Spinning string

Action

The spinning string is defined by combining the concepts of the bosonic string (subsections XIA3,B2) and the spinning (spin ½) particle (exercise IIIB1.3):

1. Extend the worldsheet "fields" to include a spacetime-vector fermion Ψ^a, and
2. Extend the constraints to include a generalized "Dirac equation" G ("superVirasoro", "superconformal generators", or "supercurrent").

$$\{\Psi^a(\sigma_1), \Psi^b(\sigma_2)\} = 2\pi\delta(\sigma_{12})\eta^{ab}$$

$$G = \Psi \cdot \hat{P}, \quad T = \frac{1}{2}(\hat{P}^2 - i\Psi' \cdot \Psi)$$

$$\{G(1), G(2)\} = 4\pi\delta(1-2)T$$

$$[T(1), G(2)] = 2\pi i\delta'(1-2)[G(1)+\frac{1}{2}G(2)]$$

$$[T(1), T(2)] = 2\pi i\delta'(1-2)[T(1) + T(2)]$$

For convenience we switch to dimensionless variables,

$$X = \sqrt{\frac{\alpha'}{2}}\,\varphi \ , \quad \hat{P} = \frac{1}{\sqrt{2}}\left(\sqrt{\alpha'}P + \frac{1}{\sqrt{\alpha'}}X'\right)$$

As part of this generalization, G generates local worldsheet supersymmetry:

$$[G(1), \varphi(2)] = -\,i2\pi\delta(1-2)\Psi$$
$$\{G(1), \Psi(2)\} = 2\pi\delta(1-2)\hat{P}$$
$$[G(1), \hat{P}(2)] = i2\pi\delta'(1-2)\Psi(1)$$

As for the bosonic case, we can have left and right algebras (using $\pm\partial/\partial\sigma$) , but now also left and right Ψ's. The action is then (summing over $\pm$)

$$S = \int \frac{d^2\sigma}{2\pi}\left(-\dot{X}\cdot P + i\tfrac{1}{2}\dot{\Psi}_\pm \cdot \Psi_\pm + g_\mp T_\pm + \chi_\mp G_\pm\right)$$

in terms of the worldsheet metric g & "gravitino" χ (supergravity).

(Notational comment: One may want to insert some i's here and in what follows for hermiticity, as redefinitions, since our index sums include terms with fermion times fermion. Technically this is not required, since "covariant" & "contravariant" fermions have opposite hermiticity properties.) The gauge transformations on the dynamic fields follow from the above commutators, as do those of the gauge fields: From subsection IIIA5,

$$\delta(q,p) = i\zeta^i[G_i,(q,p)\} \ , \quad \delta\lambda^i = \dot{\zeta}^i + \zeta^k\lambda^j f_{jk}{}^i \qquad (\ [G_i,G_j\} = -if_{ij}{}^k G_k \ )$$

where the sums are now over T & G as well as $\int d\sigma/2\pi$. Rather than extract the structure constants f, we can just multiply the explicit commutation relations of the generators by ζ & λ and integrate, pulling off the coefficients of T & G: For the left transformations (all except φ are left variables),

$$\delta\varphi = \zeta\hat{P} + \varepsilon\Psi \ , \quad \delta\Psi = (\zeta\Psi' + \tfrac{1}{2}\zeta'\Psi) + i\varepsilon\hat{P} \ , \quad \delta\hat{P} = (\zeta\hat{P} + \varepsilon\Psi)'$$

$$\delta g = \dot{\zeta} + (\zeta g' - \zeta' g) + 2i\varepsilon\chi \ , \quad \delta\chi = \dot{\varepsilon} + (\zeta\chi' - \tfrac{1}{2}\zeta'\chi) + (\tfrac{1}{2}\varepsilon g' - \varepsilon' g)$$

To find the Lagrangian, we integrate P out, and make the substitutions as before for the metric and zweibein (see exercise XIA3.1):

$$\nabla_\pm = \frac{1}{\sqrt{2}}(\nabla_0 \pm \nabla_1) = \frac{1}{\sqrt{2}}(\partial_\tau \pm g_\pm \partial_\sigma), \quad \sqrt{-g} = \frac{2}{g_+ + g_-}$$

(The Lorentz connection doesn't contribute on scalar X, and on spinor Ψ it multiplies $\Psi_\pm^2 = 0$.) Then

$$S = \int \frac{d^2\sigma}{2\pi} \sqrt{-g} \left[\frac{1}{2\alpha'}(\nabla X)^2 - i\frac{1}{\sqrt{2}}\Psi_\pm \cdot \nabla_\mp \Psi_\pm + \chi_\mp \Psi_\pm \cdot \nabla_\pm X - \frac{1}{2}\chi_+ \chi_- \Psi_+ \cdot \Psi_- \right]$$

The Hamiltonian form was manifestly Weyl scale invariant: While the scalar X was already scale invariant, Ψ was a scale invariant density. Thus in converting to the Lagrangian an appropriate factor was scaled out to make it a scalar, and something similar for χ:

$$\Psi \to (-g)^{1/4}\Psi, \quad \chi \to (-g)^{-1/4}\chi$$

It's now easy to read off both the Lorentz and Weyl weights of $\Psi_\pm$ $(\pm 1/2, 1/2)$ and $\chi_\pm$ $(\pm 3/2, 1/2)$. We can also see separation of left-handed modes into Ψ_+ and right-handed into Ψ_-.

Quantization

As in the bosonic case, we have the usual "temporal" gauge where (the τ components of) the gauge fields for the constraints are fixed, now the "superconformal gauge"

$$\chi_\pm = 0, \quad g_\pm = 1 \quad \Rightarrow \quad g_{mn} = \eta_{mn} \,, \quad \partial_\mp \zeta_\pm = \partial_\mp \varepsilon_\pm = 0$$

$$S = \int \frac{d^2\sigma}{2\pi} \left[\frac{1}{2\alpha'}(\partial X)^2 - i\frac{1}{\sqrt{2}} \Psi_\pm \cdot \partial_\mp \Psi_\pm \right] \qquad \begin{aligned} G_\pm &= \Psi_\pm \cdot \partial_\pm X / \sqrt{\alpha'} \\ T_\pm &= \frac{1}{2\alpha'}(\partial_\pm X)^2 + \frac{1}{\sqrt{2}} i \Psi_\pm \cdot \partial_\pm \Psi_\pm \end{aligned}$$

(In this gauge, $P = \dot{X}/\alpha'$.) Besides the usual boundary term for X, we also have

$$\pm \Psi_\pm \cdot \delta \Psi_\pm = 0 \quad \Rightarrow \quad \Psi_+ = \Psi_- \quad \textbf{or} \quad \Psi_+ = -\Psi_-$$

By field redefinition, we can always set $\Psi_+ = \Psi_-$ at 1 end of an open string. Then the 2 choices appear only at the other end.

A more convenient treatment, as a generalization of the open bosonic string, is to combine left and right variables, just as for X (see subsection XIB1), but now

$$\Psi(\sigma) = \begin{cases} \Psi_+(\sigma) & \textbf{for } \sigma > 0 \\ \Psi_-(-\sigma) & \textbf{for } \sigma < 0 \end{cases}$$

Thus the 2 choices of boundary conditions appear as (anti)periodicity at

$$\Psi(-\pi) = \begin{cases} \Psi(\pi) & \textbf{"Ramond"} \\ -\Psi(\pi) & \textbf{"Neveu-Schwarz"} \end{cases}$$

For the closed string, (anti)periodicity can be applied independently for Ψ_+ and for Ψ_-. (If we don't care about Lorentz invariance, it can also be applied independently for each Lorentz component of Ψ for open and closed.) Thus there are R and NS open strings, but closed strings can be R-R, R-NS, NS-R, and NS-NS.

In the lightcone gauge, besides fixing X^+ and P^+, we also fix

$$\Psi^+ = 0$$

χ and Ψ^- then fix each other as auxiliary fields. The X part of the action is then as in the bosonic case, while the Ψ part just truncates its superconformal action to the transverse components.

The fermionic contribution to spin is the obvious generalization of the particle (as was the bosonic):

$$\Delta S^{ab} = \int \frac{d\sigma}{2\pi} \tfrac{1}{2} \Psi^{[a} \Psi^{b]}$$

($[ab] = ab - ba$.)

The fermionic contribution to the mass² can be read from the integral of T. (Note that Ramond fermions have a zero-mode that doesn't contribute.)

Modes and states

The fermion self-anticommutation relations involve no derivatives, so their modes are just ordinary fermionic oscillators, without the hidden factors of $\sqrt{n}$ that appear for the bosons. But the mass operator has the same factor of n, since $\int T$ generates a worldsheet derivative. (Consider $-i\partial/\partial\sigma$ on $e^{in\sigma}$.) Ramond fermions are periodic, like the bosons, and so have a zero-mode, the usual Dirac γ-matrices:

$$\Psi(\sigma) = \sum_{n\in\mathbf{Z}} d_n e^{in\sigma}, \quad d_0 = \frac{1}{\sqrt{2}} i\gamma, \quad \{d_m, d_n\} = \delta_{m+n,0}$$

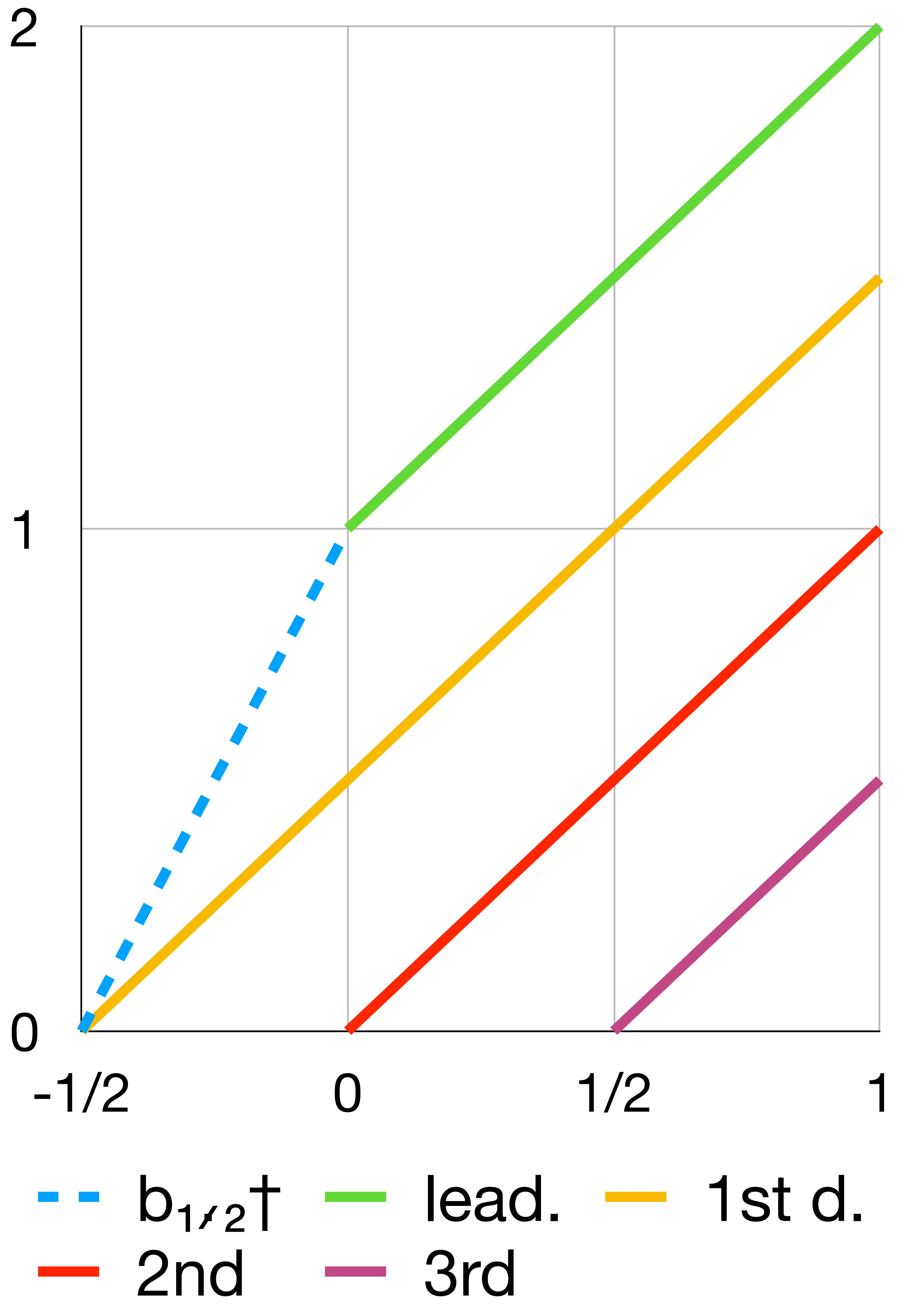
Thus R open strings, and R-NS and NS-R closed strings, carry a spacetime-spinor index, and describe fermionic states.

Neveu-Schwarz fermions are antiperiodic, and have half-(odd-)integer mode numbers, so they increase the mass² by half-integers.

$$\Psi(\sigma) = \sum_{n\in\mathbf{Z}+1/2} b_n e^{in\sigma}, \quad \{b_m, b_n\} = \delta_{m+n,0}$$

We saw the bosonic string has a "leading trajectory" with a linear relation between spin and mass², and all other states appearing integer distances below it. This trajectory was generated by powers of just the first bosonic oscillator, $n = 1$. (See subsection XIB2. Closed-string states came from direct products of left and right open-string oscillators.) Ramond strings are similar. But the ground state is a spinor, not a scalar. (Also, spinors can't be tachyons.)

On the other hand, Neveu-Schwarz strings have an oscillator with $n = 1/2$, which increases the mass more slowly than $n = 1$ would. However, since it's a fermion, powers can give only antisymmetric tensors, all of which are considered spin 1 or less, and thus can't generate a rising trajectory by themselves. This means that the leading trajectory for NS strings is generated by 1 $n = 1/2$ fermionic oscillator (from Ψ) and an arbitrary number of $n = 1$ bosonic oscillators (from X). By the same argument as for the bosonic string, the vector state must be massless. But this vector is the first state on its trajectory (slope α'): The tachyonic ground state lies on the first daughter trajectory, with intercept 1/2 lower.



NS trajectories

Graph of spin as a function of ($\alpha' \times$)mass² for the (open) NS string.

The first 4 trajectories are shown (solid lines); they are extended by repeated action of the first ($n = 1$) X oscillator.

The dashed line is the action of the first ($n = 1/2$) NS oscillator once on the tachyonic vacuum.

The massless vector is on the leading trajectory. The tachyon is on the first daughter trajectory.

Exercise: Find all states at $m^2 = 3/2$. Combine transverse states into massive ones.

Supersymmetry leads to some improvements in string theory. In particular, it gets rid of tachyons. So the spinning string as described clearly isn't supersymmetric. But there is a truncation of its spectrum that is, and is preserved by its interactions, due to an obvious discrete symmetry of the action that was originally called "G-parity":

$$\Psi \rightarrow -\Psi$$

This gives the transformation on the oscillators, but for the transformation on states, we also need to know the transformation of the ground state. On the Ψ zero-modes of the R string (Dirac matrices), this symmetry is the usual chiral symmetry generated by "γ_5". There is then the sign ambiguity of which Weyl spinor is chosen "positive" chirality. For reasons to be explained further later (related to ghosts), the NS tachyonic ground state is taken to be odd under the transformation, so the massless vector is even.

Restricting to the even G-parity sector is called "Gliozzi-Scherk-Olive projection". For the R string this kills half the chiralities, depending on how many Ψ oscillators are used. For the NS string, it simply kills all the "odd" trajectories, those whose intercept is a half-integer below the leading, because only an odd number of NS oscillators is allowed to hit the tachyonic vacuum.

BRST

We now generalize the treatment of BRST for constrained systems (subsection VIA1) for bosonic constraints to graded constraints. In terms of constraint algebra G_i , ghosts c^i, & antighosts b_i , we now have

$$[G_i, G_j\} = -if_{ij}{}^k G_k \ , \ [b_i, c^j\} = \delta_i^j \ \Rightarrow \ Q = c^i G_i - i\tfrac{1}{2}(-1)^j c^j c^i f_{ij}{}^k b_k$$

(The 1st term in Q is the usual replacement of gauge parameter with ghost.) The extra sign factor $(-1)^j$ is 1 when j is a bosonic index as before, -1 when it's fermionic. It appears because ghosts have effectively a hidden fermionic index "•" to account for statistics, $c^i = c^{\bullet i}$, $b_i = b_{i\bullet}$, and there's a sign with respect to moving the j on c past the 2nd • so that the contracted indices take the canonical ordering. (In the generalization of BRST to include "anti-BRST", this additional index becomes a 2-valued index of Sp(2), the spinor of SO(2,1). See subsection XIIA2.)

We want to apply this to the spinning string: Looking at just the left-handed algebra for the closed string (or full algebra for the open), with $Q = Q_{(+)} + Q_{(-)}$, where

$$G_i = T(\sigma), G(\sigma); \quad c^i = c, \gamma, \quad b_i = b, \beta$$

$$G = \Psi \cdot \hat{P}, \quad T = \tfrac{1}{2}(\hat{P}^2 - i\Psi' \cdot \Psi)$$

$$\{G(1), G(2)\} = 4\pi\delta(1-2)T$$

$$[T(1), G(2)] = 2\pi i\delta'(1-2)[G(1) + \tfrac{1}{2}G(2)]$$

$$[T(1), T(2)] = 2\pi i\delta'(1-2)[T(1) + T(2)]$$

we can immediately write the 1st term in Q by multiplying the constraints by ghosts & summing (including $\int d\sigma/2\pi$), & the 2nd term (as we did for gauge-field transformation laws) by multiplying 2 ghosts (times $(-1)^j$) times the right-hand side of the commutators (with 2 integrals), replacing the constraint by its antighost, & using the $2\pi\delta$ to eliminate 1 integral. Then

$$Q = \int cT + \gamma G - \gamma^2 b - ic(c'b + \tfrac{3}{2}\gamma'\beta + \tfrac{1}{2}\beta'\gamma)$$

Necessarily, c & b satisfy the same boundary conditions as T, and γ & β the same as G. (The same goes for gauge fields & parameters.)

We could also include "fields" that drop out of the gauge-invariant action, such as the determinant of the worldsheet metric (only its det = –1 part appears), or the trace of the gravitino. As usual, such fields contribute only separable, nonminimal terms to Q. E.g.,

$$\delta(\ln\sqrt{-g}) = \zeta + \ldots \equiv \hat{\zeta}$$

where ζ is the Weyl scale parameter, & $\hat{\zeta}$ is a redefinition to absorb all additional gauge transformations (...). Such nonminimal contributions are of the form "$\tilde{c}\tilde{p}$", where $\tilde{p}$ is the canonical conjugate to the trivially transforming field, & $\tilde{c}$ is its ghost. (Such terms are useful for gauge conditions that involve derivatives: See subsection VIA2.)

As usual, fixing the gauge fields to specific functions f^i (temporal gauge) is achieved by adding to H_{gi} (0 in our case) the BRST trivial term $\{Q, f^i b_i\} = f^i G_i + \ldots \equiv f^i \hat{G}_i$. In our case, the conformal gauge is (after eliminating P in S as usual)

$$\hat{T} = \{Q, b\} = T - i(2c'b - b'c + \tfrac{3}{2}\gamma'\beta + \tfrac{1}{2}\beta'\gamma)$$
$$H = \{Q, b_0\} = \hat{T}_0 = (T - ic'b - i\gamma'\beta)_0$$
$$S = \int \frac{d^2\sigma}{2\pi}\left[\tfrac{1}{2\alpha'}(\partial X)^2 - i\frac{1}{\sqrt{2}}\Psi_\pm \cdot \partial_\mp \Psi_\pm - i\sqrt{2}c_\pm \partial_\mp b_\pm - i\sqrt{2}\gamma_\pm \partial_\mp \beta_\pm\right]$$
(but remember $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/\sqrt{2}$).

Here is a preview of things we'll find later:

- After conformally transforming to the plane, NS is periodic in a circle, while R is antiperiodic (i.e., there's a cut).

- The superconformal vacuum is a ghost state in the NS sector, & is G-parity even.

- The R ground states should not be considered vacua, & are created by "nonperturbative" operators acting on the superconformal vacuum. Thus, all "fields" appearing in Q should be considered periodic (no cuts).

Wick rotation

We collect here some notational conventions for later application to conformal theory. We first Wick rotate to 2D Euclidean space $\tau \to -i\tau$, using as our complex coordinates $\rho = \tau + i\sigma$ and its complex conjugate $\bar{\rho}$. We then transform from the cylinder to the plane, $z = e^{\rho}$. Then the action, propagators (subsection VIIB5) and energy-momentum tensor (subsection XIB4) for real fields are:

$$T_+ = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}\psi_+\partial\psi_+ \ , \qquad T_- = -\frac{1}{2}(\bar{\partial}\varphi)^2 - \frac{1}{2}\psi_-\bar{\partial}\psi_-$$

$$G_+ = \psi_+\partial\varphi \ , \quad G_- = \psi_-\bar{\partial}\varphi$$

$$\{G(1), G(2)\} = -4\pi i\delta(1-2)T$$

$$[T(1), G(2)] = -2\pi i\delta'(1-2)[G(1)+\frac{1}{2}G(2)]$$

$$[T(1), T(2)] = -2\pi i\delta'(1-2)[T(1) + T(2)]$$

$$Q = \int cT + \gamma G + c(\partial c)b - \gamma^2 b + c(\frac{3}{2}\beta\partial\gamma + \frac{1}{2}\gamma\partial\beta)$$

$$\hat{T} = T + 2(\partial c)b + c\partial b + \frac{3}{2}\beta\partial\gamma + \frac{1}{2}\gamma\partial\beta$$
$$H = \{Q, b_0\} = \hat{T}_0 = (T - c\partial b - \gamma\partial\beta)_0$$
$$S = \int \frac{d^2\sigma}{2\pi}[(\partial\varphi)(\bar{\partial}\varphi) + \psi_+\bar{\partial}\psi_+ + \psi_-\partial\psi_- + 2(c_+\bar{\partial}b_+ + c_-\partial b_- + \gamma_+\bar{\partial}\beta_+ + \gamma_-\partial\beta_-)]$$
$$\langle\varphi\ \varphi\rangle = -\ln|z|^2, \quad \langle\psi_+\ \psi_+\rangle = \langle b_+\ c_+\rangle = \langle\beta_+\ \gamma_+\rangle = \frac{1}{z}\ , \quad \langle\psi_-\ \psi_-\rangle = \ldots = \frac{1}{\bar{z}}$$
where $X = \sqrt{\alpha'/2}\,\varphi$ and $\int d^2\sigma$ is over the real and imaginary parts of z. On the plane, G-parity is simply 2π rotation. (CPT is π rotation.)