp i[J_+, J_-] = 0$$

not summed over $\pm$ (but independent of the normalizations), spoiling the equations $\partial_\pm J_\mp = 0$ that would indicate decomposition into left and right-handed currents.

2-form

On the other hand, a worldsheet 2-form in a general theory (not necessarily related to groups) would contribute a term to the field equations (making only indices for the "2D fields" α^m explicit)

$$\delta B = \delta(\tfrac{1}{2} d\alpha^m \wedge d\alpha^n B_{mn}) \approx \tfrac{1}{2} d\alpha^m \wedge d\alpha^n \delta\alpha^p H_{mnp}$$

after manipulations similar to those used to derive δJ, but "$\approx$" means up to a total derivative that we drop in the action so that we can apply integration by parts to the $d\delta\alpha$ term. Here we normalize

$$H_{mnp} = \tfrac{1}{2}\partial_{[m} B_{np]}$$

where "[ ]" means to sum over all permutations with weight $+1$ for even and -1 for odd.

To compare the contribution of such a term to the equations of motion, we need to convert "curved" (coordinate-basis) indices to "flat" (group-invariant) indices by use of the current itself: Making the matrix representation of the Lie algebra generators G_a explicit,

$$J = J^a G_a \ , \quad J^a = d\alpha^m e_m{}^a$$

where the "vielbein" $e_m{}^a$ is a function of what are now the group coordinates α^m. (We could do similar for currents invariant under right group multiplication, but we'll avoid that here for simplicity.) In the Abelian case, J would just be $(d\alpha)G$, and e would be a Kronecker δ. Since the algebra is identical (δ is like d in a third dimension), we also have

$$\varDelta = \varDelta^a G_a \ , \quad \varDelta^a = \delta\alpha^m e_m{}^a$$

Using this vielbein to convert the indices on H, the field equations ($\pm$ MC) now become

$$(-2\partial_\pm J^b_\mp \mp J^c_+ J^d_- f_{cd}{}^b)\eta_{ba} - J^c_+ J^d_- H_{cda} = 0$$

where the η is the Cartan metric coming from the trace (which we can assume to be for the adjoint representation). Thus

$$H_{abc} = \pm f_{ab}{}^d \eta_{dc} \quad \Rightarrow \quad \partial_\mp J_\pm = 0$$

allowing us a left-handed current J_+ or right-handed J_- (both invariant under left group multiplication), but not both, unless vanishing of structure constants (as for Abelian or non-semisimple groups) allows the corresponding components of H to vanish. (This is a "Wess-Zumino term", a Chern-Simons term for scalars.)

The same result follows from imposing T-duality invariance (electric-magnetic self-duality in the 2D theory): Start with the Bianchi identities for J (MC equations), replace J with its Hodge dual ($J_\pm \to \pm J_\pm$, or the opposite sign), and require that those are the field equations. That determines the action as above and, again taking the sum or difference of the 2 equations, fixes the handedness of 1 component.

In a Hamiltonian analysis, we could instead analyze the current algebra: We assume the existence of a left-handed algebra

$$[J_a(1), J_b(2)] = -i\delta f_{ab}{}^c J_c - i\delta'(2-1)\eta_{ab}$$

in some convenient normalization, where the "1" and "2" refer to σ_1 and σ_2, and the δ functions include 2π's corresponding to the measure $\int d\sigma/2\pi$. This "affine Lie algebra" generalizes the usual particle Lie algebra with the metric term. The Jacobi identities are then the usual ff identity of a Lie algebra, and an identity that imposes the total antisymmetry of f after lowering the last index with η:

$$f_{[ab}{}^d f_{c]d}{}^e = 0 = f_{a(b}{}^d \eta_{c)d}$$

Finding a group-coordinate representation then leads to the previous results.

Superstring

Action

We now consider the action for Type II strings, with group coordinates X, Θ_L, Θ_R; i.e., the coordinates of N=2 superspace. (Type I follows from boundary conditions, heterotic from truncation.) Since the only nonvanishing structure constants come from $\{d, d\} \sim p$, we would only have a problem with left and right-handed Θ's. But we saw that (at least in the lightcone gauge for the superparticle) Θ satisfies a first-order differential equation, so it'll be enough to get each Θ to have a current of 1 handedness. Then the only nonvanishing H will be

$$H_{\alpha\beta c} = \pm 2\gamma_{\alpha\beta}^d \eta_{dc}$$

where the η comes from the worldsheet metric term having only the square of the p current. This H is identified with the field strength of the B superfield for background supergravity in superspace in the string frame, which is nonvanishing even in flat space because the scalar field strength gets a vacuum value. (This is related to the fact that the "structure constants" for a spacetime symmetry group are the torsion, which is also nonvanishing in flat superspace.)

We first introduce the notation

$$\chi_{L,R} \equiv i(d\Theta_{L,R})\gamma\Theta_{L,R} \ , \quad P \equiv dX + \chi_L + \chi_R$$

(where P and χ are vectors in both the worldsheet and spacetime): P is the same invariant current we used for the superparticle. (The other currents are again $d\Theta_{L,R}$.) Then the action is

$$S = \int \frac{d^2\sigma}{2\pi} \left[\tfrac{1}{2}P^2 - dX \wedge (\chi_L - \chi_R) + \chi_L \wedge \chi_R\right]$$

where the worldsheet metric is implicit for the first term. The worldsheet chirality of the Θ's is associated with the fact that the metric term, which is even in σ derivatives, is LR symmetric in Θ's, while the 2-form terms, odd in $\partial/\partial\sigma$, are antisymmetric.

Exercise 1

Show that variation of the WZ terms give

$$\delta B \approx -\tfrac{1}{2} J^A \wedge J^B \Delta^C H_{CBA}$$

with the above H.

Again, the 2-form term is required by T-duality invariance. In terms of the action, one introduces a first-order formalism for the metric term with auxiliary field F:

$$\tfrac{1}{2}P^2 \to P\cdot F - \tfrac{1}{2}F^2 : \quad \delta/\delta F \;\Rightarrow\; F = P \ , \quad \delta/\delta X \;\Rightarrow\; F = \epsilon(\partial\tilde{X} + \chi_L - \chi_R)$$

where the latter is the solution to the X field equation $\partial\cdot[F - \epsilon(\chi_L - \chi_R)] = 0$. The T-duality is performed with respect to X only (P is its supersymmetry invariant current), but invariance requires also $\Theta_L \to \Theta_L$, $\Theta_R \to i\Theta_R$ (i.e., $\tilde{X} \to X$, $\chi_L \to \chi_L$, $\chi_R \to -\chi_R$), because the transformation switches the Θ contributions to metric and 2-form terms. (This transformation is similar to CPT on real spinors.) Note that X is a contravariant vector in spacetime, while $\tilde{X}$ is a covariant one.

Another method is to use κ symmetry: It now comes in left and right versions for left and right-handed Θ's, respectively, which "square" to left and right Virasoro generators. This effectively imposes that D comes in left and right versions. Then (here $\pm$ are worldsheet indices)

$$\delta\Theta_L = \not{P}_+\kappa_L \ , \quad \delta\Theta_R = \not{P}_-\kappa_R \ , \quad \delta X = -i(\delta\Theta_L)\gamma\Theta_L - i(\delta\Theta_R)\gamma\Theta_R$$

For the lightcone gauge, we can use the usual Hamiltonian procedure for gauge fixing the bosons (subsection XIB1), and the same κ gauge for the fermions used for the superparticle. The result is (with p^+ again an independent variable)

$$S = \int \frac{d^2\sigma}{2\pi}\,[\dot{x}^- p^+ + \tfrac{1}{2}(-\dot{X}^{i2} + X'^{i2}) + i(\dot{\Theta}_L - \Theta'_L)\gamma^-\Theta_L p^+ + i(\dot{\Theta}_R + \Theta'_R)\gamma^-\Theta_R p^+]$$

(Here " $\dot{}$ " is again the τ derivative, while " $'$ " is the σ derivative.)

Current algebra

An equivalent analysis can be made by Hamiltonian methods. This still looks covariant for a 2D worldsheet because the conformal group factors into separate left and right 1D general coordinate transformations. Then we look at the current algebra, from which we can construct Virasoro and κ generators quadratically. It also suggests generalizations that can be simpler in some respects, and relates to field theory, where we need the "covariant derivatives" that generalize d and p for the particle. Because they appear in field equations, they also correspond to vertex operators for the string.

Since the left and right algebras are independent (at least for the free string in flat space), we look at just the left-handed one. (As usual, the relation of the spinors between left and right-handed algebras differs for Types IIA and IIB.) We can start with just the D current and generate the rest from it:

$$\{D_\alpha(1), D_\beta(2)\} = \delta 2\gamma^a_{\alpha\beta}P_a$$
$$[D_\alpha(1), P_a(2)] = \delta 2\gamma_{a\alpha\beta}\Omega^\beta$$
$$[P_a(1), P_b(2)] = -i\delta'(2-1)\eta_{ab}$$
$$\{D_\alpha(1), \Omega^\beta(2)\} = -i\delta'(2-1)\delta_\alpha^\beta$$

The appearance of the current Ω might be unexpected, as there was no analog in the free particle case. It's required by the Jacobi identities (discussed previously for compact groups): The $f\eta$ identity requires a metric for D by defining a "dual" current. Since the group is not (semi)simple, the metric appearing in the current algebra (and action) is not required to be related to the Cartan metric, which vanishes in this case.

Exercise 2

Find the Jacobi identity (cubic in currents) that requires the existence of Ω. (Hint: There are no Ω's appearing until commutators are evaluated.) Relate the nontriviality of this identity to the nonvanishing components of H.

However, when the superparticle is coupled to an external super Yang-Mills field, D and P are replaced by the corresponding Yang-Mills covariant derivatives, and Ω by the spinor superfield strength. In both cases, the satisfaction of one of the ff identities requires a Fierz identity:

$$\gamma_{a(\alpha\beta}\gamma^a_{\gamma)\delta} = 0$$

After contracting spinor indices with more γ's, this is found to imply spacetime dimension D=3,4,6, or 10. These are the dimensions for which Yang-Mills with simple supersymmetry has no scalars, i.e., the maximal dimensions with various numbers of supersymmetries (scalars coming from dimensional reduction). Of course, Yang-Mills is the massless sector of the open string, so this result is part of the relation of the "vertex operators" in the 2 theories.

For the bosonic string the left-handed algebra consists of just

$$P = \hat{P} \equiv \tfrac{1}{\sqrt{2}}(\overset{\circ}{P} + X')$$

where $\overset{\circ}{P}$ is the canonical conjugate to X. But for the superstring the coordinate representation of the current algebra is

$$\Omega = -i\Theta'$$
$$P = \hat{P} + \chi_\sigma$$
$$D = \Pi + (\hat{\not{P}} + \tfrac{1}{2}\not{\chi}_\sigma)\Theta$$

where $\mathit{\Pi}$ is the canonical conjugate of Θ. (χ_σ is the σ component of χ_L defined earlier.) As expected, this differs from the particle by σ-derivative terms.

Only η_{ab} and $\gamma^a_{\alpha\beta}$ appear in the current algebra, but η^{ab} (just the inverse) and $\gamma^{a\alpha\beta}$ appear in the extended Virasoro algebra,

$$T = \tfrac{1}{2}P^2 \ , \quad \kappa = \not{P}D$$

as was the case for the particle. (The complete algebra is rather complicated, and won't be discussed here.) Also as for the particle, the action can be straightforwardly reconstructed from the Virasoro constraints T and the mixed first and second-class constraints D, but now with both left and right-handed sets.

AdS

For the AdS/CFT correspondence, we need to perturb the Type IIB superstring about an $AdS_5 \times S^5$ background, whose curvature comes from flux from the selfdual 5-form RR field strength. Unfortunately, in the RNS formalism such perturbations can't be treated semiclassically, or by functional integral, since there spinors act appropriately only at the fully quantum mechanical level (all orders in α'). So we look instead at its GS formulation, where spacetime (conformal) supersymmetry is manifest.

Action

The 4D N=4 superconformal group is the supergroup PSU(2,2|4). This supergroup has bosonic subgroup SU(2,2)⊗SU(4) (the covering group of SO(4,2)⊗SO(6), conformal and R-symmetry), plus 32 fermionic generators in the defining representations of both SU(2,2) and SU(4) (4×4 complex). (The "P" means a factor proportional to the identity is gauged away.) We can then write the group generators as

$$G_{\underline{A}}{}^{\underline{B}} \quad \underline{A} = (\underline{a}, \underline{\alpha})$$

where $\underline{a}$ is an SU(4) 4-index, and $\underline{\alpha}$ is for SU(2,2). The commutation relations for U(2,2|4) (from which we can subtract traces for "PS") are

$$[G_{\underline{A}}{}^{\underline{B}}, G_{\underline{C}}{}^{\underline{D}}\} = \delta_{\underline{A}}{}^{\underline{D}} G^{\underline{B}}{}_{\underline{C}} - \delta^{\underline{B}}{}_{\underline{C}} G_{\underline{A}}{}^{\underline{D}}$$

where the reverse ordering $\delta^{\underline{B}}{}_{\underline{A}}$ or $G^{\underline{B}}{}_{\underline{A}}$ (vs. the "natural" $\delta_{\underline{A}}{}^{\underline{B}}$ and $G_{\underline{A}}{}^{\underline{B}}$) means there's an extra statistics sign when reordering 2 "fermionic" indices $\underline{\alpha}, \underline{\beta}$. An arbitrary element of the supergroup is then

$$g = e^{i\alpha_{\underline{A}}{}^{\underline{B}} G_{\underline{B}}{}^{\underline{A}}}$$

where another statistics ordering sign is understood for contracting ${}_{\underline{A}}{}^{\underline{A}}$ (the "supertrace") opposite to the natural contraction order ${}^{\underline{A}}{}_{\underline{A}}$. The currents are given by

$$-ig^{-1}dg = J = G_{\underline{A}}{}^{\underline{B}} J_{\underline{B}}{}^{\underline{A}}$$

The AdS_5 realization of SO(4,2) is the coset space G/H=SO(4,2)/SO(4,1). (For more on cosets see subsection IVA3, or my string notes on superconformal symmetry.) The "vacuum" preserves SO(4,1), meaning we use the usual 5D coordinates with manifest SO(4,1) Lorentz symmetry, while the coset comes from the translation generators, which close only on Lorentz. (In the boundary limit, the SO(4,1) is contracted to the 4D Poincaré group ISO(3,1). The resulting coset is 4D Minkowski space, plus an extra coordinate that parametrizes scale weight.) Similar remarks apply to the sphere S^5 as the coset SO(6)/SO(5). We thus are left with 10 bosonic coordinates, and keep the full 32 fermionic coordinates of Type IIB supergravity. So the superspace is G/H=PSU(2,2|4)/USp(2,2)⊗USp(4). (USp(4), USp(2,2), and Sp(4) cover SO(5), SO(4,1), and SO(3,2).) This means that the only symmetry invariant (because symmetry acts to the left of g) and gauge covariant currents are

$$J_{\langle \underline{ab} \rangle}, J_{\langle \underline{\alpha\beta} \rangle}, J_{\underline{a\beta}}, J_{\underline{\alpha b}}$$

(the former bosonic, the latter fermionic), where the $\underline{B}$ index on $J_{\underline{A}}{}^{\underline{B}}$ has been lowered with the USp(4) and USp(2,2) metrics $C_{\underline{ab}}$ and $C_{\underline{\alpha\beta}}$, and $\langle\ \rangle$ means we keep only the antisymmetric, (Sp-)traceless parts. (The symmetric parts were eliminated by the Sp gauge groups, the traces by the P and S.)

The action can then be written by methods similar to those we have discussed. By a fortunate accident, the WZ term in this action can be written in manifestly supersymmetric form, as the square of currents (but using the ϵ tensor, not the worldsheet metric). (Actually this can also be done for the GS action in general, but not without the introduction of extra fermionic coordinates.) The Lagrangian is then proportional to

$$J^2_{\langle \underline{ab} \rangle} - J^2_{\langle \underline{\alpha\beta} \rangle} + \tfrac{1}{2} J^{\underline{a\beta}} \wedge J_{\underline{a\beta}} - \tfrac{1}{2} J^{\underline{\alpha b}} \wedge J_{\underline{\alpha b}}$$

(κ symmetry can be used to determine the coefficient of the WZ terms, but also T-duality, see below.)

Coordinates

For purposes of a nice coordinate realization, we Wick rotate to the real super-coset GL(4|4)/[GL(1)⊗Sp(4)]^2. (The bosonic subcoset is (SO(3,3)/SO(3,2))^2.) Then, rather than annoying cosh's and sinh's, we can write things in terms of fractions, because we can write a group element of GL(4|4) as an arbitrary matrix instead of an exponential.

We then use matrix decompositions, writing the group element in factorized form, each factor being either block diagonal, or the identity plus block off-diagonal. This makes the currents polynomial in the off-diagonal variables. For example, consider a decomposition of the form, for 2 arbitrary size diagonal blocks,

$$g_{\underline{M}}{}^{\underline{A}} = \begin{pmatrix} I & w \\ 0 & I \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \bar{u}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -v & I \end{pmatrix} \qquad \underline{A} = (A, A')$$

(We now use indices from the middle of alphabets to indicate the action of the symmetry group G from the left, and indices from the beginning for the gauge group H acting from the right.) It's clear that w will appear in the currents $g^{-1}dg$ only as dw, since it will cancel in the middle if undifferentiated. This allows T-duality transformations to be transformed easily on w, for which there's an Abelian translation invariance. (w is the coordinates for a torsion-free superspace.)

As an example, consider a term for AdS_5 (or S^5) alone, taken as the coset GL(4)/GL(1)⊗Sp(4). In a triangular Sp(4) gauge we can write the group element X as

$$X_{\underline{\mu}}{}^{\underline{\alpha}} = \begin{pmatrix} I & x \\ 0 & I \end{pmatrix} \begin{pmatrix} Ix_0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} Ix_0 & x \\ 0 & I \end{pmatrix} \quad \Rightarrow \quad X^{-1} = \begin{pmatrix} I/x_0 & -x/x_0 \\ 0 & I \end{pmatrix}$$

in terms of a real 2×2 matrix x and a fifth coordinate x_0. Then the current is

$$X^{-1}dX = \frac{1}{x_0} \begin{pmatrix} dx_0 & dx \\ 0 & 0 \end{pmatrix}$$

Squaring gives the metric in Poincaré coordinates

$$\frac{dx^2 + dx_0^2}{x_0^2}$$

where dx^2 is the determinant of dx, giving that part signature $--++$. (These coordinates are globally sufficient when working with Euclidean AdS.)

For the full superconformal group, 2 particular choices for which the action is invariant under T-duality are (for w as "chiral" and "projective" superspaces)

$$\begin{array}{lll} (1) & A = \alpha\ , & A' = (a, a', \dot{\alpha}) \\ (2) & A = (a, \alpha)\ , & A' = (a', \dot{\alpha}) \end{array}$$

where $a, a', \alpha, \dot{\alpha}$ are SL(2) 2-indices (or SU(2) and SL(2,C) before Wick rotating). In both these cases we can choose triangular gauges

$$v = 0$$

by using part of the Sp(4)'s for the bosons (as above for AdS_5) and part of κ symmetry for the fermions. Then we can also write (dropping the "i" because we already dropped a lot of them for Wick rotation)

$$g^{-1}dg = \begin{pmatrix} J_u & J_w \\ -J_v & -J_{\bar{u}} \end{pmatrix}$$

$$J_w = u^{-1}(dw)\bar{u}^{-1}\ , \quad J_u = u^{-1}du\ , \quad J_{\bar{u}} = (d\bar{u})\bar{u}^{-1}\ , \quad J_v = 0$$

For purposes of w T-duality we can then write the Lagrangian as

$$F \cdot \partial w - C(uF_+\bar{u})C(uF_-\bar{u})^T + J_u^2 + J_{\bar{u}}^2$$

where again varying F gives the original form while varying w gives the dual.

T-duality

T-duality is then almost trivial for w: Effectively it just replaces

$$w_M{}^{M'} \to \tilde{w}_{M'}{}^M \ , \quad u_M{}^A \to (u^{-1})_A{}^M \ , \quad \bar{u}_{A'}{}^{M'} \to (\bar{u}^{-1})_{M'}{}^{A'}$$

after which the action has the same form. (Note that again covariant and contravariant indices have been switched on w vs. $\tilde{w}$.) There is a fine point: When we dualized $dw \to \epsilon d\tilde{w}$ there was a factor $u \otimes \bar{u}$ (because of the 2 indices on w), which would appear in a Jacobian determinant, and so could contribute α' corrections. Using the identity

$$sdet(e^A) = e^{str\,A} \quad \Rightarrow \quad sdet(A \otimes B) = (sdet\,A)^{str\,I_B}(sdet\,B)^{str\,I_A}$$

(for identities I in the A and B spaces), and using the "S" condition $sdet\,u = sdet\,\bar{u}$, we find the Jacobian

$$sdet(u \otimes \bar{u}) = (sdet\,u)^{str\,I_{\bar{u}}}(sdet\,\bar{u})^{str\,I_u} = (sdet\,u)^{str\,I_u + str\,I_{\bar{u}}} = 1$$

since $str\,I_u + str\,I_{\bar{u}} = 0$ for PSU(2,2|4).

In case (1) we dualize 4 coordinates of AdS_5, in case (2) we also dualize 4 from S^5; in both cases also 8 fermionic coordinates. Of course, we could have modified case (1) by dualizing 4 from S^5 and none from AdS; case (2) then follows from doing both types of case (1). These dualities have interesting effects on the D3-branes often associated with AdS_5: Depending on which of these dualities we consider, we can get D−1-branes, D7-branes, or again D3-branes, where the brane spaces are associated with either 0 or 4 dimensions of both AdS_5 and S^5.

This invariance implies the existence of PSU(2,2|4) symmetry on the dual space. As we saw previously for the bosonic string (subsection XIA7), T-duality for the string corresponds to Fourier transformation for the Feynman diagrams of the "partons" of which it's composed. In the AdS/CFT correspondence, the partons are identified with the 4D N=4 Yang-Mills CFT, while the string is identified with "hadrons" (or whatever the analog is for color-singlet composites in this Yang-Mills theory). Thus the new PSU(2,2|4) symmetry resulting after dualizing the 4 coordinates of the AdS boundary (and perhaps simultaneously 4 of S^5) is a "dual superconformal symmetry" acting in some way on this CFT. This symmetry, combined with the usual superconformal, closes on an infinite-dimensional "Yangian symmetry" (not directly related to Yang-Mills symmetry, but the same guy) that places strong restrictions ("integrability") on S-matrices in the theory.

Limits

There are various limits one can take to simplify the AdS action, corresponding to group contractions. One type of contraction acts on only the gauge group, not the symmetry group. In the above triangular language, it corresponds to defining the limit with respect to flat and not curved indices:

$$g_{\underline{M}}{}^{A} \to \epsilon g_{\underline{M}}{}^{A} \ , \quad g_{\underline{M}}{}^{A'} \to \epsilon^{-1} g_{\underline{M}}{}^{A'} \ ; \quad \epsilon \to 0$$

and thus on the currents

$$J_w \to \epsilon^{-2} J_w \ , \quad J_u \to J_u \ , \quad J_{\bar{u}} \to J_{\bar{u}} \ , \quad J_v \to \epsilon^2 J_v$$

or in terms of the block coordinates,

$$w_M{}^{M'} \to w_M{}^{M'} \ , \quad u_M{}^A \to \epsilon u_M{}^A \ , \quad \bar{u}_{A'}{}^{M'} \to \epsilon \bar{u}_{A'}{}^{M'} \ , \quad v'_A{}^A \to \epsilon^2 v'_A{}^A$$

For a simple example, look at just bosonic AdS: There the effect of

$$x \to x \ , \quad x_0 \to \sqrt{\epsilon} x_0$$

(taking into account the GL(1) gauge) is to reduce the metric, up to an overall factor, as

$$\frac{dx^2 + dx_0^2}{x_0^2} \quad \to \quad \frac{dx^2}{x_0^2}$$

The limit $x_0 \to 0$ for the bulk coordinate x_0 is the boundary limit. The metric is still SO(4,2) conformal invariant, but the coset is now SO(4,2)/ISO(3,1): The gauge group SO(4,1) has been contracted. Similar results hold for the full supergroup, yielding Minkowski-space representations from the AdS ones.

Another interesting limit contracts also the symmetry group. The "Penrose limit" is one that generates simple wave solutions from any metric. In this case it's the limit

$$w \to \epsilon w \ , \quad v \to \epsilon v$$

combined with a scaling of the ratio "x^-" of the x_0 from AdS_5 and the corresponding one from S^5:

$$ln(x_0/y_0) \to \epsilon^2 ln(x_0/y_0)$$

Taken on just the (Wick rotated) $AdS_5 \times S^5$ metric, the simplest form comes from the redefinitions

$$x \to \epsilon e^{x^+} x \ , \quad y \to \epsilon e^{x^+} y \ , \quad x_0 \to e^{x^+ + \epsilon^2 (x^- + y^2 - x^2)/4} \ , \quad y_0 \to e^{x^+ - \epsilon^2 (x^- + y^2 - x^2)/4}$$

Then the $\epsilon \to 0$ limit is proportional to

$$\frac{dy^2 + dy_0^2}{y_0^2} - \frac{dx^2 + dx_0^2}{x_0^2} \quad \Rightarrow \quad -dx^+ dx^- + dy^2 - dx^2 + (y^2 - x^2) dx^{+2}$$

In the lightcone gauge $x^+ = \tau$, this gives a harmonic oscillator action (also for the fermions).