Vertex operators

Open-string operator products

For later calculations, we'll need a short list of operator products. But first we need to emphasize some differences and similarities for open and closed strings. For the closed string we have left- and right-handed modes $\widehat{X}_L$ and $\widehat{X}_R$, while for the open string $\widehat{X}_L = \widehat{X}_R$:

$$X(z,\bar{z}) = \begin{cases} \sqrt{\frac{\alpha'}{2}}[\widehat{X}_L(z) + \widehat{X}_R(\bar{z})] & \text{for closed} \\ \sqrt{\frac{\alpha'}{2}}[\widehat{X}(z) + \widehat{X}(\bar{z})] & \text{for open} \end{cases}$$

All these $\widehat{X}$'s have conveniently normalized propagators

$$\langle \widehat{X}(z)\ \widehat{X}(z')\rangle = \langle \widehat{X}_L(z)\ \widehat{X}_L(z')\rangle = \langle \widehat{X}_R(z)\ \widehat{X}_R(z')\rangle = -ln(z-z')$$

from which follows directly those for X itself:

$$\frac{2}{\alpha'}\langle X(z,\bar{z})\ X(z',\bar{z}')\rangle = \begin{cases} -ln(|z-z'|^2) & \text{for closed} \\ -ln(|z-z'|^2) - ln(|z-\bar{z}'|^2) & \text{for open} \end{cases}$$

where the open string has extra contributions from crossterms, now involving the same $\widehat{X}$.

For open-string amplitudes involving only open-string external states, all the vertex operators will be on the boundary,

$$z = \bar{z} \quad \Rightarrow \quad X(z,\bar{z}) = \sqrt{2\alpha'}\widehat{X}(z)$$

Therefore, when Fourier transforming wave functions we use the exponentials

$$e^{ik\cdot X(z,\bar{z})} = \begin{cases} e^{i\hat{k}\cdot\widehat{X}(z)} & \text{for open} \\ e^{i\hat{k}\cdot\widehat{X}_L(z)}e^{i\hat{k}\cdot\widehat{X}_R(\bar{z})} & \text{for closed} \end{cases} \quad \Rightarrow \quad \hat{k} = k \times \begin{cases} \sqrt{2\alpha'} & \text{for open} \\ \sqrt{\frac{\alpha'}{2}} & \text{for closed} \end{cases}$$

Closed-string vertex operators are the product of left- and right-handed ones, which are functions of z and $\bar{z}$, respectively, and thus take the form of the product of 2 independent open-string vertex operators.

Working directly in terms of $\widehat{X}$, we then have the "operator products"

$$(i\partial\widehat{X})(z')\ e^{i\hat{k}\cdot\widehat{X}(z)} \approx \hat{k}\ \frac{1}{z'-z}\ e^{i\hat{k}\cdot\widehat{X}(z)}$$

$$or \quad (\partial\widehat{X})(z')\ f(X(z,\bar{z})) \approx -\ \frac{1}{z'-z}\ (\partial f)(X(z,\bar{z})) \times \begin{cases} \sqrt{2\alpha'} & \text{for open} \\ \sqrt{\frac{\alpha'}{2}} & \text{for closed} \end{cases}$$

$$(i\partial\widehat{X})(z')\ (i\partial\widehat{X})(z) \approx \frac{1}{(z'-z)^2}$$

(Note the context: $\partial\widehat{X}$ is a z derivative, ∂f is an x derivative. The "i" associated with $\partial\widehat{X}$ is from Wick rotation.)

For example, we can use these results to determine the proper normalization of massless vertex operators, by comparison with that of tachyons: For the tachyon,

$$W_{\hat{k}}(z) = e^{i\hat{k}\cdot\widehat{X}(z)}, \quad \hat{k}^2 = 2 \quad \Rightarrow \quad W_{\hat{k}}(z')\ W_{-\hat{k}}(z) \approx \frac{1}{(z'-z)^2} e^{i\hat{k}\cdot[\widehat{X}(z')-\widehat{X}(z)]}$$

(For the closed string, we have the product of left and right versions of the above. Note this correctly gives $k^2 = 1/\alpha'$ for the open string tachyon and $4/\alpha'$ for the closed, where α' is the slope of the open-string Regge trajectory, and the parameter that appears in the action that describes both open- and closed-string states.) The z factors are canceled in string field theory by considering the gauge-fixed kinetic term $\langle 0|V(c_0\Box)V|0\rangle$, where $V = cW$.

Gauge-independent vertex operators

When ghosts are included, vertex operators can be generalized to arbitrary gauges for the external gauge fields. (This result follows from the same method applied to relate integrated and unintegrated vertices in subsection XIIB8 of *Fields*. We'll do a better job of that here.) The main point is the existence of integrated and unintegrated vertex operators: Integrated ones are natural from adding backgrounds to the gauge-invariant action; unintegrated ones from adding backgrounds to the BRST operator. We'll relate the two by going in both directions. The following discussion will be for general quantum mechanics (except in the relativistic case we use τ in place of t), but we'll add some special comments for open strings at the end.

The action can be written as

$$S \sim \int d\tau\, H_I$$

plus the usual terms for converting Hamiltonian to (first-order) Lagrangian, where the interacting Hamiltonian consists of the free part plus linearized vertex

$$H_I = H_0 + W$$

BRST invariance with respect to the free BRST operator then implies

$$\begin{aligned} & [Q_0, S] \approx 0 \\ \Rightarrow \quad & [Q_0, \textstyle\int d\tau\, W] \approx 0 \\ \Rightarrow \quad & [Q_0, W] \approx \partial_\tau V \\ \Rightarrow \quad & \{Q_0, V\} \approx 0 \end{aligned}$$

for some V, where "$\approx$" means "at the linearized level". The BRST invariants $\int W$ and V are thus our integrated and unintegrated vertex operators, respectively.

Going in the other direction, we start with interacting BRST

$$Q_I = Q_0 + V$$

where fully interacting BRST invariance implies at the linearized level

$$Q_I^2 = 0 \quad \Rightarrow \quad \{Q_0, V\} \approx 0$$

The full gauge-fixed action is then defined (in relativistic quantum mechanics, or otherwise in the ZJBV formalism) by

$$H_I = \{Q_I, b\} \approx H_0 + W$$

It then follows that

$$0 = [Q_I, H_I] \approx [Q_0, H_0] + ([Q_0, W] + [V, H_0])$$

which agrees with the above, since H_0 gives the (free) time development:

$$[H_0, V] = \partial_\tau V$$

The only modifications for the open string are eliminating σ dependence:

$$Q_0 = \int \frac{d\sigma}{2\pi} J, \qquad H_0 = \int \frac{d\sigma}{2\pi} T, \qquad b \to \int \frac{d\sigma}{2\pi} b \quad (0 - mode)$$

$$V \to V|_{\sigma=0}, \qquad W \to W|_{\sigma=0}$$

After combining the left and right-handed modes into functions of just z over the whole plane, as usual, we can then replace σ and τ with z in our definitions in an appropriate way.

Vector vertex

The simplest case is the massless vector. The choice for integrated vertex was obvious from the gauge transformation of the external field:

$$W = \dot{X} \cdot A(X), \quad \delta A(x) = -\partial\lambda(x)$$

$$\Rightarrow \quad \int d\tau\ W = \int dX \cdot A(X), \quad \delta \int d\tau\ W = -\int d\lambda(X) = 0$$

As usual, the τ integral gets converted into a z integral over the boundary (real axis).

Besides this "background" gauge invariance, we also need the "quantum" BRST invariance. The unintegrated vertex V and the BRST invariance of $\int W$ then follow from the same calculation:

$$[Q,W] = \partial V \quad \Rightarrow \quad Q\int W = QV = 0$$

We use the BRST operator

$$Q = \int \frac{dz}{2\pi i}\, J, \quad J = cT + c(\partial c)b, \quad T = \tfrac{1}{2}(i\partial\widehat{X})^2, \quad [Q,W(z)] = \oint_z \frac{dz'}{2\pi i}\, J(z')\, W(z)$$

(For the open string, this is all of Q; the closed string has $Q = Q_L + Q_R$, with Q_L and Q_R given by the above, with "L" or "R" subscripts on everything. For now, we stick to the open string. There is a sign convention change from *Fields* for Q and T.)

For $T(z')W(z)$, we get "single-contraction" (tree/classical) terms from the singular part of either ∂X with W (one propagator), and nonsingular (ordinary) product of the other ∂X (no propagator). So we evaluate

$$(\partial\widehat{X}^a)(z')\ [(\partial\widehat{X})\cdot A(X)](z) \approx -\frac{1}{(z'-z)^2}\, A^a(X(z)) - \sqrt{2\alpha'}\,\frac{1}{z'-z}\,[(\partial\widehat{X})^b\partial^a A_b(X)](z)$$

We also get "double-contraction" (1-loop) terms from the singular part of the product of the second ∂X with the above:

$$(\partial\widehat{X})(z')\cdot(\text{right-hand side of above}) \approx$$
$$2\sqrt{2\alpha'}\frac{1}{(z'-z)^3}\ \partial\cdot A(X(z)) + 2\alpha'\frac{1}{(z'-z)^2}(\partial\widehat{X}\cdot\Box A)(z)$$

We then need to integrate, using

$$\oint_z \frac{dz'}{2\pi i}\ \frac{1}{(z'-z)^{n+1}}\ f(z') = \frac{1}{n!}\partial^n f(z)$$

Putting it all together,

$$W = (i\partial\widehat{X})\cdot A \quad \Rightarrow \quad [Q,W] = \partial V - \alpha'(i\partial c)(\partial\widehat{X}^a)\partial^b F_{ba}$$
$$V = c(i\partial\widehat{X})\cdot A - \sqrt{\frac{\alpha'}{2}}(i\partial c)\partial\cdot A$$

(We have repeatedly used the identity $\partial_z f(X) = (\partial X)\cdot\partial f = \sqrt{2\alpha'}(\partial\widehat{X})\cdot\partial f$.)

Thus BRST invariance of $\int W$ and V requires the background satisfy only the (free) gauge-covariant field equations $\partial^b F_{ba} = 0$. This was to be expected, since quantum BRST invariance of Yang-Mills in a Yang-Mills background requires the same in field theory. We also find an order α' correction to the vertex operator V: This can be explained by noting that, while $c\partial X$ creates a Yang-Mills state from the vacuum, ∂c creates its Nakanishi-Lautrup field plus $\partial\cdot A$, in a combination that vanishes by that field's equation of motion.

RNS trees

Here are the method and some simple examples for tree graphs in the Ramond-Neveu-Schwarz formalism. We'll restrict ourselves to just external vectors: Tachyons are easier, but irrelevant for the superstring; fermions are much easier in manifestly supersymmetric formalisms.

Zero-modes

Calculations are pretty straightforward, as for the bosonic string. However, while ghosts for the bosonic string were almost trivial, and could almost be ignored, ghosts in RNS affect the non-ghost factors of vertex operators. Such complications ("pictures") can be discussed without ghosts, but the ghosts make the existence of pictures clearer. As for the bosonic string, for tree amplitudes with external bosons, ghosts contribute only their zero-modes.

Oscillators

We first give a general definition of zero-modes for tree graphs using conformal field theory. To describe the ground-state wave function for a string, we begin by considering ground states in a more general context. Define a general set of variables in terms of coordinates and momenta:

$$[p^i, x^j\} = -i\eta^{ij}; \qquad x^{i\dagger} = x^i, \qquad p^{i\dagger} = p^i$$

$$I = \int \left(\prod \frac{dx^i}{\sqrt{2\pi}} \right) |x\rangle\langle x|$$

for some arbitrary indices i, j and arbitrary (constant) metric η^{ij}.

Then define general harmonic oscillators:

$$a^i = \tfrac{1}{\sqrt{2}}(x^i + ip^i), \qquad a^{i\dagger} = \tfrac{1}{\sqrt{2}}(x^i - ip^i)$$

$$\Rightarrow \quad [a^i, a^{j\dagger}\} = \eta^{ij}$$

The ground state has the usual Gaussian wave function:

$$a^i|0\rangle = 0, \qquad a^{i\dagger}|0\rangle \neq 0$$

$$\Rightarrow \quad \langle x|0\rangle = e^{-\eta_{ij}x^i x^j/2}, \qquad \langle 0|0\rangle = 1$$

But the string also has non-oscillator modes, i.e., zero-modes. A general set of zero-modes satisfies instead

$$p^i|0\rangle = 0$$
$$\Rightarrow \quad \langle x|0\rangle = 1, \qquad \left\langle 0 \left| \prod \delta(x^i) \right| 0 \right\rangle = 1$$

in terms of the same kind of coordinates and momenta as above (with some $\sqrt{2\pi}$'s defined into our δ functions), since the vacuum expectation value is now just an integral over the x's (without Gaussians).

Since the string has both oscillators and zero-modes, its vacuum will be the direct product of the above types of ground states. The type to which any mode belongs is determined by conformal field theory: We saw that for a general conformal field χ of weight w,

$$\begin{cases} \chi_n|0\rangle = \chi_{-n}|0\rangle = 0 & \text{for } -w < n < w \\ \chi_n|0\rangle = 0,\ \chi_{-n}|0\rangle \neq 0 & \text{for } n \geq |w| \end{cases}$$

where $\chi_{-n} = {\chi_n}^\dagger$. We also have the commutation relations

$$[\chi_m, \tilde{\chi}_n\} = \delta_{m+n,0}$$

up to some normalization, where $\tilde{\chi}$ has weight $1 - w$, and may or may not be linearly related to χ. The vacuum is then "saturated" by

$$\left\langle 0 \left| \prod_{\substack{w \leq n \leq -w \\ i}} \delta(\chi_n^i) \right| 0 \right\rangle = 1$$

Note that only fields with $w > 0$ have zero-mode "momenta", so only fields with $w \leq 0$ have zero-mode "coordinates".

In practice, vertex operators will be evaluated at specific coordinates z, not for specific mode numbers. So integration over the zero-modes of δ functions of those fields at different z's will give the Jacobian determinant for the change of the $1 - 2w$ variables

$$\chi_n \to \chi(z_i) = \sum_n \chi_{-n}(z_i)^{-w+n} \approx \chi_{-w} + \chi_{-w-1} z_i + ... + \chi_w (z_i)^{-2w}$$

Explicitly, we have

$$\langle \delta[\chi(z_{1-2w})] \cdots \delta[\chi(z_2)]\ \delta[\chi(z_1)] \rangle = \begin{cases} J & \text{for fermions} \\ |J|^{-1} & \text{for bosons} \end{cases}$$

where

$$J = det \begin{pmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_{1-2w} \\ \vdots & \vdots & \ddots & \vdots \\ (z_1)^{-2w} & (z_2)^{-2w} & \dots & (z_{1-2w})^{-2w} \end{pmatrix} = \prod_{i<j}^{1-2w} (z_j - z_i)$$

can be evaluated by subtracting columns and taking factors out, or noting its zeros and evaluating the coefficient of one (e.g., the diagonal) term. Note that we order the z's so $z_j - z_i > 0$ for $i < j$, so actually $|J| = J$. For fermions, this result was obtained by choosing the ordering

$$\langle \chi_w \ \chi_{w+1} \cdots \chi_{-w} \rangle = 1$$

remembering that a δ function of a fermion is equal to that fermion.

Now we specialize to the cases of interest, all the fields of the RNS string. Let's list just the zero-mode "coordinates", which have to go inside δ functions to give nonvanishing vacuum values: Since this involves only fields with nonpositive weights, we have just

$$\begin{aligned} w(X) = 0: &\quad x \\ w(c) = -1: &\quad c_{-1}, c_0, c_1 \\ w(\gamma) = -\tfrac{1}{2}: &\quad \gamma_{-1/2}, \gamma_{1/2} \end{aligned}$$

(For X, really only ∂X is a conformal field, so we picked out its zero-mode momentum p, and then identified its canonical conjugate x.) We then have

$$\langle 0|\delta^{10}(x) c_{-1} c_0 c_1 \delta(\gamma_{-1/2})\delta(\gamma_{1/2})|0\rangle = 1$$

In practice, only the ghost zero-modes need to be saturated this way; for x we use instead, from the $e^{ik\cdot X(z)}$'s in the vertex operators.

$$\int dx \ \prod_i e^{ik_i\cdot x} = \delta\left(\sum_i k_i\right)$$

Pictures

This vacuum might seem to be a funny state, so let's relate it to "physical" states. We already saw such a relation for the bosonic string: There the tachyon $|t\rangle$ was the physical ground state; treating it the same way as a particle, whose wave function was independent of the particle ghost c_0, we had

$$\langle t|c_0|t\rangle = 1$$

(From here on we ignore the $\delta(x)$'s.) We could then identify the conformal vacuum as the only constant in the BRST cohomology, a constant Yang-Mills ghost (corresponding to a global Yang-Mills symmetry transformation):

$$|0\rangle = b_{-1}|t\rangle \quad \Rightarrow \quad \langle 0|c_{-1}c_0c_1|0\rangle = 1$$

(As usual, the physical Yang-Mills fields were given by $a_{-1}|t\rangle$, while the Yang-Mills anti-ghost was given by $c_{-1}|t\rangle$.) This was applied by picking out the 3 c's, at different points on (the boundary of) the worldsheet, coming from the 3 unintegrated vertex factors, and evaluating the corresponding z dependence.

We can try the same construction for the NS (sector of the RNS) string, since this vacuum has a nice interpretation in terms of BRST, which is so important in terms of constructing physical vertex operators. The tachyon is now unphysical, at least for the superstring, but we still need it to find the Yang-Mills states (at least in RNS formulations of the superstring). The relevant state to examine is then again the Yang-Mills ghost

$$|1\rangle = \beta_{-1/2}|t\rangle \quad \Rightarrow \quad \langle 1|c_0\gamma_{-1/2}\gamma_{1/2}|1\rangle = 1$$

(Now the physical Yang-Mills fields are $\psi_{-1/2}|t\rangle$ and the Yang-Mills anti-ghost is $\gamma_{-1/2}|t\rangle$.)

To see how this vacuum $|1\rangle$ relates to the "superconformal vacuum" $|0\rangle$, we need to look at a simple picture changing operator. In general, any picture changing operator can be written as

$$\delta(f)\delta([Q,f\})$$

since Q commutes with the argument of the latter factor, but also with the former factor by virtue of the latter factor. In fact, this is simply BRST gauge fixing with nonminimal fields removed: Integration over the Nakanishi-Lautrup field in the action for the gauge $f=0$ gives the former factor, while integration over the anti-ghost gives the latter (like the Faddeev-Popov determinant, but with the ghost still attached). So saturating the zero-modes is just gauge fixing for the invariances corresponding to their absence. In string theory, we generally choose f to be a function of conformal fields at some position z: The value of z is irrelevant; changing it is simply a change in the choice of gauge, the difference is BRST trivial.

In this case, the new invariance for the RNS string is for the ghost γ. So we would like to fix this invariance by introducing 2 factors of

$$\delta(c)\delta(\{Q,c\}) \sim c\delta(\tfrac{1}{2}\gamma^2)$$

where we have used only the $\gamma^2 b$ term in Q because the c kills the contribution of the $c(\partial c)b$ term. (The 1/2 is a convenient normalization factor.) As a byproduct, this also fixes 2 of the 3 gauge invariances for c. (The 2 factors are usually introduced at 2 different points, although they can effectively be introduced at the same point by taking a limit, which introduces derivatives, along with canceling powers of the difference in position.)

We then note that

$$1 \sim \langle 0|ccc\delta(\gamma)\delta(\gamma)|0\rangle \sim [\langle 0|c\delta(\tfrac{1}{2}\gamma^2)]c\gamma\gamma[c\delta(\tfrac{1}{2}\gamma^2)|0\rangle]$$

This allows us to define a new BRST-invariant vacuum

$$|1\rangle = c\delta(\tfrac{1}{2}\gamma^2)|0\rangle \quad \Rightarrow \quad \langle 1|c\gamma\gamma|1\rangle \sim 1$$

which we can identify with the Yang-Mills ghost as above. (As usual, we take the limit of the operators going to $z = 0$ or ∞ for the initial and final vacuua.) As for the bosonic string, these 3 ghost factors will come from the 3 unintegrated vertex operators.

With the Yang-Mills vacuum $|1\rangle$, integrated and unintegrated vertex operators can be defined as potential terms for the gauge-fixed action and BRST operator, respectively, as for the bosonic string. (Other pictures for these operators can be found by applying various picture-changing operators.) As before, we have

$$Q \to Q + V, \qquad \textstyle\int T \to \int T + W$$

$$W = \left\{ \textstyle\int b, V \right\}, \qquad [Q, W] = \partial V$$

where V is the unintegrated vertex operator and $\int W$ is the integrated one. However, now $V \neq cW$, but has extra terms, just as Q has more terms than just cT.

An alternative to putting this picture changing into the states (vacua) is to put it into the vertex operators: In terms of the original vertex operator, which we temporarily call "V_0" for distinction, we then replace

$$V_0 \to V_1 = c\delta(\tfrac{1}{2}\gamma^2)V_0$$

for 2 of the 3 unintegrated vertex operators V. Any physical state is then defined by a vertex operator acting on the corresponding vacuum:

$$|\text{state}\rangle = V_0|1\rangle = V_1|0\rangle$$

Amplitudes

Vertex operators

For the case of the vector, the vertex operators have the same form as for the particle, found from minimal coupling for the Dirac equation. In that case, the BRST operator is

$$Q = c\tfrac{1}{2}p^2 + \gamma\psi\cdot p - \gamma^2 b$$

where

$$\{\psi^a, \psi^b\} = \eta^{ab} \quad \Rightarrow \quad (\psi\cdot p)^2 = \tfrac{1}{2}p^2$$

The generalization to minimal coupling is

$$\psi\cdot p \to \psi\cdot(p+A) \quad \Rightarrow \quad (\psi\cdot p)^2 \to [\psi\cdot(p+A)]^2 = \tfrac{1}{2}(p+A)^2 - i\tfrac{1}{2}\psi^a\psi^b F_{ab}$$

so the terms linear in the external field give the vertex operators

$$Q \to Q + V, \quad T = \tfrac{1}{2}p^2 \to T + W$$

as

$$V = cW + \gamma\psi\cdot A, \quad W = p\cdot A - i\tfrac{1}{2}\psi^a\psi^b F_{ab}$$

(The p and A in $p\cdot A$ are "normal ordered".)

To apply conformal field theory techniques to the (open) string, we can use the identification

$$p \to \tfrac{1}{2\alpha'}i\partial X = \tfrac{1}{\sqrt{2\alpha'}}i\partial\widehat{X}$$

as seen most easily from the commutator of "p" and X derived from the operator products given previously:

$$\langle\partial X(z)\ X(z',\bar{z}')\rangle = -2\alpha'\ \frac{1}{z-z'}$$

Note the difference in normalization from that derived previously (for the bosonic case): The vertex operator *without coupling* is $i\partial\widehat{X}\cdot A + \cdots$; it creates a normalized state from the tachyonic vacuum, and so excludes a factor of the (open) string coupling. The vertex operators as derived above exclude only the *Yang-Mills* coupling (from rescaling A; we could have put it in from the beginning). We thus conclude

$$g_{YM} = \sqrt{2\alpha'}g_{string}$$

This is easily checked to agree with dimensional analysis, since the string coupling applied to the tachyon gives the usual ϕ^3 interaction. Corresponding to this relation, we note that the above identification for p introduces spurious powers of $2\alpha'$ into the definition of the free BRST operator, which we can fix by the rescalings

$$c \to 2\alpha' c, \qquad \gamma \to \sqrt{2\alpha'}\gamma$$

(and the inverse for b and β), which effectively replaces $p \to i\partial\widehat{X}$ in Q. To do the same for the vertex operators, we also replace

$$A \to \tfrac{1}{\sqrt{2\alpha'}} A$$

which effectively replaces $g_{YM} \to g_{string}$.

So the string vertex operators (less g_{string}) are

$$V = cW + \gamma\psi\cdot A, \qquad W = (i\partial\widehat{X})\cdot A - \sqrt{2\alpha'} i \tfrac{1}{2} S^{ab}F_{ab} \qquad (S^{ab} = \tfrac{1}{2}\psi^{[a}\psi^{b]})$$

(Again we use the gauge $\partial\cdot A = 0$; otherwise, as shown previously, V would get an extra term $-\frac{1}{2}(i\partial c)\partial\cdot A$.) If we use the Yang-Mills vacuum $|1\rangle$, since $\langle 1|c\gamma\gamma|1\rangle \sim 1$, the unintegrated vertex operators will contribute 1 W and 2 $\psi\cdot A$'s, in all 3 permutations.

If we use the superconformal vacuum $|0\rangle$, we also need the picture-changed vertex operators

$$V_1 = c\delta(\tfrac{1}{2}\gamma^2)V_0 = c\delta(\gamma)\psi\cdot A$$

(The W term vanishes, as compared to the $\psi\cdot A$ term, in the limit where the picture-changing operator approaches V, since $c(z)c(z') \sim (z-z')c\partial c$.) Then, since $\langle 0|ccc\delta(\gamma)\delta(\gamma)|0\rangle \sim 1$, we use the 2 V_1's and only the cW part of the 1 V_0.

In a ghost-free way of thinking, the 2 $\psi\cdot A$'s come from not integrating over 2 of the fermionic coordinates of the vertices in 2D worldsheet superspace, just as 3 of the z's of the vertices aren't integrated over. But ghosts are still useful to determine the z dependence of the measure.

There is also another type of picture changing, defined at least for physical fields, where the two $\psi\cdot A$ vertices are moved toward the vacua, and the vacua are changed so that W acting on the new vacua is the same as $\psi\cdot A$ acting on the old ones. Thus all vertices become W. However, the ghost and BRST structure of this picture is not well understood. (It was the first picture to be introduced historically.)

3-point vertex

The simplest example of a tree amplitude with external vectors is the 3-point vertex. This is actually simpler for the superstring than for the bosonic string, since supersymmetry forbids the appearance of F^3 terms in the action.

As for the bosonic string, the ghosts contribute only to the measure, as a function of only the 3 unintegrated z's. In this case, those are all the z's, so there is no integration. So we only need to evaluate

$$\mathcal{A} = \langle \psi \cdot A(z_3)\ W(z_2)\ \psi \cdot A(z_1) \rangle$$

(We use units $\alpha' = \frac{1}{2}$, and leave off the factor of the coupling g. We have put the non-W vertices on the ends so when changing pictures we won't have to worry about signs from pushing picture-changing operators past the unchanged vertex.) Because all the particles are massless, and because of momentum conservation, all inner products of momenta vanish: E.g.,

$$2k_1 \cdot k_2 = (k_1 + k_2)^2 - k_1^2 - k_2^2 = (-k_3)^2 - k_1^2 - k_2^2 = 0$$

Therefore, we don't need to consider $\langle X\ X \rangle$ propagators connecting two $e^{ik\cdot X}$ factors.

Thus, the only propagators we need to evaluate are

$$(i\partial X)(z)\ e^{ik\cdot X(z')} \approx k\ \frac{1}{z - z'}\ e^{ik\cdot X(z')}$$

coming from an $(i\partial X) \cdot A$ in the 1 W and an $e^{ik\cdot X}$ in one of the 2 $\psi \cdot A$'s, and

$$\psi(z)\ \psi(z') \approx \frac{1}{z - z'}$$

among the $\psi \cdot A$'s and SF term in W. We use these matrix elements to kill all the ψ's and the 1 ∂X, leaving only the exponentials, whose matrix element is 1 because of the vanishing momentum inner products. (We ignore the x zero-mode integral of the exponentials, which gives the momentum conservation δ function.) Thus there are only 3 terms total: (1) 1 term from contracting the 2 ψ's in S with the other 2 vertices; (2) 2 terms from contracting the ψ's in the 2 $\psi \cdot A$ vertices with each other, then the ∂X with the exponential of either of those 2 vertices. All these are linear in momenta, giving the usual Yang-Mills vertex. This should be compared with the bosonic string case, which gives 14 terms: (1) 6 terms from contracting 2 ∂X's, and then the final ∂X with 1 of the 2 other exponentials; (2) 8 terms from contracting each of the ∂X's with either of other 2 exponentials, giving the F^3 terms, cubic in momenta.

We then find for the above amplitude

$$\mathcal{A} = \frac{1}{(z_2 - z_1)(z_3 - z_2)}\ tr[A(3) \cdot A(2) k_3 \cdot A(1) + \text{ cyclic permutations }]$$

where the trace is for group theory, and we have used the gauge condition $\partial \cdot A = 0$ (because we dropped the $(\partial c)\partial{\cdot}A$ term from V). Up to the z-dependent normalization, this is the 3-point vertex following from the Lagrangian F^2. (There is also the usual factor of the coupling constant.)

The z-dependence can be canceled by including the ghost contributions. If we use the Yang-Mills vacuum, we get

$$\langle 1|\gamma(z_3)\ c(z_2)\ \gamma(z_1)|1\rangle = (z_1 + z_3) z_2$$

However, in that picture the cW term can come from any of the 3 V's, so we must sum over those 3 permutations. But we saw in $\mathcal{A}$ that the factor multiplying the z dependence was already cyclically symmetric, so we only need to sum

$$\frac{(z_1 + z_3) z_2}{(z_2 - z_1)(z_3 - z_2)} + \text{cyclic permutations } = 1$$

If we use the superconformal vacuum, there is no sum, and according to the discussion above for zero-mode Jacobians, we get

$$\langle 0|[c\delta(\gamma)](z_3)\ c(z_2)\ [c\delta(\gamma)](z_1)|0\rangle = \frac{(z_2 - z_1)(z_3 - z_1)(z_3 - z_2)}{(z_3 - z_1)} = (z_2 - z_1)(z_3 - z_2)$$

for cyclic ordering of the vertices $z_1 < z_3$, canceling the z dependence in $\mathcal{A}$, giving the same result.

Ramond spinors

As we have seen for lightcone spinors (subsection XIB5 of *Fields*; see also VIIB5), Ramond spinors are constructed from exponentials of free bosons, which in turn are obtained by bosonization from the usual (spacetime vector) RNS fermions. In particular, this means Lorentz invariance is not manifest. Also, these spinors require summing over an infinite number of boson-loops, i.e., are nonperturbative (with respect to first-quantization), and thus preclude methods perturbative in α'. Although products of 2 such spinors are easy to handle, the difficulty increases rapidly with the number of spinors, and few such amplitudes have been calculated. (In general in interacting field theory, the "operator product expansion" is useless for the product of more than 2 operators.) Various alternative methods have been developed replacing the RNS free spacetime-vector fermion with a free spacetime-spinor fermion, for the usual superspace coordinates, so not only Lorentz symmetry but also supersymmetry are manifest. These fermions are much easier for such calculations: For loop amplitudes there is no comparison even for lower-point amplitudes; 2-loop calculations with external spinors have been calculated in the "pure spinor" formalism, but not in RNS. (Similar remarks apply for Ramond-Ramond bosons.) Here we will consider the construction of fermion vertex operators in the RNS formalism, as an example of a dog walking on its hind legs.

Bosonizing bosons

The lightcone (SO(8)) Ramond spinor was obtained from 4 bosons, which were required for: (1) the right propagator (and thus anticommutation relations), (2) the right conformal weight (given the bosons came from fermions with weight 1/2), and (3) triality. For the covariant SO(9,1) case the analog of triality is not clear, but the right propagator easily follows by adding equal numbers of bosons of opposite metric (so their contributions to the propagator cancel); the conformal weight can then be fixed if the bosons of unphysical metric come from bosonization of fields that do not have the usual fermionic propagator.

Specifically, we had the general equations for propagator and conformal weight:

$$e^{ia\cdot\phi(z)}e^{i\tilde{a}\cdot\phi(z')} \approx (z-z')^{a\cdot\tilde{a}}e^{ia\cdot\phi(z)+i\tilde{a}\cdot\phi(z')}$$

$$w(e^{ia\cdot\phi}) = \tfrac{1}{2}a\cdot a + i\mu\cdot a$$

Here the metric is chosen to be diagonal with eigenvalues ± 1, while μ is determined from the original conjugate "vector" components for

$$\textit{vector}: \qquad a = (0,...0,\pm 1,0,...,0)$$

for use in the "spinor" components for

$$spinor: \qquad a = (\pm 1/2, ..., \pm 1/2)$$

(with all ±'s independent). All this followed from the energy-momentum tensor

$$T = \tfrac{1}{2}(i\partial\phi)\cdot(i\partial\phi) + \mu\cdot\partial\partial\phi$$

(remember our sign change from *Fields*) where the latter term comes from an $R\phi$ term in the action. The reality of ghosts is funny, so we usually require the redefinition of ghost components of ϕ (and μ) by factors of i: For example, if we were to bosonize the usual fermionic ghosts of the bosonic string, we would make such a redefinition because c and b are both real rather than complex conjugates. Then, *for ghosts only*, we write

$$a\cdot\phi(z)\tilde{a}\cdot\phi(z') \approx a\cdot\tilde{a}\ ln(z-z') \quad \Rightarrow \quad e^{a\cdot\phi(z)}e^{\tilde{a}\cdot\phi(z')} \approx (z-z')^{a\cdot\tilde{a}}e^{a\cdot\phi(z)+\tilde{a}\cdot\phi(z')}$$

$$T = \tfrac{1}{2}(\partial\phi)\cdot(\partial\phi) - \mu\cdot\partial\partial\phi \quad \Rightarrow \quad T(z)\phi(z') \approx \frac{1}{z-z'}\partial\phi(z') + \mu\frac{1}{(z-z')^2}$$

$$T(z)\chi(z') \approx \frac{1}{z-z'}\partial\chi(z') + w(\chi)\frac{1}{(z-z')^2}\chi(z'), \qquad w(e^{a_i\phi^i}) = \tfrac{1}{2}\eta^{ij}a_i a_j + \mu^i a_i$$

For the fermionic ghosts, the boson constructed from the (antihermitian) ghost-number current (as for general fermions discussed previously,

$$J_F = cb = \partial\sigma, \qquad c(z)b(z') \approx \frac{1}{z-z'} \approx b(z)c(z')$$

is sufficient to reconstruct the ghosts,

$$c = e^{\sigma}, \qquad b = e^{-\sigma}$$

However, for the bosonic ghosts the corresponding current,

$$J_B = \gamma\beta = \partial\phi, \qquad \beta(z)\gamma(z') \approx \frac{1}{z-z'} \approx -\gamma(z)\beta(z')$$

is not enough. The bosonization construction we used for fermions breaks down because of the above sign change:

$$J_F(z)J_F(z') \approx \frac{1}{(z-z')^2}, \qquad J_B(z)J_B(z') \approx -\frac{1}{(z-z')^2}$$

This tells us that ϕ has the opposite metric of σ (i.e., negative). The reason for the sign change is simple: This term in the current algebra comes from 1 loop, and fermions give an extra sign. (The structure constants, 0 for this Abelian case, come from the tree, which is classical.) Thus the product of e^{ϕ} and $e^{-\phi}$ is $(z-z')^{+1}$.

To simplify, let's use a general notation for the 2 cases (fermionic/bosonic) as

$$\mathcal{C} = (c, \gamma), \quad \mathcal{B} = (b, \beta), \quad \Sigma = (\sigma, \phi); \qquad J = \mathcal{C}\mathcal{B} = \partial\Sigma$$

To find the value of μ for ϕ, we need to find the action of T on it, which follows from the action of T on J_B, which can be calculated directly with just γ and β. The calculation is the same as for J_F, except for that 1-loop sign again. Using

$$T = -\mathcal{C}\tfrac{1}{2}\overleftrightarrow{\partial}\mathcal{B} - (w - \tfrac{1}{2})\partial(\mathcal{C}\mathcal{B})$$

where w is the weight of $\mathcal{C}$ (-1 and $-\frac{1}{2}$ for the usual c and γ), we find

$$T(z)J(z') \approx \frac{1}{z-z'}\partial J(z') + \frac{1}{(z-z')^2}J(z') - (w - \tfrac{1}{2})\eta\partial\frac{1}{(z-z')^2}$$

(where η is the eigenvalue of the metric) since in the 1-loop contribution the former term in T cancels, while the latter term gives a derivative of the product of currents. Thus

$$T(z)\Sigma(z') \approx \frac{1}{z-z'}\partial\Sigma(z') + (w - \tfrac{1}{2})\eta\frac{1}{(z-z')^2} \quad \Rightarrow \quad \mu = (w - \tfrac{1}{2})\eta$$

$$\Rightarrow \quad w(e^{a\Sigma}) = \tfrac{1}{2}\eta a^2 + (w - \tfrac{1}{2})\eta a$$

Thus for $a = \pm 1$, one gets the expected answer w and $1 - w$ for fermions, but for bosons one is off by -1.

Vertex operator

We then look at a spinor vertex operator of the form

$$W_{1/2} = \Psi^\alpha(X) e^{\phi/2} S_\alpha$$

where $\Psi^\alpha(X)$ is the external spinor wave function (field), and S_α is the SO(9,1) Weyl spinor constructed from just the 5 bosons found from bosonizing the original SO(9,1) fermionic vector variables of RNS. As for the vector vertex, we expect BRST invariance to imply the (free Weyl) wave equation for Ψ. Since Ψ is massless, it has vanishing conformal weight. S_α has conformal weight 5/8, while $e^{\phi/2}$ has weight

$$w(e^{\phi/2}) = \tfrac{1}{2}(-1)(\tfrac{1}{2})^2 + (-1)(-1)\tfrac{1}{2} = \tfrac{3}{8}$$

Thus W has the desired weight 1 for an integrated vertex operator; but BRST invariance needs to be checked.

We use

$$Q = \int cT + \gamma\psi \cdot i\partial X - \gamma^2 b$$

(T is modified for its cb term, but we won't need that here.) The cT term gives the right result if the operator has the right conformal weight. For the other terms we need to know the operator product of γ with $e^{\phi/2}$, and ψ^a with S_α.

The former product we can do by the same method as for T with $e^{\phi/2}$:

$$\gamma(z)J(z') = -\frac{1}{z-z'}\gamma(z') + :\gamma(z)J(z'): \approx -\frac{1}{z-z'}\gamma(z) + :\gamma(z)J(z'):$$

$$\Rightarrow \quad \gamma(z)\phi(z') \approx ln(z-z')\gamma(z) + :\gamma(z)\phi(z'):$$

$$\Rightarrow \quad \gamma(z)e^{a\phi(z')} \approx (z-z')^a :\gamma(z)e^{a\phi(z')}:$$

(We dropped a regular term in the first line that contributes a less singular term in the last line.) Since here $a = \frac{1}{2}$, we see γ's make things *less* singular. So we can forget the last term in T.

For the latter product, we're looking at

$$e^{ia\cdot\phi(z)} e^{i\tilde{a}\cdot\phi(z')} \approx (z-z')^{a\cdot\tilde{a}} e^{ia\cdot\phi(z) + i\tilde{a}\cdot\phi(z')}$$

for *physical* ϕ^i: With a_i from ψ^a, $\tilde{a}_i$ from S_α, and $\eta^{ij} = \delta^{ij}$, we have

$$a \cdot \tilde{a} = \pm\tfrac{1}{2}$$

So any divergence from here can be canceled by the $(z-z')^{+1/2}$ from γ. But the second term in T also has a ∂X, which contributes

$$(\partial X^a)(z)\Psi(X(z')) \approx -\frac{1}{z-z'}(\partial^a\Psi)(X(z'))$$

So we need to know the Lorentz covariant form of the product of ψ^a with S_α. The general form can be determined from Lorentz covariance itself: Since the divergent pieces come from $a\cdot\tilde{a} = -\frac{1}{2}$, $a+\tilde{a}$ in the resulting exponent will still have all components $\pm\frac{1}{2}$, and thus still represent a spinor, but the product of all the signs will change sign: The spinor has opposite chirality. Thus

$$\psi^a(z)S_\alpha(z') \approx (z-z')^{-1/2}\gamma^a_{\alpha\beta}S^\beta + (z-z')^{+1/2} :\psi^a S_\alpha:$$

where the γ-matrix elements are normalized to 1 in the representation of spinors and vectors implied by bosonization. (They are expressed as the direct products of 2D γ-matrices, which is the same as dividing up the γ's in complex pairs as coordinates and momenta.)

Putting the pieces together, the divergence cancels, and thus the integrated vertex operator is BRST invariant, if the wave function (external field) satisfies the Weyl equation

$$\gamma^a_{\alpha\beta}\partial_a\Psi^\beta = 0$$

Since only the first term in Q contributes a divergence (and thus commutator term), the corresponding unintegrated vertex is simply $V = cW$.

Pictures

Bosonizing picture changing

Since we are forced to deal with exponentials of the form $e^{\pm\phi/2}$ in constructing spinor vertex operators, it's useful to represent picture changing also in terms of such exponentials. We then need the identifications

$$e^{\phi} = \delta(\gamma), \qquad e^{-\phi} = \delta(\beta)$$

which are similar to the relations for fermionic ghosts

$$e^{\sigma} = \delta(c) = c, \qquad e^{-\sigma} = \delta(b) = b$$

These can be derived by noting that they satisfy the same first-order differential equation with respect to z:

$$\partial\mathcal{O} = \, : J\mathcal{O} :$$

For example, for e^{ϕ} we have

$$\partial(: e^{\phi} :) = \, : Je^{\phi} :$$

while for $\delta(\gamma)$ we can compare

$$\partial[\delta(\gamma)] = (\partial\gamma)\delta'(\gamma)$$

with the $z' \to z$ limit of

$$: (\beta\gamma)(z) : \delta(\gamma(z')) \approx \beta(z)(z-z')(\partial\gamma)(z')\delta(\gamma(z')) \approx (z-z')(\partial\gamma)(z')\frac{1}{z-z'}\delta'(\gamma(z'))$$
$$\to (\partial\gamma)\delta'(\gamma)$$

(There is still an initial condition to fix, but this is the same zero-mode ambiguity we encountered before when defining ϕ from $J = \partial\phi$: Conformal invariance, and locality of operator products, fixes it to be a normalization constant.)

The relevant picture-changing operators are

$$\mathcal{Y} \equiv \delta(c)\delta(\{Q,c\}) = c\delta(\gamma^2)$$
$$\mathcal{X} \equiv \delta(\beta)\delta([Q,\beta]) = \delta(\beta)(-\psi\cdot i\partial X + c\partial\beta + \tfrac{3}{2}\beta\partial c + 2\gamma b)$$

(One can motivate the choice of c and β in the gauge-fixing conditions by noting that they are the bottom components in the conjugate superfields $C = c + \theta\gamma$ and $B = \beta + i\theta b$.) $\mathcal{Y}$ was introduced earlier, and we can check that $\mathcal{X}$ can be considered its inverse:

$$\mathcal{Y}(z)\mathcal{X}(z') \approx 1$$

The only relevant term in this product is from the last term in $\mathcal{X}$: The pole in cb is canceled by

$$\delta(\beta(z))\delta(\gamma(z')) \approx z - z'$$

We can completely bosonize γ and β (not just their current) by defining

$$\gamma = e^{-\phi}\eta, \qquad \beta = e^{\phi}\partial\xi$$

where ξ and η are conjugate fermions of weights 0 and 1. (The derivative of a field of weight 0 is still conformally covariant, of weight 1.) This can be useful in the operator formalism, but working with γ and β is more useful in the path-integral formalism, since then most properties (like worldsheet supersymmetry) are valid classically. Unfortunately, in the RNS formalism spinors can't be treated classically (even with respect to the physical bosonized fermions), and at least ϕ is necessary.

We also note that

$$\mathcal{X} = \{Q, \xi\}$$

Not only is the physical interpretation of picture changing as gauge fixing not clear in this form, but it might seem that it's BRST trivial. However, ξ is undefined in terms of the original ghosts γ, β; only $\partial\xi$ appears there. (Alternatively, one could introduce the zero-mode of ξ as a new variable; but then one would also need an extra term in Q to kill it.)

Supersymmetry

As noted earlier, picture-changing operators can be moved around the worldsheet freely without affecting physical matrix elements. As we saw for vector vertices, they can also be combined with vertex operators by "colliding" them. Now consider the product of 2 spinor vertex operators: This involves

$$(e^{\phi/2}S_\alpha)\cdot(e^{\phi/2}S_\beta) \approx (z-z')^{-1/4}e^{\phi}(z-z')^{-3/4}\gamma^a_{\alpha\beta}\psi_a$$

by arguments similar to the above. We recognize

$$W_1 = \delta(\gamma)\psi^a A_a(X)$$

as the (integrated version of) the picture-changed vector vertex operator found earlier. This suggests hitting one of the spinor vertex operators with an $\mathcal{X}$, to yield a picture-changed spinor vertex whose product with the unchanged spinor vertex will yield the original picture-unchanged vector vertex. We thus find

$$W_{-1/2} = \mathcal{X}W_{1/2} \sim e^{-\phi/2}\left[\gamma^a_{\alpha\beta}(\partial X_a)S^\beta\Psi^\alpha(X) - \psi^a S_\alpha \partial_a \Psi^\alpha\right] + ...$$

where the subscripts on W we have been using denote the power of $e^\phi = \delta(\gamma)$. (We dropped total-derivative c terms that drop out after integration, or in V; "..." denotes a b term that won't contribute to amplitudes with $|0\rangle$, at least at tree level.)

Since we need 2 $\delta(\gamma)$'s to get a finite superconformal-vacuum expectation value, we'll generally need both $W_{1/2}$'s and $W_{-1/2}$'s for multi-spinor amplitudes.

An interesting way to think of the momentum operator (generator of translations) is as the zero-mode (integral) of the vector vertex operator at vanishing external momentum. Similarly, we can find the supersymmetry operator as the zero-mode of the spinor vertex operator. (This was first understood in superspace.) Thus

$$q_{1/2,\alpha} = \int e^{\phi/2} S_\alpha, \qquad q_{-1/2,\alpha} = \int e^{-\phi/2} \gamma^a_{\alpha\beta} (\partial X)_a S^\beta$$

Symmetry generators can carry different pictures, just as the vertex operators. For translations we then have

$$p_{1,a} = \int e^\phi \psi_a, \qquad p_{0,a} = \int \partial X_a$$

As we saw previously, we have

$$\{q_{1/2,\alpha}, q_{1/2,\beta}\} = \gamma^a_{\alpha\beta} p_{1,a}, \qquad \{q_{1/2,\alpha}, q_{-1/2,\beta}\} = \gamma^a_{\alpha\beta} p_{0,a}$$

etc.

Spinor-spinor-vector vertex

As a simple example, consider the vertex for 2 spinors and a vector. For the 2 vacua, we evaluate

$$\langle 0 | V_{1/2} V_{1/2} V_1 | 0 \rangle = \langle 1 | V_{-1/2} V_{1/2} V_0 | 1 \rangle$$

For the non-ghost part, we get in both cases

$$\langle \Psi^\alpha S_\alpha(1) \Psi^\beta S_\beta(2) A^a \psi_a(3) \rangle$$

We first contract the 2 S's to get a ψ times a γ matrix, then that ψ with the other one to get a Minkowski metric. The result is thus a γ matrix, as expected.

There are also some z factors, which are canceled by the ghosts: Picking out the terms with the right 0-modes, we find for the superconformal vacuum

$$\langle c e^{\phi/2}(1) c e^{\phi/2}(2) c e^\phi(3) \rangle$$

and for the Yang-Mills vacuum

$$\langle \gamma e^{-\phi/2}(1) c e^{\phi/2}(2) \gamma(3) \rangle$$

(The first factor comes from hitting the $e^{-\phi}\gamma b$ term in $\mathcal{X}$ on the $ce^{\phi/2}$ in $V_{1/2}$.)