Lightcone algebra

Bosonic string

The Lorentz algebra in the lightcone formalism is the analog of the BRST algebra in the covariant formalism. The calculations are very similar, as having 2 "anticommuting directions" (for vector indices; subsection XIIIB8) to give $X \rightarrow C, B$ and $\Psi \rightarrow \gamma, \beta$. In particular, 2D "field theory" and current algebra methods are again the most convenient. However, the lightcone gauge fixes conformal invariance: As a result, transforming from the cylinder to the plane introduces explicit $z$ dependence.

For convenience, we work with the closed string; the open string is obtained by the usual identification between left and right. The only quantum (nonclassical) lightcone Lorentz algebra comes from that of 2 currents cubic in oscillators, $J^{i-}$. Before Wick rotation and transformation to the complex plane, these generators are (see subsection XIB1; for the hermitian form):
\[ J^{i-} = \frac{1}{\alpha'} \oint \frac{d\sigma}{2\pi} X[i\dot{X}^-] \]

Separating into left and right modes, and using the corresponding linear equations of motion,

\[ X = x + \alpha' p \tau + \sqrt{\frac{\alpha'}{2}} (Y_+(\tau) + Y_-(\tau)) \implies J^{i-} = x[i\dot{p}^-] + S^{i-} \]

\[ S^{i-} = \frac{1}{2} \oint (Y_+(\tau) + Y_-(\tau))[i(Y_+(\tau) - Y_-(\tau))'] \]

Integrating by parts to keep \( \partial \) on \( Y^- \),

\[ S^{i-} = \frac{1}{2} \oint (Y_+(\tau) + Y_-(\tau))i(Y_+(\tau) - Y_-(\tau))' - + (Y_+(\tau) - Y_-(\tau))i(Y_+(\tau) + Y_-(\tau))' \]

\[ = \oint \pm Y^i_{(\pm)} Y'_{(\pm)} = \oint Y^i_{(\pm)} \dot{Y}_{(\pm)}^- \]

(summed over \( \pm \)). Only the oscillators \( Y \) contribute to the "loop" correction. We could have done the same for the zero-modes, except that \( X_{(\pm)} \) are not periodic:

\[ X = X_{(+)} + X_{(-)} \ , \quad X_{(\pm)} = \frac{1}{2} x + \frac{\alpha'}{2} p(\tau \pm \sigma) + \sqrt{\frac{\alpha'}{2}} Y_{(\pm)} \]
Then

\[ J_i^- = -x^- p_i^+ + \oint \left( \frac{2}{\alpha'} X_i^{(\pm)} + p^i \sigma \right) \dot{X}^{(\pm)} \]

where the \( x^- \) term is required because \( X^- \) appears only with a derivative. In the analogous BRST case, there is no \( \sigma \) term in \( C^{(\pm)} \) to cancel.

In the Wick-rotated 2D Euclidean complex plane \( z = e^{i \tau + i \sigma} \), the action, propagators (subsection VII B5) and energy-momentum tensor (subsection XIB4) for real fields are:

\[ S = \int \frac{d^2 \sigma}{2\pi} \left[ (\partial \phi)(\bar{\partial} \phi) + \psi_+ \bar{\partial} \psi_+ + \psi_- \bar{\partial} \psi_- \right] \]

\[ T_+ = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} \psi_+ \bar{\partial} \psi_+ \quad \quad \quad \quad \quad T_- = -\frac{1}{2} (\bar{\partial} \phi)^2 - \frac{1}{2} \psi_- \bar{\partial} \psi_- \]

\[ \langle \phi \; \phi \rangle = -\ln |z|^2 \quad \quad \quad \langle \psi_+ \; \psi_+ \rangle = \frac{1}{z} \quad \quad \langle \psi_- \; \psi_- \rangle = \frac{1}{\bar{z}} \]

where \( X = \sqrt{\alpha'/2} \phi \). The zero-mode terms in \( X \) are

\[ X = x - i \frac{\alpha'}{2} p \ln |z|^2 + \ldots \quad \Rightarrow \quad p = \frac{2}{\alpha'} \oint \frac{dz}{2\pi i} i \partial X \]
(As usual, we extend the open string into the lower-half complex plane to work with only $\partial$ and not $\bar{\partial}$. The $i\partial$ is from Wick rotation $\tau \to -i\tau$. We’ll set $\alpha' = 2$ for the rest of these notes, to normalize $X$ like a standard boson.)

We then plug in the gauge condition and solution to the Virasoro constraints,

$$i\partial X^- = \frac{(i\partial X^i)^2}{2i\partial X^+}, \quad i\partial X^+ = \frac{p^+}{z}$$

(and similar for $\bar{\partial}$) to get the final expression for the Lorentz generators

$$J^i^- = x^i \frac{1}{2p^+} a - x^- p^i + \frac{1}{2p^+} \oint (X^i_{(\pm)} + ip^i \ln z) z (i\partial X^i_{(\pm)})^2$$

($\bar{z}$ and $\bar{\partial}$ for $X_{(-)}$ are understood.) We added a normal-ordering constant as

$$(p^i)^2 \to (p^i)^2 + a/\alpha';$$

it was prohibited in the conformal gauge by conformal invariance, and since the conformal vacuum is necessarily massless. It’s a quantum correction whose trees contribute at 1 loop. For all of the loop calculation we can deal separately with left and right-handed modes. (We then drop the (±).)
Looking at just "loop" terms, we get 3 types, from double contractions between $X(dX)^2$ terms of

$$\langle(dX)^2 (dX)^2\rangle \quad \langle(dX)^2 XdX\rangle \quad \langle XdX XdX\rangle$$

Ultimately the result for $[J^i^-, J^j^-]$ must come out antisymmetric in $[ij]$, but it saves steps to antisymmetrize by hand: In particular, we can ignore direct contraction of $\langle X^i X^j \rangle$ in the last term:

$$\langle X^i X^j \rangle = - \delta^{ij} \ln(z - z')$$

(for $X_{(+)}$, with $\bar{z}$ for $(−)$). The 3 terms then give operator products of the 2 currents proportional to $1/(z - z')$ to powers 4,3,2 respectively. Integration then gives terms $-1/4(p^+)^2$ times

$$\oint (zX^i) \partial^3(zX^j) \quad \text{and} \quad \oint X^i \partial X^j$$

(the latter of which will be canceled by $a$ trees) after using the rule

$$A(z)B(z') \approx a(z)b(z') \frac{1}{(z - z')^{n+1}} \quad \Rightarrow \quad [\oint A, \oint B] \approx \frac{(-1)^n}{n!} \oint ab^{(n)}$$
Carefully keeping track of all signs, the $1/n!$’s, and the various permutations (2,8,4, respectively), the result for the $\partial^3$ terms comes out proportional to, adding the 3 types of contractions,

$$\frac{D - 2}{3} - 4 - 4 = \frac{D - 26}{3}$$

(The $D - 2$ comes from summing $\delta_{ij}\delta_{ij}$ over transverse modes.) The generated term is not part of the algebra, so it must die. This implies $D = 26$. The result for the $\partial$ terms is

$$0 - 4 - 4 - 2a$$

where the $a$ contribution comes from the ordinary commutator $[x^i, p^j]$ between the $ax$ term and the $p$ in $\partial X$. Thus

$$a = -4$$

**Exercise 1:** Consider replacing $X$ with just its oscillator part $Y$, with propagators sans zero-modes

$$\langle Y Y \rangle = -\ln(z - z') + \ln z + \ln z'$$

Show by explicit calculation that the extra terms don’t contribute to these loops.
Strings with fermions

The bosonic particle describes a scalar, so its Lorentz generators have only orbital pieces. In the lightcone gauge we can set $x^+ = 0$ (at $\tau = 0$), and solve the Klein-Gordon equation for $p^-$. 

For a relativistic quantum mechanical system with spin, there is also a spin piece, 

$$J^{ab} = x^{[a} p^{b]} + S^{ab}$$

But relativistic wave functions/fields satisfy more than just the KG equation: It can be summarized (for field strengths) as (for the massless case) 

$$S^{ab} p_b = 0$$

(There’s also a "normal-ordering term" $\sim p^a$, which we’ll neglect, and can be transformed away in the lightcone formalism. See subsections IIB1-4.) This constraint has the lightcone gauge and solution
\[ S^{i+} = S^{+-} = 0, \quad S^{i-} = \frac{1}{p^+} S^{ij} p_j \]

Alternatively, we can find \( S^{i-} \) from the simple generators by closure of the algebra:

\[ i [ J^{i+}, J^{j-} ] = J^{ij} - \delta^{ij} J^{+-} \]

**Exercise 2:** Show the above choices for longitudinal components of spin satisfy this commutation relation.

Both spinning and super strings are generalizations of this to the worldsheet: Their contributions to the spin take the generic form

\[ \Delta S^{ab} = \oint S^{ab}, \quad \hat{S}^{ab} = \frac{1}{2} F^T s^{ab} F \]

\[ i [ s_{ab}, s^{cd} ] = \delta^{[c} s_{a]} d] \]

for some real, self-conjugate fermionic worldsheet field \( F \), where \( s^{ab} \) is its matrix representation of the Lorentz group. Specifically, for these strings we have
\[ i S^{ab} = \begin{cases} 
|^{[a}\langle b]| & (RNS) \\
-\frac{1}{4} \gamma^{[a} \gamma^{b]} & (GS) 
\end{cases} \]

The analog of the \( Sp \) constraint then comes from
\[ \Psi^a[\Psi \cdot (\partial X)] \quad \text{or} \quad \Theta \gamma^a[\gamma \cdot PD] \sim \hat{S}^{ab} i \partial X_b \]
(after subtracting a singular \( \partial X^a \) term). Using gauge symmetry generated by the super-Virasoro or \( \kappa \)-symmetry constraint, respectively, we then find
\[ \Delta S^{i-} = \frac{1}{p^+} \oint z \hat{S}^{ij} i \partial X_j, \quad \Delta S^{i+} = \Delta S^{+-} = 0 \]
with \( \hat{S}^{ij} \) as above, but for the reduced fermions \( F \) of the lightcone. This includes the term \( \Delta S^{ij} p_j / p^+ \). The other modification to \( J^{i-} \) is in \( \partial X^- \), the transverse part of Virasoro:
\[ (\partial X^j)^2 \rightarrow (\partial X^j)^2 + F^T \partial F \]
The final Lorentz generators are then
\[ J^{i-} = x^i \frac{a}{2p^+} - x^- p^i \]
\[ + \frac{1}{2p^+} \sum_\pm \oint z \left\{ - (X^i + ip^i \ln z) [(\partial X^j)^2 + F^T \partial F] + F^T s^{ij} F_i \partial X_j \right\} \]

**Exercise 3:** Verify the part of the Lorentz algebra in Exercise 2 for these generators.

The contribution of \( XX \langle F \partial F \ F \partial F \rangle \) to the closure of the algebra is simple. (Compare to the first of the 3 contraction terms of the bosonic case).

**Exercise 4:** Show that the result of the crossterm \( X(\partial X) \langle F \partial F \ F s F \rangle \) vanishes. (Hint: What happens to the matrix indices on \( s \)?)

But for the \((\partial X)(\partial X) \langle F s F \ F s F \rangle \) contribution, we’ll need to evaluate a double contraction for 2 \( \hat{S}^{ij} \)'s. The result is easily found to be

\[ \langle \hat{S}^{ij} \hat{S}^{kl} \rangle = \frac{1}{2} \frac{1}{z^2} tr \left( s^{ij} s^{kl} \right) \]
The Dynkin index $c$ of the $\text{SO}(D - 2)$ representation is (see subsection VIII A3):

$$- \text{tr} (s^{ij} s^{kl}) = c \delta^{i[k} \delta^{l]}i,$$

$$c = \begin{cases} 
2 & \text{(vector)} \\
\frac{D'}{4} & \text{(spinor)} 
\end{cases}$$

where the vector representation is for the RNS spinning string, and the spinor (dimension $D'$) is for the GS superstring. For the superstring, defined classically for $D = 3, 4, 6, 10$, we have $D' = D - 2$. The modification to the coefficient in the bosonic string for the $\partial^3$ term is then

$$\frac{D - 2}{3} - 4 - 4 + \frac{D - 2}{6} + 2c = \begin{cases} 
\frac{D - 10}{2} & \text{(RNS)} \\
D - 10 & \text{(GS)} 
\end{cases}$$

So in either formalism we find $D = 10$. The modification for the $\partial$ term is

$$0 - 4 - 4 + 0 + 2c - 2a = \begin{cases} 
-4 - 2a & \text{(RNS)} \\
-4 - 2a + \frac{D - 10}{2} & \text{(GS)} 
\end{cases}$$
so $a = -2$, half the result for the bosonic string.

Finally, besides the $XX$ terms also found in the bosonic string, we get $FF$ terms from 1 contraction of $X$’s and 1 of $F$’s. Specifically, we get a term of the form $-1/4(p^+)^2$ times

$$-i \oint (zF) s^{ij} \partial^2(zF)$$

This is required by worldsheet/spacetime supersymmetry to accompany the similar $XX$ term; it’s like spin but with an extra $(z\partial)^2$. Between the $XF\partial F$ and $FsF\partial X$ terms, we get 1 contribution from the latter with itself, and 1 from the crossterms. The coefficients are

$$(D - 4) - 6$$

again requiring $D = 10$. (The former comes from tracing $\delta$’s from $[s, s]$.) We also get a spin term $-i \frac{1}{2} FsF$ to go with the $X\partial X$ term, with coefficient

$$0 - 4 - 2a$$