Lightcone algebra

Bosonic string

The Lorentz algebra in the lightcone formalism is the analog of the BRST algebra in the covariant formalism. The calculations are very similar, as having 2 "anticommuting directions" (for vector indices; subsection XIIB8) to give $X \to C, B$ and $\Psi \to \gamma, \beta$. In particular, 2D "field theory" and current algebra methods are again the most convenient. However, the lightcone gauge fixes conformal invariance: As a result, transforming from the cylinder to the plane introduces explicit z dependence.

For convenience, we work with the closed string; the open string is obtained by the usual identification between left and right. The only quantum (nonclassical) lightcone Lorentz algebra comes from that of 2 currents cubic in oscillators, J^{i-}. Before Wick rotation and transformation to the complex plane, these generators are (see subsection XIB1; for the hermitian form):

$$J^{i-} = \frac{1}{\alpha'}\oint \frac{d\sigma}{2\pi} X^{[i}\dot{X}^{-]}$$

Separating into left and right modes, and using the corresponding linear equations of motion,

$$X = x + \alpha' p\tau + \sqrt{\frac{\alpha'}{2}}(Y_{(+)} + Y_{(-)}) \quad \Rightarrow \quad J^{i-} = x^{[i}p^{-]} + S^{i-}$$

$$S^{i-} = \frac{1}{2}\oint (Y_{(+)} + Y_{(-)})^{[i}(Y_{(+)} - Y_{(-)})'^{-]}$$

Integrating by parts to keep ∂ on Y^-,

$$S^{i-} = \frac{1}{2}\oint (Y_{(+)} + Y_{(-)})^{i}(Y_{(+)} - Y_{(-)})'^{-} + (Y_{(+)} - Y_{(-)})^{i}(Y_{(+)} + Y_{(-)})'^{-}$$

$$= \oint \pm Y^{i}_{(\pm)} Y'^{-}_{(\pm)} = \oint Y^{i}_{(\pm)} \dot{Y}^{-}_{(\pm)}$$

(summed over $\pm$). Only the oscillators Y contribute to the "loop" correction. We could have done the same for the zero-modes, except that $X_{(\pm)}$ are not periodic:

$$X = X_{(+)} + X_{(-)}\ , \qquad X_{(\pm)} = \frac{1}{2}x + \frac{\alpha'}{2}p(\tau \pm \sigma) + \sqrt{\frac{\alpha'}{2}}Y_{(\pm)}$$

Then

$$J^{i-} = -x^- p^i + \oint \left(\frac{2}{\alpha'} X^i_{(\pm)} \mp p^i \sigma \right) \dot{X}^-_{(\pm)}$$

where the x^- term is required because X^- appears only with a derivative. In the analogous BRST case, there is no σ term in $C_{(\pm)}$ to cancel.

In the Wick-rotated 2D Euclidean complex plane $z = e^{\tau + i\sigma}$, the action, propagators (subsection VIIB5) and energy-momentum tensor (subsection XIB4) for real fields are:

$$S = \int \frac{d^2\sigma}{2\pi} [(\partial\varphi)(\bar{\partial}\varphi) + \psi_+ \bar{\partial}\psi_+ + \psi_- \partial\psi_-]$$

$$T_+ = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}\psi_+ \partial\psi_+, \qquad T_- = -\frac{1}{2}(\bar{\partial}\varphi)^2 - \frac{1}{2}\psi_- \bar{\partial}\psi_-$$

$$\langle \varphi \ \varphi \rangle = -\ln |z|^2, \qquad \langle \psi_+ \ \psi_+ \rangle = \frac{1}{z}, \quad \langle \psi_- \ \psi_- \rangle = \frac{1}{\bar{z}}$$

where $X = \sqrt{\alpha'/2}\, \varphi$. The zero-mode terms in X are

$$X = x - i\frac{\alpha'}{2} p \ln |z|^2 + \ldots \qquad \Rightarrow \qquad p = \frac{2}{\alpha'} \oint \frac{dz}{2\pi i} i\partial X$$

(As usual, we extend the open string into the lower-half complex plane to work with only ∂ and not $\bar{\partial}$. The $i\partial$ is from Wick rotation $\tau \to -i\tau$. We'll set $\alpha' = 2$ for the rest of these notes, to normalize X like a standard boson.)

We then plug in the gauge condition and solution to the Virasoro constraints,

$$i\partial X^- = \frac{(i\partial X^i)^2}{2i\partial X^+}, \qquad i\partial X^+ = \frac{p^+}{z}$$

(and similar for $\bar{\partial}$) to get the final expression for the Lorentz generators

$$J^{i-} = x^i \frac{1}{2p^+}\frac{a}{\alpha'} - x^- p^i + \frac{1}{2p^+}\oint (X^i_{(\pm)} + ip^i \ln z) z (i\partial X^j_{(\pm)})^2$$

($\bar{z}$ and $\bar{\partial}$ for $X_{(-)}$ are understood.) We added a normal-ordering constant as $(p^i)^2 \to (p^i)^2 + a/\alpha'$; it was prohibited in the conformal gauge by conformal invariance, and since the conformal vacuum is necessarily massless. It's a quantum correction whose trees contribute at 1 loop. For all of the loop calculation we can deal separately with left and right-handed modes. (We then drop the $(\pm)$.)

Looking at just "loop" terms, we get 3 types, from double contractions between $X(\partial X)^2$ terms of

$$\langle(\partial X)^2\ (\partial X)^2\rangle \qquad \langle(\partial X)^2\ X\partial X\rangle \qquad \langle X\partial X\ X\partial X\rangle$$

Ultimately the result for $[J^{i-}, J^{j-}]$ must come out antisymmetric in $[ij]$, but it saves steps to antisymmetrize by hand: In particular, we can ignore direct contraction of $\langle X^i\ X^j\rangle$ in the last term:

$$\langle X^i\ X^j\rangle = -\delta^{ij}\ln(z - z')$$

(for $X_{(+)}$, with $\bar{z}$ for $(-)$). The 3 terms then give operator products of the 2 currents proportional to $1/(z - z')$ to powers 4,3,2 respectively. Integration then gives terms $-1/4(p^+)^2$ times

$$\oint (zX^i)\partial^3(zX^j) \quad and \quad \oint X^i\partial X^j$$

(the latter of which will be canceled by a trees) after using the rule

$$A(z)B(z') \approx a(z)b(z')\frac{1}{(z - z')^{n+1}} \quad \Rightarrow \quad [\oint A, \oint B\} \approx \frac{(-1)^n}{n!}\oint ab^{(n)}$$

Carefully keeping track of all signs, the $1/n!$'s, and the various permutations (2,8,4, respectively), the result for the ∂^3 terms comes out proportional to, adding the 3 types of contractions,

$$\frac{D-2}{3} - 4 - 4 = \frac{D-26}{3}$$

(The $D-2$ comes from summing $\delta_{ij}\delta_{ij}$ over transverse modes.) The generated term is not part of the algebra, so it must die. This implies $D=26$. The result for the ∂ terms is

$$0 - 4 - 4 - 2a$$

where the a contribution comes from the ordinary commutator $[x^i, p^j]$ between the ax term and the p in ∂X. Thus

$$a = -4$$

Exercise 1: Consider replacing X with just its oscillator part Y, with propagators sans zero-modes

$$\langle Y\ Y\rangle = -\ln(z-z') + \ln z + \ln z'$$

Show by explicit calculation that the extra terms don't contribute to these loops.

Strings with fermions

The bosonic particle describes a scalar, so its Lorentz generators have only orbital pieces. In the lightcone gauge we can set $x^+ = 0$ (at $\tau = 0$), and solve the Klein-Gordon equation for p^-.

For a relativistic quantum mechanical system with spin, there is also a spin piece,

$$J^{ab} = x^{[a}p^{b]} + S^{ab}$$

But relativistic wave functions/fields satisfy more than just the KG equation: It can be summarized (for field strengths) as (for the massless case)

$$S^{ab}p_b = 0$$

(There's also a "normal-ordering term" $\sim p^a$, which we'll neglect, and can be transformed away in the lightcone formalism. See subsections IIB1-4.) This constraint has the lightcone gauge and solution

$$S^{i+} = S^{+-} = 0, \qquad S^{i-} = \frac{1}{p^+} S^{ij} p_j$$

Alternatively, we can find S^{i-} from the simple generators by closure of the algebra:

$$i[J^{i+}, J^{j-}] = J^{ij} - \delta^{ij} J^{+-}$$

Exercise 2: Show the above choices for longitudinal components of spin satisfy this commutation relation.

Both spinning and super strings are generalizations of this to the worldsheet: Their contributions to the spin take the generic form

$$\Delta S^{ab} = \oint \hat{S}^{ab}, \qquad \hat{S}^{ab} = \tfrac{1}{2} F^T s^{ab} F$$

$$i[s_{ab}, s^{cd}] = \delta_{[b}^{[c} s_{a]}{}^{d]}$$

for some real, self-conjugate fermionic worldsheet field F, where s^{ab} is its matrix representation of the Lorentz group. Specifically, for these strings we have

$$is^{ab} = \begin{cases} |^{[a}\rangle\langle^{b]}| & (RNS) \\ -\frac{1}{4}\gamma^{[a}\gamma^{b]} & (GS) \end{cases}$$

The analog of the Sp constraint then comes from

$$\Psi^a[\Psi \cdot (\partial X)] \quad or \quad \Theta\gamma^a[\gamma \cdot PD] \;\sim \hat{S}^{ab} i\partial X_b$$

(after subtracting a singular ∂X^a term). Using gauge symmetry generated by the super-Virasoro or κ-symmetry constraint, respectively, we then find

$$\Delta S^{i-} = \frac{1}{p^+}\oint z\hat{S}^{ij} i\partial X_j \;, \qquad \Delta S^{i+} = \Delta S^{+-} = 0$$

with $\hat{S}^{ij}$ as above, but for the reduced fermions F of the lightcone. This includes the term $\Delta S^{ij}p_j/p^+$. The other modification to J^{i-} is in ∂X^-, the transverse part of Virasoro:

$$(\partial X^j)^2 \rightarrow (\partial X^j)^2 + F^T\partial F$$

The final Lorentz generators are then

$$J^{i-} = x^i \frac{1}{2p^+} \frac{a}{\alpha'} - x^- p^i$$
$$+\frac{1}{2p^+} \sum_\pm \oint z\{ -(X^i + ip^i \ln z)[(\partial X^j)^2 + F^T \partial F] + F^T s^{ij} F i \partial X_j\}$$

Exercise 3: Verify the part of the Lorentz algebra in Exercise 2 for these generators.

The contribution of $XX\langle F\partial F \ F\partial F\rangle$ to the closure of the algebra is simple. (Compare to the first of the 3 contraction terms of the bosonic case).

Exercise 4: Show that the result of the crossterm $X(\partial X)\langle F\partial F \ FsF\rangle$ vanishes. (Hint: What happens to the matrix indices on s?)

But for the $(\partial X)(\partial X)\langle FsF \ FsF\rangle$ contribution, we'll need to evaluate a double contraction for 2 $\hat{S}^{ij}$'s. The result is easily found to be

$$\langle \hat{S}^{ij} \hat{S}^{kl} \rangle = \frac{1}{2} \frac{1}{z^2} tr\,(s^{ij} s^{kl})$$

The Dynkin index c of the SO($D-2$) representation is (see subsection VIIIA3):

$$-tr\,(s^{ij}s^{kl}) = c\delta^{j[k}\delta^{l]i}, \qquad c = \begin{cases} 2 & (vector) \\ D'/4 & (spinor) \end{cases}$$

where the vector representation is for the RNS spinning string, and the spinor (dimension D') is for the GS superstring. For the superstring, defined classically for D = 3,4,6,10, we have $D' = D-2$. The modification to the coefficient in the bosonic string for the ∂^3 term is then

$$\frac{D-2}{3} - 4 - 4 + \frac{D-2}{6} + 2c = \begin{cases} \frac{D-10}{2} & (RNS) \\ D-10 & (GS) \end{cases}$$

So in either formalism we find $D = 10$. The modification for the ∂ term is

$$0 - 4 - 4 + 0 + 2c - 2a = \begin{cases} -4 - 2a & (RNS) \\ -4 - 2a + \frac{D-10}{2} & (GS) \end{cases}$$

so $a = -2$, half the result for the bosonic string.

Finally, besides the XX terms also found in the bosonic string, we get FF terms from 1 contraction of X's and 1 of F's. Specifically, we get a term of the form $-1/4(p^+)^2$ times

$$-i\oint(zF)s^{ij}\partial^2(zF)$$

This is required by worldsheet/spacetime supersymmetry to accompany the similar XX term; it's like spin but with an extra $(z\partial)^2$. Between the $XF\partial F$ and $FsF\partial X$ terms, we get 1 contribution from the latter with itself, and 1 from the crossterms. The coefficients are

$$(D-4)-6$$

again requiring $D = 10$. (The former comes from tracing δ's from $[s, s]$.) We also get a spin term $-i\frac{1}{2}FsF$ to go with the $X\partial X$ term, with coefficient

$$0-4-2a$$