Group coordinates

Symmetry generators

A Lie group is a space, so we generally want to introduce some coordinates. Since it's a curved space, the choice of coordinates generally varies according to application. A simple choice is the exponential one,

$$g = e^{i\alpha^I G_I}$$

but it's usually not the most convenient one. For coset spaces, we often use

$$g = e^{i\alpha^i T_i} e^{i\alpha^\iota H_\iota}$$

since under the gauge group $g' = gh$, so h will transform only the α^ι, not the α^i. For various other purposes (see below), we may want to further factorize g. Group multiplication of such exponentials can be performed using the Baker-Campbell-Hausdorff theorem.

Generally, it's convenient to eliminate exponentials as much as possible, since it may be difficult to evaluate them explicitly in closed form. For example, we might use

$$g = e^{i\alpha^+ G_+} e^{i\alpha^0 G_0} e^{i\alpha^- G_-}$$

where the generators have been divided up into "raising operators" G_+, "lowering operators" G_-, and those of the "Cartan subalgebra" (a maximal Abelian subalgebra) G_0. (We here take $+, 0, -$ as multivalued indices.) Since G_0 is Abelian, its exponential is easily evaluated as phase factors. The expansions of the rest will terminate, leaving polynomials.

Exercise

Evaluate this group element in these coordinates using the definining representation of SU(2) for the generators. What are the reality properties of the coordinates?

Another possibility for classical groups is to work in the defining representation, and then solve the constraints on the group matrices in terms of some rational expression. We have already seen (incomplete) examples of this above for cosets represented as projective spaces.

Once a coordinate representation has been chosen, we also want such a representation for the action of the symmetry group on this space, i.e., a translation into coordinate language of $g' = g_0 g$. For many purposes it will be sufficient to evaluate the infinitesimal transformation (using, e.g., the BCH theorem)

$$\delta g \equiv i\epsilon^I G_I g = (e^{i\epsilon^I G_I} - I)g(\alpha) = i\epsilon^I \widehat{G}_I g$$

where $\widehat{G}_I$ is a differential operator. Since it generates an infinitesimal coordinate transformation, we can write

$$i\widehat{G}_I = L_I{}^M(\alpha)\partial_M, \qquad \delta\alpha^M = \epsilon^I L_I{}^M$$

where $\partial_M \equiv \partial/\partial\alpha^M$. We thus have

$$G_I G_J g = G_I \widehat{G}_J g = \widehat{G}_J G_I g = \widehat{G}_J \widehat{G}_I g$$

$$[G_I, G_J] = -if_{IJ}{}^K G_K \quad \Rightarrow \quad [\widehat{G}_I, \widehat{G}_J] = +if_{IJ}{}^K \widehat{G}_K$$

so technically it's $-\widehat{G}_I$ that's a coordinate representation of G_I. (Cf. subsection IC1, where we saw coordinate representations of the generators on spaces other than the group space.)

Equivalently, we can solve the "dual" equation, in terms of differential forms instead of derivatives,

$$(dg)g^{-1} \equiv [g(\alpha + d\alpha) - g(\alpha)]g^{-1}(\alpha) = i\, d\alpha^M L_M{}^I G_I$$

where $L_M{}^I$ is the matrix inverse of $L_I{}^M$. (As usual, when expressing transformations in terms of coordinates it's often convenient to eliminate all i's in the above equations by absorbing them into the G's and working with antihermitian operators.)

Covariant derivatives

If symmetry (known by mathematicians as "isometry") generators are defined by the left action of group generators on a group element, then generators of the gauge (known by mathematicians as "isotropy") group are defined by right action. The latter are known as "covariant derivatives" because they commute with the symmetry generators. (Commutativity of left and right multiplication is equivalent to associativity of multiplication.) From the same arguments as above, we have

$$gG_I = D_I g, \qquad iD_I = R_I{}^M(\alpha)\partial_M$$

$$g^{-1}dg = i\, d\alpha^M R_M{}^I G_I$$

$$[\widehat{G}_I, D_J] = 0$$

We now have

$$gG_IG_J = D_IgG_J = D_ID_Jg \quad \Rightarrow \quad [D_I, D_J] = -if_{IJ}{}^K D_K$$

There is a very simple relation between the symmetry generators and covariant derivatives. Consider the coordinate transformation that switches each group element with its inverse; then

$$g' = g_0gh \quad \Rightarrow \quad (g^{-1})' = h^{-1}g^{-1}g_0^{-1}$$

$$g \leftrightarrow g^{-1} \quad \Rightarrow \quad g_0 \leftrightarrow h^{-1} \quad \Rightarrow \quad G_I \leftrightarrow -D_I$$

(For the sake of this argument we need not distinguish between global and local groups, and h can be taken as in the full group.) This relation can also be seen from the explicit expressions for L and R as $(dg)g^{-1} \leftrightarrow -g^{-1}dg$. Thus, in the exponential coordinate system, we have simply $L(\alpha) = R(-\alpha)$ (with the extra "$-$" canceling the sign change of $\partial/\partial\alpha$).

We can "integrate" the (symmetry) invariant differentials $d\alpha^M R_M{}^I$ to get finite differences. But the result can be guessed directly:

$$g(\alpha_{12}) \equiv g^{-1}(\alpha_2)g(\alpha_1) = g^{-1}(\alpha_{21})$$

Thus the group element $g(\alpha_{12})$, and hence α_{12} itself, is symmetry invariant. α_{12} reduces to the above differential in the infinitesimal case. In coordinates where $g^{-1}(\alpha) = g(-\alpha)$ (for example, parametrization with a single exponential), we have also $\alpha_{21} = -\alpha_{12}$. The action of the covariant derivatives on the symmetry invariants is given by (using $d(g^{-1}) = -g^{-1}(dg)g^{-1}$)

$$D_I(\alpha_1)g(\alpha_{12}) = g(\alpha_{12})G_I, \qquad D_I(\alpha_2)g(\alpha_{12}) = -G_Ig(\alpha_{12})$$

The invariant differentials can also be used to define a group-invariant ("Haar") measure: The wedge product of all the differentials $d\alpha^M R_M{}^i$ (i ranges over the coset) is not only invariant under the symmetry group, but also under the gauge group, since the determinant of the gauge group element is 1 for the coset representation (even for GL(1), if we use the exponential parametrization).

Exercise

Evaluate all the above (L, R, α_{12}) for the coset U(1)/I.

Wave functions and spin

To define a Hilbert space for wave functions, we begin with a vacuum state defined to be invariant under the gauge group:

$$H_\iota|0\rangle = \langle 0|H_\iota = 0$$

(For some purposes, we can think of the gauge generators as "lowering operators". In general, we don't need a Hilbert space for this construction, but only a vector space; the bras then form the dual space to the kets, as described in subsection IB1.) A coordinate basis for the coset can then be defined as

$$|\alpha\rangle = g(\alpha)|0\rangle, \qquad \langle\alpha| = \langle 0|g^{-1}(\alpha)$$

(where $g(0) = I$) and thus invariant under a gauge transformation

$$g'|0\rangle \equiv gh|0\rangle = g|0\rangle$$

The wave function is then defined with respect to this basis as

$$\psi(\alpha) \equiv \langle\alpha|\psi\rangle = \langle 0|g^{-1}(\alpha)|\psi\rangle$$

from which it follows that its covariant derivative with respect to the gauge group vanishes:

$$-D_\iota\psi(\alpha) = \langle 0|H_\iota g^{-1}(\alpha)|\psi\rangle = 0$$

On the other (right) hand, the symmetry generators act in the expected way:

$$-\hat{G}_I\psi(\alpha) = \langle 0|g^{-1}(\alpha)G_I|\psi\rangle = (G_I\psi)(\alpha)$$

So far we have analyzed only coordinate representations. But usually in quantum mechanics we want to consider more general representations by adding "spin" to such "orbital" generators. This is accomplished by first introducing spin degrees of freedom, and then tying them to the group by modifying the gauge-group constraints. So we first introduce a basis $|^A\rangle$ (and its dual $\langle_A|$) for a matrix representation $\tilde{H}_\iota$ for the gauge group,

$$\langle_A|H_\iota = \tilde{H}_{\iota A}{}^B\langle_B|, \qquad H_\iota|^A\rangle = |^B\rangle\tilde{H}_{\iota B}{}^A$$

then define a basis for the Hilbert space by using this gauge group basis as our new (degenerate) vacuum,

$$|^A,\alpha\rangle \equiv g(\alpha)|^A\rangle$$

to get the generalizations of the previous

$$\psi_A(\alpha) \equiv \langle {}_A, \alpha|\psi\rangle \quad \Rightarrow \quad -D_\iota \psi_A(\alpha) = \tilde{H}_{\iota A}{}^B \psi_B(\alpha), \qquad -\hat{G}_I \psi_A(\alpha) = (G_I \psi)_A(\alpha)$$

The wavefunction now depends on the gauge-group coordinates, but this dependence is fixed independent of the state: For example, in the 2-exponential coordinate system

$$\psi_A(\alpha) = \langle {}_A|e^{-i\alpha^\iota H_\iota} e^{-i\alpha^i T_i}|\psi\rangle = (e^{-i\alpha^\iota \tilde{H}_\iota})_A{}^M \langle {}_M|e^{-i\alpha^i T_i}|\psi\rangle \equiv e_A{}^M(\alpha^\iota)\psi_M(\alpha^i)$$

where $e_A{}^M$ is a "vielbein" depending on only the gauge coordinates, and can be gauged to the identity, while ψ_M depends on only the coset coordinates. Since we know D in terms of derivatives, $D_\iota = -\tilde{H}_\iota$ can be solved to replace partial derivatives with respect to gauge-group coordinates with matrices, in both D_I and $\hat{G}_I$. We'll see applications of this to the conformal group (and thus also the Poincaré group) later.

The commutation relations of the surviving covariant derivatives

$$[D_i, D_j\} = f_{ij}{}^k D_k + f_{ij}{}^\kappa D_\kappa$$

then identify $f_{ij}{}^k$ as the "torsion", while $f_{ij}{}^\kappa$ is the "curvature".