Introduction to AdS/CFT

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4D N = 4 Yang-Mills

Prerequisites	in *Fields*
4D 2-component spinor notation	IIA
supersymmetry, superspace	IIC, IVC
classical groups	IB4
10D IIB supergravity	
$1/N$ expansion	VIIC4

4D N = 4 super Yang-Mills has interesting quantum properties: Because of its 4 (times the minimal number of) supersymmetries, it has more spacetime symmetry than any other renormalizable 4D theory. But it's also conformal, so its N = 4 superconformal symmetry gives it an even larger symmetry. Whereas any 4D, scale-invariant theory is conformal classically, when such theories are ultraviolet finite order by order in perturbation theory, this (super)conformal invariance is preserved at the quantum level. (This also implies some nonperturbative advantages, such as absence of renormalons.) N = 4 Yang-Mills was the first known example of this behavior. (This requires at least N = 1 supersymmetry.) Having the largest symmetry group gives it the simplest quantum S-matrix elements of any interacting 4D theory. Partly due to the Anti-de Sitter/Conformal Field Theory correspondence (see below), its 4-point amplitude is known to all orders in Ng^2, to leading order in $1/N$ (colors for U(N), not to be confused with the (S)U(N) internal symmetry for N = 4 supersymmetries). As a consequence of these properties, the N = 4 theory is often used as a starting point or approximation for calculations in the Standard Model. It is also the CFT with the most number of explicit calculations for testing AdS/CFT. There are even methods motivated by AdS/CFT that don't explicitly make use of the string: For example, Yangian symmetry and the dual Wilson loop are two approaches to N = 4 Yang-Mills that use only the T-duality property of string theory to imply useful methods to determine properties and perform calculations entirely on the CFT side, but with methods that might not have been found otherwise. For these reasons, most of what is discussed below relates specifically to this theory, but some generalizations are also outlined.

4D N = 4 Yang-Mills can be described in superspace by the commutation relations of derivatives covariant with respect to both the Yang-Mills gauge group and supersymmetry. These derivatives are with respect to both the spacetime coordinates $x^{\alpha\dot{\alpha}}$ (in SL(2,C) spinor notation) and fermionic coordinates $\theta^{\underline{a}\alpha}$ and their hermitian conjugates $\bar{\theta}_{\underline{a}}{}^{\dot{\alpha}}$ (with also SU(4) "R-symmetry" indices). (In general we'll use underlined indices for those for defining representations that will be broken up finally into two smaller indices as direct sums.) The superfield strengths that appear give the usual component field strengths when evaluated at $\theta = \bar{\theta} = 0$: For all N $\leq$ 4,

$$
\begin{aligned}
\{\nabla_{\underline{a}\alpha}, \overline{\nabla}^{\underline{b}}{}_{\dot{\beta}}\} &= -i\delta_{\underline{a}}^{\underline{b}}\nabla_{\alpha\dot{\beta}} \\
\{\nabla_{\underline{a}\alpha}, \nabla_{\underline{b}\beta}\} &= C_{\alpha\beta}\bar{\phi}_{\underline{ab}} \\
\{\overline{\nabla}^{\underline{a}}{}_{\dot{\alpha}}, \overline{\nabla}^{\underline{b}}{}_{\dot{\beta}}\} &= \bar{C}_{\dot{\alpha}\dot{\beta}}\phi^{\underline{ab}} \\
[\nabla_{\underline{a}\alpha}, \nabla_{\beta\dot{\beta}}] &= C_{\alpha\beta}\bar{\lambda}_{\underline{a}\dot{\beta}} \\
[\overline{\nabla}^{\underline{a}}{}_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] &= \bar{C}_{\dot{\alpha}\dot{\beta}}\lambda^{\underline{a}}{}_{\beta} \\
[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] &= \bar{C}_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta} + C_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}}
\end{aligned}
$$

from which we can see the 6 scalars $\phi^{\underline{ab}} = -\phi^{\underline{ba}}$, 4 (Weyl) spinors λ, and selfdual and antiselfdual Yang-Mills field strengths $f^{\alpha\beta} = f^{\beta\alpha}$ and $\bar{f}$. (Of course, all fields are in the adjoint representation of the Yang-Mills group, by supersymmetry.) Here we have used the SL(2,C) metrics

$$
C_{\alpha\beta} = -C^{\alpha\beta} = \bar{C}_{\dot{\alpha}\dot{\beta}} = -\bar{C}^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
$$

The Jacobi identities imply the supersymmetry relations between these field strengths (through covariant spinor derivatives, i.e., Taylor expansion in θ), and that N $\leq$ 4. For N = 4, they also imply their field equations, and the reality condition

$$
\bar{\phi}_{\underline{ab}} = \tfrac{1}{2}\epsilon_{\underline{abcd}}\phi^{\underline{cd}}
$$

up to a phase whose choice breaks U(4) to SU(4). (N = 3 implies equivalent fields and field equations, but doesn't need a reality condition.)

In this notation, the action can be written in components as

$$
\begin{aligned}
S = \frac{1}{g^2}tr\int d^4x\, \{ &-\tfrac{1}{2}f^{\alpha\beta}f_{\alpha\beta} + \tfrac{1}{8}[\nabla^{\alpha\dot{\alpha}}, \bar{\phi}_{\underline{ab}}][\nabla_{\alpha\dot{\alpha}}, \phi^{\underline{ab}}] - \tfrac{1}{32}[\bar{\phi}_{\underline{ab}}, \bar{\phi}_{\underline{cd}}][\phi^{\underline{ab}}, \phi^{\underline{cd}}] \\
&+ \bar{\lambda}_{\underline{a}}{}^{\dot{\alpha}}[-i\nabla_{\alpha\dot{\alpha}}, \lambda^{\underline{a}\alpha}] + \tfrac{1}{2}i(\lambda^{\underline{a}\alpha}[\bar{\phi}_{\underline{ab}}, \lambda^{\underline{b}}{}_{\alpha}] + \bar{\lambda}_{\underline{a}}{}^{\dot{\alpha}}[\phi^{\underline{ab}}, \bar{\lambda}_{\underline{b}\dot{\alpha}}]\ \}
\end{aligned}
$$

The simplest derivation is by dimensionally reducing 10D super Yang-Mills, which has no scalars: Then the first line comes from the F^2 term, while the second comes from the $\lambda\nabla\!\!\!\!/\,\lambda$ term. (The covariant derivative commutators can also be done there.)

Later we'll consider a smaller superspace that includes only $\nabla_{a'\alpha}$ and $\overline{\nabla}^a{}_{\dot\alpha}$ as nontrivial on the field strength that lives there. (For N = 4, $\underline{a}$ is 4-valued, a and a' are each 2-valued.) For N = 4 only 1 complex scalar appears explicitly (at $\theta = 0$) in the above commutators of the *other* spinor covariant derivatives:

$$\{\nabla_{a\alpha}, \overline{\nabla}^{b'}{}_{\dot\beta}\} = 0$$
$$\{\nabla_{a\alpha}, \nabla_{b\beta}\} = C_{\alpha\beta}C_{ab}\phi$$
$$\{\overline{\nabla}^{a'}{}_{\dot\alpha}, \overline{\nabla}^{b'}{}_{\dot\beta}\} = \bar{C}_{\dot\alpha\dot\beta}C^{a'b'}\phi$$

The same ϕ appears in both equations (and not $\bar\phi$, because of the reality condition), and because of the Jacobi identities it satisfies

$$\nabla_{a\alpha}\phi = \overline{\nabla}^{a'}{}_{\dot\alpha}\phi = 0$$

Correspondence

The AdS/CFT *conjecture* is that (10D) superstring theory is equivalent to 4D N = 4 super Yang-Mills. (In string theory, the identification of physical dimensions can be ambiguous because of compactification/bosonization, branes, etc.) This equivalence is nonperturbative, as is the conjectured equivalence between all superstring theories (I, IIA, IIB, and both heterotic) and the suspected "M-theory". In particular, this equivalence is seen perturbatively about the vacuum of 5D anti-de Sitter space $\times$ the 5-sphere, where these 2 spaces have the same radius of curvature $\mathcal{R}$ (so the 10D curvature scalar cancels).

The basic idea is that we live on the boundary of AdS_5, so we don't see the fifth dimension "x_0" directly as a dimension, in the same way that we don't see the other 5 dimensions of S^5. In terms of the boundary being "branes", the fields with which we are most familiar are located there. The idea that the boundary conditions determine all fields in the "bulk" is called "holography". Specifically, the relation is between arbitrary fundamental fields of 10D IIB superstring theory and color-singlet composite operators on the boundary, which act as sources for the string fields (or vice versa).

The corresponding vacuum on the YM side is the usual YM vacuum. Remember that a vacuum does not define a theory, but is only a particular state in a theory, usually for purposes of perturbation. For example, giving some YM scalars vacuum values corresponds to moving some D-branes away from the boundary. These and other modifications to the vacuum, such as modifying the S^5, can be useful for more general correspondences, like "AdS/QCD". In practice, such modifications make the theory more realistic, but more difficult to calculate.

The AdS/CFT relation is summarized by the equation

$$Z_{string}[\phi(x)] = \left\langle e^{\int dx\, \phi \mathcal{O}} \right\rangle_{CFT}$$

On the lefthand side Z is the S-matrix generating functional as calculated in string theory, where ϕ is a background string field satisfying the "free" (on AdS) field equation, evaluated at the boundary. (For most of what we do, these string fields will be taken to be those of 10D IIB supergravity; "stringy" contributions come from the massive fields.) This is the same as the S-matrix element as calculated in ordinary field theory, where external line factors are always fields (not sources). However, unlike ordinary field theory, we keep boundary terms in Z_{string}, so there are quadratic terms, which give 2-point correlators. (For example, $\int (\partial\phi)^2$ is nonzero on shell, because of boundary terms surviving integration by parts. In usual field theory there is no analogous "2-point S-matrix".) Since there really isn't any sensible "momentum space" for AdS (not all translations commute), this is generally done in coordinate space, at least for the extra dimension x_0 of the bulk. Also, "initial/final" states are now states on the AdS boundary x, so the free external field $\phi(x, x_0)$ attached as an external line factor to a vertex is related to the boundary field $\phi(x)$ by a free propagator.

On the righthand side the calculation is performed in a corresponding conformal field theory on the boundary. It's conformal because the conformal group SO(D,2) in D dimensions is the same as the AdS symmetry group in D+1 dimensions. (In particular that means the scale weights of ϕ and $\mathcal{O}$ must add up to D.) The x integral is of course performed on the boundary, where the CFT lives. (There will also be powers of x_0 that cancel.) In that calculation $\mathcal{O}$ is some color-singlet composite of the fundamental fields of the CFT, while ϕ appears only as a source. Of course, one must determine which fundamental field ϕ of the string theory corresponds to which composite of the CFT (including normalization): This is usually done by comparing $\phi\phi$ propagators on the string side with 2-operator correlators $\langle \mathcal{O}\, \mathcal{O} \rangle$ on the CFT side. Just being able to find a match is some confirmation of the AdS/CFT correspondence: It involves only free strings on the string side, but (generally) interacting fields on the CFT side.

The most calculable case is the $AdS_5 \times S^5$ solution to the supergravity sector of the Type IIB superstring. The solution comes from constant Riemann curvature tensor and constant selfdual 5-form field strength for the 4-form gauge field (both with flat indices, for coordinate independence); the other fields vanish (or the dilaton is constant, depending on how you define it). Like the metric, the 5-form has a "direct product" form, being proportional to the 5D Levi-Civita tensor in each sector. As a result, it is effectively a pseudoscalar in each 5D space, and hence also covariantly constant, so its curl and divergence vanish, which are its field equations.

The 2 (dimensionless) couplings g_{YM} and $N$of (super) Yang-Mills are related to the 2 (dimensionless) couplings g_s and $\mathcal{R}^2/\alpha'$ of string theory on AdS by (up to some normalization conventions)

$$g_s = g_{YM}^2 \ , \qquad \left(\frac{\mathcal{R}^2}{\alpha'}\right)^2 = 4\pi N g_{YM}^2$$

Thus perturbation in α' on the string side does not relate directly to that in the 't Hooft coupling on the YM side: Weak coupling in one is strong in the other. The former equation is based on identifying the CFT fields with (some of the) open string states whose ends are attached to the boundary. (The fields are said to "Reggeize": Quantum corrections are expected to show that the massless fields lie on Regge trajectories, giving the other states of the open string.) The motivation is to consider the boundary as N (D$-$1)-branes. (The "string fields" are then the usual IIB closed string fields.)

The latter equation comes from solving the equations of motion of 10D IIB supergravity for an $AdS_5 \times S^5$ solution using just the metric and 4-form. Since the 4-form is the only source of energy-momentum, its charge is related to the radius of curvature: As it appears in the field equations,

$$\frac{1}{g_s^2 \alpha'^4}\frac{1}{\mathcal{R}^2} = \left(\frac{4\pi N}{\mathcal{R}^5}\right)^2 \quad \Rightarrow \quad N = \frac{1}{4\pi g_s}\left(\frac{\mathcal{R}^2}{\alpha'}\right)^2$$

Everything comes from the way g_s appears in the action (with α' to make it dimensionless), dimensional analysis, and the definition of charge (flux on S^5), except the 4π: Because of selfduality the charge N is both electric and magnetic, and therefore (Dirac) quantized to be integer. (Consider separate sets of branes with charges e_1 and e_2: Then $e_1 e_2$ always integer implies the same for e_1 and e_2 separately.) Actually N can have a sign (branes vs. "anti-branes"), and this corresponds to the fact that open string ends can be associated with the N or $\overline{N}$ representation of U(N).

Notation

There are way too many kinds of indices and coordinates. We'll try to maintain some degree of consistency. Much of the following won't make sense till you get to it, but at least you can see some pattern.

General:

When the distinction is relevant, we use indices from

beginning of alphabet :	"flat": tangent space, coset gauge group
middle of alphabet :	"curved": base space(time), coset symmetry group

Super:

For supersymmetric theories (and often nonsupersymmetric ones) with bosonic symmetries that have factors of SO(6) or smaller, spinor notation (indices of defining representations) are usually more convenient. The rules for indices are

Capital :	super (graded)
lower-case roman :	bosonic, internal
lower-case gr$\epsilon\epsilon\kappa$:	fermionic, spacetime
$\underline{\text{underlined}}$:	direct sum of 2
dotṫed, primed$'$:	complex conjugate, other

where "super" or "graded" means an index is interpreted as running over both bosonic and fermionic "values", with the statistics of the object carrying them changing accordingly. Thus (with groups to be defined later)

A :	(P)SU(2,2|N), GL(N|4) (superconformal)
$\underline{a}$:	SU(N) (full internal)
$\underline{\alpha}$:	SU(2,2) (conformal)
$\underline{A}, \underline{A}'$:	GL(2|N/2)^2, GL(2|n)$\otimes$GL(2|N$-$n) (super projective)
a, a' :	SU(2)^2, SU(n)$\otimes$SU(N$-$n) (projective internal)
$\alpha, \dot{\alpha}$:	SL(2,C) (Lorentz)

(An exception is 10D spinor indices α, since $10 > 6$.) Coordinates then tend to carry pairs of indices:

$Z_M{}^A, Z_A{}^M$:	superconformal
$z_{\underline{A}}{}^M, \bar{z}_M{}^{\underline{A}'}$:	rectangles
$w_{\underline{M}}{}^{\underline{N}'}$:	projective
$u_{\underline{M}}{}^{\underline{A}}, \bar{u}_{\underline{A}'}{}^{\underline{M}'}$:	gauge
$x_\mu{}^{\dot{\nu}}$:	spacetime
$y_m{}^{n'}$:	internal (projective)
$\theta_\mu{}^{\underline{n}}, \bar{\theta}_{\underline{m}}{}^{\dot{\nu}}$:	fermionic (supersymmetry)
$\zeta^M, \bar{\zeta}_M$:	supertwistor
$x_{\underline{\alpha}}{}^{\underline{\nu}}, X^{\underline{\mu\nu}}$:	AdS_5
$y_{\underline{m}}{}^{\underline{b}}, Y^{\underline{mn}}$:	S^5
$\theta_{\underline{m}}{}^{\underline{\nu}}(\theta^\alpha), \bar{\theta}_{\underline{\mu}}{}^{\underline{n}}(\bar{\theta}^\alpha)$:	IIB fermions (on $AdS_5 \times S^5$)

(For 4D supertwistors there is a hidden U(1) index for the little group; in D = 6 it becomes an explicit SU(2) index.)

Bosonic:

A small minority of the following text will be about AdS or CFT without reference to the correspondence, supersymmetry, number of dimensions, etc. Rather than waste symbols or alphabets, we'll use some conventions that contradict the above, restricted to bosonic discussions (which won't last long). Single indices are then carried by coordinates:

I	$\mathcal{Y}$	SO(p+1,q+1)
$\mathcal{A}$	X	SO(D,2), SO(p+1,q), SO(p,q+1)
A	$\mathcal{X}$	SO(D,1), SO(p,q)
a	x	SO(D−1,1)
$+,-$	$\mathcal{Y}, X$	SO(1,1)
0	x, y	bulk

General Lie:

Even fewer indices for general algebras (but not the above special cases):

I :	symmetry (adjoint)
ι :	gauge (adjoint)
i :	coset (adjoint)
A :	gauge, arbitrary representation

Miscellaneous symbols:

g :	group element
G :	algebra generator
G :	group
d :	spinor covariant derivative
D :	general covariant derivative
D :	dimension
N :	Yang-Mills U(N)
N :	R-symmetry (S)U(N), number of supersymmetries
$\mathcal{R}$:	radius of curvature, gravitino field strength
R :	curvature
Δ :	dilatation generator, propagator
$\triangleleft$:	scale weight ("spin" piece of dilatation Δ)
K :	conformal boost generator
k :	conformal boost covariant derivative ("spin" piece of K)
S_{ab} :	first-quantized spin
M_{ab} :	second-quantized spin

Grading signs:

We hide signs due to statistics in ordering of graded indices. The analogy is to signs in the Minkowski metric when separating vector indices into time and space pieces: We don't put in those signs explicitly when doing calculations with covariant equations, only just before doing the separation; we even avoid putting in explicit metric tensors by raising and lowering indices. The signs are important when doing the separation: E.g., a traceless, symmetric tensor is a matrix that is traceless with one index up and one down, but symmetric with both the same. In our case, the signs are made explicit just before the separation into bosons and fermions (or when dealing with just one or the other).

The rules are:

1) Moving a fermionic index past a fermionic index produces a minus sign. (Since moving a fermion past a fermion does.)

2) The usual Einstein summation convention applies when adjacent identical indices are ordered upper-left with lower-right, as in $V^A W_A$. (This is a sign convention.) Any adjacent indices contracted the opposite way imply a $-$ sign for the part of the sum over the fermionic part of the index.

3) The ordering of indices on a matrix relevant for matrix multiplication is $M_A{}^B$. (This follows from the previous, since $(MV)_A = M_A{}^B V_B$, $(MN)_A{}^C = M_A{}^B N_B{}^C$, etc.) In particular, the usual Kronecker δ is the one with indices ordered as $\delta_A{}^B$. (So $\delta_A{}^B V_B = V_A$.)

An alternative notation, which is more explicit but messier, would include sign factors such as $(-1)^A$ or $(-1)^{AB}$, where bosonic indices are given the value 0 and fermionic ones are given the value 1. For example, we could write graded antisymmetrization of indices as

$$M_{[AB)} \equiv M_{AB} - M_{BA} \quad \to \quad M_{AB} - (-1)^{AB} M_{BA}$$

We could also write the graded commutator in general as

$$[\mathcal{A}, \mathcal{B}\} \equiv \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \quad \to \quad \mathcal{A}\mathcal{B} - (-1)^{\mathcal{A}\mathcal{B}} \mathcal{B}\mathcal{A}$$

by writing the symbols for the operators themselves as the indices for the operators.

An obvious consequence of the above is the "supertrace":

$$str\, M \equiv M_A{}^A \quad \to \quad (-1)^A M_A{}^A$$

is required for consistency with $str\, MN = str\, NM$. It's the same sign that appears in fermion loops in quantum field theory. (Consider an operator $\bar\psi^A M_A{}^B \psi_B$.)

Cosets

General

The simplest way to study representations of spacetime symmetry groups is on the group space, which includes the usual spacetime coordinates as those for translations. (Here it is the coordinates that are elements of the group, not the fields themselves.) The alternative is to use a Hilbert space approach, but we know that wave functions are more useful in general from quantum mechanics, and especially in quantum field theory, where they become interacting fields. Later we'll focus on applying our results for conformal representations and their properties specifically to the case of 4D N = 4 Yang-Mills. Note that here we are discussing the coordinates themselves living on a group, not the fields living there, so this group should not be confused with the Yang-Mills group or other internal symmetry groups in general. Also, the groups we now have in mind will also be spacetime symmetry groups. However, these distinctions can be moot in some cases, e.g.: (1) R-symmetry is an internal symmetry that's part of the superconformal group. (2) In Kaluza-Klein reduction, the Yang-Mills fields that come from the vielbein have Killing vectors as Yang-Mills group generators. So there will always be close analogies.

We'll begin with a general discussion of wave functions on group spaces. To relate to the more common (at least for quantum mechanics) Hilbert space approach, we will use a Hilbert space notation; but we can think of this space more generally as an infinite dimensional vector space. (The bras then form the dual space to the kets.) We begin by choosing an arbitrary state $\langle 0|$ in this space to call the "vacuum", and define the group G in terms of operators $g(\alpha)$ acting on this space, parametrized by coordinates α. Then a coordinate basis for the group space is given by

$$\langle \alpha| = \langle 0|g(\alpha)$$

where $g(0) = I$, so the vacuum is identified with the identity operator, and thus the origin of the group space. (But the choice of vacuum can be changed by a group transformation, corresponding to a coordinate transformation that changes the choice of origin.) The wave function is then defined with respect to this basis as

$$\psi(\alpha) \equiv \langle \alpha|\psi\rangle = \langle 0|g(\alpha)|\psi\rangle$$

Next we notice that there are 2 different ways for the group to act on these wave functions, corresponding to left and right group multiplication: In terms of the Lie group generators $\widehat{G}_I$ that act on the Hilbert space (as does g), we define differential operators G_I and D_I that act on the wave function:

$$D_I\psi(\alpha) \equiv \langle 0|\widehat{G}_I g(\alpha)|\psi\rangle \ , \qquad G_I\psi(\alpha) \equiv \langle 0|g(\alpha)\hat{G}_I|\psi\rangle = (\hat{G}_I\psi)(\alpha)$$

Thus the derivatives G_I generate an infinitesimal group transformation directly on the state $|\psi\rangle$, while the derivatives D_I act on the vacuum. Thus G_I should be interpreted as symmetry (or "isometry") generators. At this point, we note only that D_I commute with G_I,

$$[G_I, D_J\} = 0$$

since left and right multiplication in the Hilbert space commute, because multiplication there is associative. We therefore call D_I "covariant derivatives" (but actually they are *invariant* under symmetry transformations at this point of our discussion.) Similarly we can define symmetry invariant products and differentials

$$g(\alpha') = g(\alpha)(g_0)^{-1} \ , \quad g(\alpha_{12}) \equiv g(\alpha_1)g^{-1}(\alpha_2) \quad \Rightarrow \quad \alpha'_{12} = \alpha_{12}$$

$$i\, d\alpha^M E_M{}^I(\alpha)\hat{G}_I = (dg)g^{-1} \quad \Rightarrow \quad (d\alpha^M E_M{}^I)' = d\alpha^M E_M{}^I$$

(The latter is the infinitesimal version of the former.) These are related to the covariant derivatives as

$$D_I = E_I{}^M \partial_M$$

(where $\partial_M = \partial/\partial\alpha^M$ and $E_I{}^M$ is the inverse of $E_M{}^I$), and similarly for explicit coordinate representations for the symmetry derivatives G_I from $g^{-1}dg$. (As usual, this "vielbein" can be used to define an integration measure invariant under the symmetry group.)

However, we know from general group theory constructions (e.g., for the simple case of SU(2) that you learned in your graduate quantum mechanics course), that it's sufficient to start with a vacuum that is a "lowest-weight state", annihilated by "lowering operators" (or maybe "highest" and "raising", depending on your preference) that are themselves generators of the group. In general, these lowering operators generate a subgroup of the original group. In an appropriate basis, we can then divide up the group index as

$$I = (\iota, i) \ , \qquad \langle 0|\hat{G}_\iota = 0$$

where $\hat{G}_\iota$ generate a subgroup H of G. (We don't yet specify whether G_ι are hermitian.) It then follows that the corresponding covariant derivatives vanish on the wave function:

$$D_\iota\psi(\alpha) = \langle 0|\hat{G}_\iota g(\alpha)|\psi\rangle = 0$$

(In CFT parlance, these have recently been dubbed "shortening" conditions, at least in the fermionic case.) In an appropriate coordinate system, this condition eliminates the dependence of the wave function on the coordinates of H; in group theory language we call the space parametrized by the remaining coordinates the "coset space" G/H. If we then write left and right group multiplication as

$$g'(\alpha) = h(\alpha)g(\alpha)g_0^{-1}$$

we can choose $h(\alpha)$ to be an element of the subgroup H, but otherwise to have arbitrary dependence on the coordinates α (since it annihilates the vacuum), while g_0 is independent of α. Thus $h(\alpha)$ describes a "gauge" ("isotropy", "stability group", or "little group") transformation that can be used to eliminate dependence on the H coordinates, while g_0 yields a symmetry transformation on the state $|\psi\rangle$ that's basis independent.

The gauge group constraints cause *first-quantized, spontaneous* symmetry breaking: first-quantized because the vacuum and all other states are single-particle states (so no Goldstone particles), spontaneous because we consider the action of $\hat{G}_I$ on the vacuum $\langle 0|$, not the state $|\psi\rangle$ (so the symmetry is still realized, but nonlinearly in the coordinates, at least when a unitary gauge is chosen).

This is sufficient for coordinate representations. But usually in quantum mechanics we want to consider more general representations by adding "spin" to such "orbital" generators. This is accomplished by first introducing spin degrees of freedom, and then tying them to the group by modifying the gauge-group constraints. So we first introduce a basis $\langle_A|$ for a matrix representation $\tilde{G}_\iota$ for the gauge group (since that's all we can fix on the vacuum),

$$\langle_A|\hat{G}_\iota = \tilde{G}_{\iota A}{}^B\langle_B|$$

We then define a basis for the Hilbert space by using this gauge group basis as our new (degenerate) vacuum,

$$\langle_A, \alpha| \equiv \langle_A|g(\alpha)$$

to get the generalizations of the previous

$$\psi_A(\alpha) \equiv \langle_A, \alpha|\psi\rangle \quad \Rightarrow \quad D_\iota\psi_A(\alpha) = \tilde{G}_{\iota A}{}^B\psi_B(\alpha), \qquad G_I\psi_A(\alpha) = (\hat{G}_I\psi)_A(\alpha)$$

The wave function now depends also on the gauge-group coordinates, but this dependence is fixed independent of the state: For example, in the 2-exponential coordinate system $\alpha^I = (\beta^i, \gamma^\iota)$ where dependence on the gauge and coset generators is explicitly factorized,

$$\psi_A(\alpha) = \langle_A| e^{i\gamma^\iota \hat{G}_\iota} e^{i\beta^i \hat{G}_i} |\psi\rangle = (e^{i\gamma^\iota \tilde{G}_\iota})_A{}^M \langle_M| e^{i\beta^i \hat{G}_i} |\psi\rangle \equiv e_A{}^M(\gamma)\psi_M(\beta)$$

where $e_A{}^M$ is a "vielbein" depending on only the gauge coordinates γ and independent of the state ψ, and can be gauged to the identity in a "unitary" gauge, while ψ_M depends on only the coset coordinates β. Since we know D in terms of derivatives, $D_\iota = \tilde{G}_\iota$ can be solved to replace partial derivatives with respect to gauge-group coordinates with matrices, in both D_I and G_I.

A special case of this treatment is when H consists of the Cartan (maximal abelian) subgroup and the lowering operators it defines. Then only the Cartan subalgebra needs a nontrivial matrix representation, but since this algebra is abelian, the vacuum needs only 1 component, and the "matrix" is just a set of eigenvalues. This feature makes this choice preferable, except when we want to make more of the symmetry manifest. In an appropriate coordinate system, the vielbein factor above is then simply

$$e(\gamma) = e^{i\gamma^\iota n_\iota}$$

where A and M have been dropped because that take only a single value, ι runs over only the Cartan subalgebra, and n_ι are their eigenvalues.

SU(2)

A simple example of all these concepts is the group SU(2). (In fact, one method of analysis for general groups is to consider SU(2) subgroups.) Here we'll look at SU(2) in some detail to illustrate the concepts discussed so far, as well as motivate the analysis to come for the special case of superconformal groups.

If we make the usual division of the generators into D_0 and $D_\pm$, we can pick D_0 and D_- to generate the gauge group H, by setting

$$D_-\psi = 0 \ , \quad D_0\psi = -j\psi$$

on the wave function. After the first constraint eliminates 1 of the 3 coordinates of SU(2), the 2 remaining coordinates can be taken as the usual ones for the sphere. (The eliminated coordinate was the usual third Euler angle of that parametrization of SU(2).) The second constraint then fixes the dependence on the azimuthal angle ϕ. This gives a representation, but a reducible one: It still has arbitrary dependence on the polar angle θ. There are various methods (but all with the same result) to limit this dependence to give a finite-dimensional representation: One is to fix, for some nonnegative integer n,

$$(D_+)^{n+1}\psi = 0$$
$$\Rightarrow \quad 0 = D_-(D_+)^{n+1}\psi \sim (n+1)(n-2j)(D_+)^n\psi$$

bu pushing D_- and all subsequent D_0's to the right. So recursively we find: If $n < 2j$, $\psi = 0$; if $n > 2j$, the same constraint holds for lower n at least down to $2j$; and only for $n = 2j$ can the constraints terminate. Thus we can impose nontrivially just

$$(D_+)^{2j+1}\psi = 0$$

This additional constraint is not a coset constraint (except for the trivial case $j = 0$); its analog for the superconformal group will be analyzed soon: They will be interpreted as field equations. In particular, the case $j = \frac{1}{2}$ satisfies a quadratic condition, which is related to the quadratic nature of free field equations. In general, such equations will be satisfied only by field strengths on shell, not by gauge fields nor off-shell field strengths, which will contain infinite-dimensional representations of SU(2) (or a larger R-symmetry group): For extended 4D supersymmetry, superfields or their gauge parameters have infinite numbers of auxiliary fields.

For the usual reasons it may be convenient to use various different sets of coordinates. Here for pedagogical purposes we'll derive a particular set of somewhat unusual coordinates from a more familiar one. (For the superconformal case, we'll just choose our coordinates directly.) The chief unusual feature is that our coordinates are defined by Wick rotation: As we know from the usual field theory method of Wick rotation to Euclidean space, this technique can wreak havoc with reality conditions. For SU(2), a standard space for defining SU(2) is the sphere, which is the coset SU(2)/U(1) (i.e., SO(3)/SO(2)). For algebra and analysis, it's often more convenient to use stereographic coordinates than the usual spherical (trigonometric) ones. Then the metric and volume are given by similar expressions:

$$ds^2 = \frac{dz\, d\bar{z}}{(1+\frac{1}{2}z\bar{z}/\mathcal{R}^2)^2}\ , \quad d^2V = \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1+\frac{1}{2}z\bar{z}/\mathcal{R}^2)^2}$$

where $\mathcal{R}$ is the radius of the sphere.

If we perform some sort of Wick rotation, z and $\bar{z}$ no longer need be complex conjugates: Thus we throw caution to the wind and freely make the coordinate change

$$z \to \mathcal{R}z\ , \quad \bar{z} \to -2\mathcal{R}/\bar{z}$$

which implies

$$ds^2 \to 2\mathcal{R}^2\frac{dz\, d\bar{z}}{(\bar{z}-z)^2}\ , \quad d^2V \to i\mathcal{R}^2\frac{dz\wedge d\bar{z}}{(\bar{z}-z)^2}$$

(This looks more like Wick rotation in string theory for the usual coordinate $\rho = ln\, z$, so $\bar{\rho} \to$ constant $-\bar{\rho}$.) Except for normalization, these are recognized as (Liouville-Beltrami-)Poincaré coordinates for the hyperbolic plane (the Poincaré half-plane). This is the coset SU(1,1)/U(1) (i.e., SO(2,1)/SO(2), or maybe SO(2,1)/SO(1,1)), so clearly we have Wick rotated. It's now convenient to separate z and $\bar{z}$ into (what would have been) its real and imaginary parts as

$$z = y + i\mathcal{R}y_0 \ , \quad \bar{z} = y - i\mathcal{R}y_0$$

to give these coordinates in their usual form

$$ds^2 = -\tfrac{1}{2}\frac{dy^2 + \mathcal{R}^2 dy_0^2}{y_0^2} \ , \quad d^2V = -\tfrac{1}{2}\mathcal{R}\frac{dy \wedge y_0}{y_0^2}$$

where $y_0 > 0$ for the upper half-plane (which is disconnected from the lower half).

Obviously we can realize SU(2) (or SU(1,1), depending on where we put our i's) on these coordinates. But it will be simpler to consider the limit $\mathcal{R} \to 0$, which takes the hyperbolic plane to a "projective" cone:

$$ds^2 \to \frac{dy^2}{y_0^2} \ , \quad d^2V \to \frac{dy \wedge dy_0}{y_0^2}$$

The realization of SU(2) on this space (at least in the way that generalizes to higher-dimensional spheres) is a bit simpler; we'll give a nice derivation later, but for this simple case it's easy to check that the answer is

$$y \to \frac{ay+b}{cy+d} \ , \quad y_0 \to \frac{y_0}{(cy+d)^2} \ ; \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2)$$

which leaves both the degenerate metric and the volume element invariant.

But already we can realize SU(2) on y alone; why do we need y_0? The coset with just y is SO(2,1)/ISO(2). (ISO(2) is rotations and translations in D = 2. This can also describe SO(2,1)/ISO(1,1).) Because of the contraction, it's actually the "I" that came from contracting the SO(2) of SO(2,1)/SO(2); the new SO(2) is what eliminates y_0. There is a simple relation to our Hilbert space discussion of SU(2) previously: D_- killed the first coordinate, while D_0 fixes y_0. Explicitly, we have

$$D_0 = y_0\partial_0 \ , \quad D_+ = y_0\partial_y$$

These are invariant under SU(2) except for D_+ under the "c" part, because we eliminated the third coordinate (which would appear in the D's) with D_-. Then the wave function for spin j takes the form

$$\psi_{(j)}(y, y_0) = (y_0)^{-j}\sum_{n=0}^{2j}\binom{2j}{n}\psi_{(j)n}y^n$$

(The $(y_0)^{-j}$ is the "vielbein" factor.) If we look at infinitesimal transformations, we see that the "b" part gives translations in y, giving $\delta c_n \sim c_{n+1}$. The "a, d" part scales $\delta c_n \sim (n-j)c_n$; this is a phase for SU(2). Finally, the "c" part gives $\delta c_n \sim c_{n-1}$. Explicitly, we have

$$G_+ = \partial_y \ , \quad G_0 = y\partial_y + y_0\partial_0 \ , \quad G_- = \tfrac{1}{2}y^2\partial_y + yy_0\partial_0$$

The coordinate y_0 was necessary to implement SU(2) completely as a coordinate transformation; otherwise we would need a "spin" operator to give the action of D_0.

Next, we consider complex conjugation and reality. Unitarity of SU(2) (along with the unit determinant condition) give

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

which tells us that the pair $(-1/y^*, (y_0/y^2)^*)$ transforms the same way as (y, y_0). So we define the charge conjugate of a field as

$$(\mathcal{C}\psi)(y, y_0) \equiv \left[\psi\left(-\frac{1}{y^*}, \frac{{y_0}^*}{y^{2*}}\right)\right]^*$$

where $\mathcal{C}\psi$ depends only on y and y_0 because of the double complex conjugation. In terms of components, we have

$$(\mathcal{C}\psi)_n = (-1)^{2j-n}{\psi_{2j-n}}^*$$

This nonlinear realization can also be derived from a projective approach, as CP(1). Starting from the defining representation

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \ , \quad z' = gz$$

we define

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{\sqrt{y_0}}\begin{pmatrix} y \\ 1 \end{pmatrix}$$

which yields the above transformation laws for y and y_0. The defining representation of SU(2) is pseudoreal, so we know we also have

$$\mathcal{C}z = \begin{pmatrix} -{z_2}^* \\ {z_1}^* \end{pmatrix} = \frac{y^*}{\sqrt{{y_0}^*}}\begin{pmatrix} -1/y^* \\ 1 \end{pmatrix}$$

as above. If we want to eliminate y_0, we do something similar to a coset by gauging by an overall complex scale ($\lambda(z)$ has one component), and finding a gauge invariant:

$$z \to \lambda(z)z \quad \Rightarrow \quad y = \frac{z_1}{z_2}$$

If we define a spin-j state as totally symmetric in $2j$ spinor indices, and contract each spinor index with a factor of the above spinor z, we arrive at exactly the wave function $\psi_{(j)}(y, y_0)$ found above.

Because of the Wick rotation and singular limit $\mathcal{R} \to 0$, integration is funny. We can define an SU(2)-invariant bilinear inner product by starting with an ordinary-looking integral and inserting a δ function in y to make a double integral:

$$\langle A|B\rangle = \oint \frac{dy_0}{y_0}\frac{dy_0'}{y_0'} \int \frac{dy}{y_0}\frac{dy'}{y_0'} \left[\sqrt{y_0 y_0'}\,\delta(y - y')\right] A(y, y_0) B(y', y_0')$$

where we have arranged factors into invariant combinations. (We won't need locality in y_0.) Then we substitute

$$\sqrt{y_0 y_0'}\,\delta(y - y') \to \frac{y_0 y_0'}{(y - y')^2 + y_0 y_0'}$$

up to some normalization, valid for small y_0 and y_0', which we can take for our y_0 contours. If we then Taylor expand the "δ" in $y_0 y_0'$, these contour integrals match up equal spins ($\sim y_0^{-j}$) in A and B. (This is a cheat, but similar to how δ's will appear in the AdS/CFT correspondence, so we can define our source term there accordingly.) The y integrals then give an invariant inner product for each spin,

$$\int \frac{dy\, dy'}{(y - y')^{2j+2}} A_{(j)}(y) B_{(j)}(y')$$

This can be evaluated by appropriate choices of contours, e.g., the first open between arbitrary points a and b, and the second as a closed figure 8 about those points (giving a result independent of a, b). The result is proportional to

$$\sum_{n=0}^{2j} (-1)^n \binom{2j}{n} A_{(j)n} B_{(j)2j-n}$$

(as also expected from treating A and B as having $2j$ spinor indices).

Conformal

Before looking at any explicit representations (even for just coordinates) of conformal groups, we want to give the algebras, and identify which generators are familiar from massive theories and which are new. Let's begin with the ordinary conformal group (no supersymmetry), in arbitrary spacetime dimensions (really just D > 2), where the group is SO(D,2). We choose a lightcone basis for the extra (with respect to

SO(D−1,1)) space and time "dimensions": In terms of a vector index, $\mathcal{A} = (+, a, -)$. So the metric is

$$\eta^{\mathcal{AB}} = \begin{matrix} & - & b & + \\ - & 0 & 0 & -1 \\ a & 0 & \eta^{ab} & 0 \\ + & -1 & 0 & 0 \end{matrix}$$

Then the SO(D,2) generators can be divided up as

$$G_{\mathcal{A}}{}^{\mathcal{B}} = \begin{matrix} & - & b & + \\ - & G_-{}^- & G_-{}^b & 0 \\ a & G_a{}^- & G_a{}^b & G_a{}^+ \\ + & 0 & G_+{}^b & G_+{}^+ \end{matrix}$$

$$= \begin{pmatrix} \text{scale} & \text{translation} & 0 \\ \text{conformal boost} & \text{Lorentz} & \text{translation} \\ 0 & \text{conformal boost} & \text{scale} \end{pmatrix}$$

Note that $G^{\mathcal{AB}} = -G^{\mathcal{BA}}$ as usual, but we have lowered one index with the metric (which only really matters for the ± indices) for comparison to later results, and so the dimensionless stuff lies along the diagonal. The algebra then follows as usual from the SO(D,2) algebra. So we have the familiar Poincaré group, as well as scale transformations (or "dila(ta)tions"), but also "conformal boosts", which we'll say more about later. (SO(D,2)⊃SO(D−1,1)⊗SO(1,1), and the scale generator G^{+-} of the SO(1,1) counts the number of +'s minus −'s on the generators, giving the engineering dimensions.) At this point we only note that they allow us to imbed the usual massless symmetries into a classical (orthogonal) group, which is easier for certain applications. (Notational point: Because of a shortage of alphabets, when discussing supersymmetry, these letters will not represent the types of indices indicated here.)

SO(D,2) is also the anti-de Sitter group in D+1 dimensions; the appropriate decomposition is then to separate 1 spatial index to get just translations (now nonabelian) and Lorentz. (We'll make a few passing remarks on AdS from time to time, but consider it in detail much later.)

For the conformal case we have the full set of covariant derivatives

$$D^{+a} = P^a \ , \quad D^{ab} = S^{ab} \ , \quad D^{+-} = \triangleleft \ , \quad D^{-a} = k^a$$

for momentum P, spin S, scale weight $\triangleleft$, and *covariant derivative* for conformal boosts k. Out of these we usually choose the gauge group to be generated by all generators of nonpositive engineering dimension:

$$\text{conformal:} \quad D_\iota = (S, \triangleleft, k)$$

Specifically, we set $k = 0$ without loss of generality (for physics), while $\triangleleft$ is fixed to some constant and S gets the usual matrix representations. Then the only arbitrary coordinate dependence is on the translation coordinates.

Note that we are here considering SO(D,2) as the conformal group in D dimensions, not the AdS group in D+1 dimensions, for which a related analysis can be made. There k is replaced by $k - P$, giving the gauge group SO(D,1) with S; $\triangleleft$ must then be dropped from the gauge group, hence the extra dimension: $P + k$ and $\triangleleft$ (no longer just a number) are then covariant, nonabelian translations in D+1 dimensions.

$$\text{AdS:} \quad D_\iota = (S, k - P)$$

This corresponds to separating out a timelike index as $\mathcal{A} = (0, A)$, where

$$G_{AB} = D_\iota \ , \qquad G_{0A} = (P + k, \triangleleft)$$

So, SO(D,2)/SO(D,1) is AdS$_{D+1}$, while SO(D,2)/ISO(D−1,1) is Minkowski space in D dimensions (more on this later). (Notational note: "0" will generally be used differently below, as a spacelike index.)

We will not give more detail on this approach now, since a nicer and more useful form of the algebra in 4 dimensions (especially for spinors) comes from noting that SO(4,2) is the same algebra as SU(2,2).

CFT

Groups

The cases where there are infinite sets of superconformal groups, corresponding to graded generalizations of classical groups, are for D = 3,4,6, where the groups are essentially the same except for being over the real, complex, or quaternionic numbers. (Octonions are a problem, but would in principle be for D = 10. There is a single "exceptional" supergroup for D = 5.) Sticking to the usual representation over just real or complex numbers, we have:

$$\mathrm{D} = \begin{cases} 3: & \mathrm{OSp(N|4)} \\ 4: & \mathrm{(P)SU(N|2,2)} \\ 6: & \mathrm{OSp^*(8|2N)} \end{cases}$$

"OSp(*)" means that the group preserves a graded metric, orthogonal (symmetric) over the first set of indices, symplectic (antisymmetric) over the second. OSp has a real defining representation, OSp* has pseudoreal; thus the first case has bosonic subgroup SO(N)⊗Sp(4), while the last has SO*(8)⊗USp(2N). For the algebras, we have Sp(4) = SO(3,2), SU(2,2) = SO(4,2), SO*(8) = SO(6,2). These are also AdS supergroups in D = 4,5,7. Most of the time we'll focus on D = 4.

The superconformal group with N supersymmetries in 4 dimensions is the graded group (P)SU(N|2,2) (or equivalently, (P)SU(2,2|N); we use the other order because it will prove convenient to list first the index we treat as bosonic). This makes for more convenient algebra, since we can treat it as GL(N|4) with a funny reality condition (hermiticity, or unitarity for the group) and 1 (S) or 2 (PS, for N = 4) elements missing. So we begin by defining GL(N|4) in terms of the defining representation, an N+4 vector with N components of one statistics and 4 of the other: In accordance with the usual spin-statistics relation of physics, we treat the 4-spinor index as anticommuting and the N-valued internal index as bosonic. This representation of GL(N|4) then consists of otherwise arbitrary matrices that preserve these statistics. So we can divide up the graded matrices, or the corresponding generators, into their bosonic and fermionic parts:

$$G_A{}^B = \begin{array}{c} \\ \underline{a} \\ \underline{\alpha} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad\underline{b} & \quad\underline{\beta} \end{array} \\ \begin{pmatrix} G_{\underline{a}}{}^{\underline{b}} & G_{\underline{a}}{}^{\underline{\beta}} \\ G_{\underline{\alpha}}{}^{\underline{b}} & G_{\underline{\alpha}}{}^{\underline{\beta}} \end{pmatrix} \end{array}$$

where underlined Latin indices are bosonic internal R-symmetry GL(N) indices and underlined Greek are fermionic spacetime GL(4) spinor indices. But SL(4) (which over the reals is actually SO(3,3)), which will become SU(2,2) after we apply reality, is the 4D conformal group, so we should really further divide the GL(4) spinor indices into 2 Lorentz GL(2) Weyl spinor indices, since $SL(2)^2$ (which over the reals is SO(2,2)) will become SL(2,C), which is the same algebra as SO(3,1). After reordering as determined by dimensional analysis (again fixed by SO(4,2)⊃SO(3,1)⊗SO(1,1), or now $SL(4)\supset SL(2)^2\otimes GL(1)$, where the 2 kinds of SL(2) indices have opposite GL(1) weight),

$$G_A{}^B = \begin{array}{c} \\ \alpha \\ \underline{a} \\ \dot{\alpha} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \beta & \underline{b} & \dot{\beta} \end{array} \\ \begin{pmatrix} G_\alpha{}^\beta & G_\alpha{}^{\underline{b}} & G_\alpha{}^{\dot{\beta}} \\ G_{\underline{a}}{}^\beta & G_{\underline{a}}{}^{\underline{b}} & G_{\underline{a}}{}^{\dot{\beta}} \\ G_{\dot{\alpha}}{}^\beta & G_{\dot{\alpha}}{}^{\underline{b}} & G_{\dot{\alpha}}{}^{\dot{\beta}} \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} \text{Lorentz + scale} & \text{supersymmetry} & \text{translation} \\ \text{S-supersymmetry} & \text{internal} & \text{supersymmetry} \\ \text{conformal boost} & \text{S-supersymmetry} & \text{Lorentz} - \text{scale} \end{pmatrix}$$

where we now have also internal symmetry, supersymmetry, and "Special"-supersymmetry. Again the scale weights (engineering dimensions) increase from lower-left to upper-right. The internal symmetry is U(N), except for N = 4, SU(4) (from the P).

The commutation relations, except for statistics, are those expected for multiplying general matrices:

$$[G_A{}^B, G_C{}^D\} = \delta_C^B G_A{}^D - \delta_A^D G_C{}^B$$

In the alternative notation, this would be

$$[G_A{}^B, G_C{}^D\} = (-1)^B \delta_C^B G_A{}^D - (-1)^{(B+C)D+BC} \delta_A^D G_C{}^B$$

The "S" condition is then written as

$$str\, G = G_A{}^A \equiv G_{\underline{a}}{}^{\underline{a}} - G_{\underline{\alpha}}{}^{\underline{\alpha}} = 0$$

in terms of the "supertrace", where we have used the implicit signs for the A superindex but never do so for the subindices. Since group elements g are exponentials of elements of the Lie algebra, this condition can also be written in terms of the superdeterminant as

$$sdet\, g = 1\ , \quad sdet(e^G) \equiv e^{str\, G}$$

(Conversely, if *det* is defined by Gaussian integration, *sdet* can be defined as for integrals over graded coordinates, and then the definition of *str* follows.) The "P" condition is the gauge invariance

$$\delta G_A{}^B = \delta_A{}^B \mathcal{O}$$

This is independent from S only if $str(I) = 0$; in our case that means N = 4. In that case, the S generator acts on the other generators but is not produced from them by commutators, while for the P generator the opposite is true.

In general, S must be treated as a gauge invariance (like P), for which $str\, G = 0$ is a gauge condition. This is because $str\, D$ is the "superhelicity" (more on this later), which in general takes a nonvanishing value. For example, for N = 4, Yang-Mills is superhelicity 0, while supergravity is superhelicity 1.

Field equations

We have reduced arbitrary representations of groups (general functions defined on the group space) by imposing constraints linear in the covariant derivatives, which does not interfere with symmetry transformations. To define irreducible representations of groups, we may need further conditions, quadratic or higher order in the covariant derivatives. (For people who call the linear conditions "shortening", the higher order ones are called "semi-shortening".) This approach is effectively what is done in the usual analysis of the ordinary conformal group, or just the Poincaré group: For example, in Wigner's analysis of 4D Poincaré representations, the spin is defined essentially as the covariant derivative left over when orbital angular momentum is subtracted from the full Lorentz generators. Then the Klein-Gordon and Pauli-Lubański equations, expressed in terms of group generators, directly reduce to expressions quadratic in covariant derivatives.

We first briefly discuss some examples for the ordinary conformal group in arbitrary dimensions; then we give a more detailed analysis for the more interesting case of the 4D superconformal group. The generators thus carry SO(D,2) vector indices, and we can decompose expressions quadratic in covariant derivatives as

$$\mathrm{SO(D,2)} : \left(\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \right)_S = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \oplus \begin{array}{|c|c|}\hline & \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \oplus \bullet$$

Not all of these should be constrained to vanish on the wavefunction, or we get trivial representations. A closer analysis reveals that we should only consider subsets of the second representation, such that their algebra closes with themselves and with the linear constraints. This is the conformal representation that includes the usual Klein-Gordon equation. The first yields trivial representations. The last 2 describe only scalars if combined with the second. (When applied separately they describe anti-de Sitter space, but only for the massless case:. The last is the massless AdS Klein-Gordon equation.)

Decomposing SO(D,2) indices to SO(D−1,1), we find:

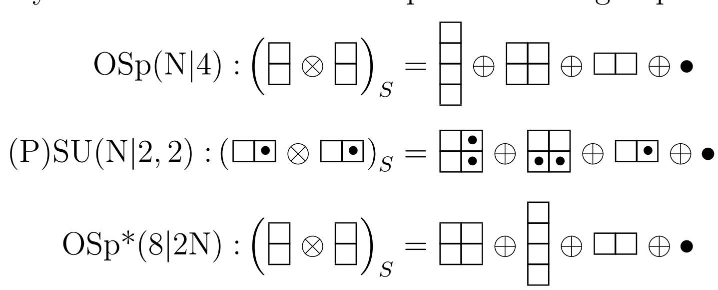
dim	$\square\square$	DD
2	$++$	P^2
1	$+a$	$S^{ab}P_b + \triangleleft P^a$
0	ab	$S^{ac}S_c{}^b + k^{(a}P^{b)} - tr$
0	$+-$	$S^2 + (\mathrm{D}+1)\triangleleft^2 + (\mathrm{D}-2)k\cdot P$
−1	$-a$	$S^{ab}k_b - \triangleleft k^a$
−2	$--$	k^2

(All these expressions should be symmetrized with anticommutators.) Since S is part of the gauge group, any quadratic constraints we choose must be covariant with respect to it. K is also part, and it takes any item in the table to ones of lower dimension. If we impose $P^2 = 0$, all the rest follow, and we find general free massless field equations (for field strengths: Dirac equation, Maxwell's equations, Einstein's equations in terms of the Weyl tensor, etc.), with $\triangleleft$ determined by S, from the $SP + \triangleleft P$ constraint, and S constrained to rectangular Young tableaux of height D/2 from the next. (In odd dimensions only spin 0 and 1/2 are allowed.)

Besides the linear constraints, which define the gauge group, and the quadratic constraints, which give field equations, there can also be constraints higher-order in derivatives: E.g., conservation laws for currents appear at third order as $S_a{}^b S_b{}^c P_c + ...$. The constraints at any number of derivatives always imply those of higher derivatives, so the general pattern is that these constraints are always consistent subsets (with respect to their algebra closing) of

$$D_{A_1}{}^{A_2} D_{A_2}{}^{A_3} D_{A_3}{}^{A_4} ... D_{A_n}{}^{A_{n+1}} - tr = 0$$

A similar analysis can be made for the superconformal group in D = 3,4,6:



The second representation of each is the relevant one for the conformal group in flat space. (But again the last 2 are relevant for AdS.)

We use graded symmetrization, so "symmetric" in the tableaux means symmetric in the former label of the group, since in the first and last cases that has the symmetric metric. For the unitary case, dots in boxes refer to the complex conjugate representation; ordering of the dotted block with respect to the undotted block is arbitrary. We thus find a supersymmetrization of the results described above for the bosonic case, only vector index notation has been replaced by spinor index notation: For example, since P now appears as $P_\alpha{}^{\dot\beta}$ in D = 4 (cf. above matrix decomposition for $G_A{}^B$), the massless Klein-Gordon operator appears in D = 4 as $P_{[\alpha}{}^{[\dot\beta}P_{\gamma]}{}^{\dot\delta]}$. (Antisymmetrization effectively contracts SL(2) indices, so Lorentz indices get contracted in these equations.)

In general, we can apply subsets of these quadratic field equations: E.g., just the first 2 in the list of conformal equations give those are arbitrary massless (but not necessarily conformal) representations of the Poincaré group. Or we can apply some of the lower equations, consistently with $k = 0$, to get conformal representations; we then need also higher-derivative constraints to get irreducible ones. But if we apply the full (super)conformal set (at any order in derivatives), to get "shorter" conformal representations, we can express them in terms of just symmetry generators as well as just covariant derivatives, since D and G are (coordinate-dependent) linear combinations of each other. (This can be seen, e.g., by their derivation in terms of $g^{-1}dg$ and $(dg)g^{-1}$.)

To see that this classification of (quadratic) constraints is consistent with the usual identification of the superconformal mass shell, we evaluate them in the supertwistor representation. (Supertwistors are basically a covariantization of the lightcone gauge. They have received a lot of attention recently in the search for simpler methods to calculate 4D scattering amplitudes, especially for N = 4 Yang-Mills.) Supertwistors also exist for the special values D = 3,4,6: The generators G in terms of supertwistors ζ are

$$\begin{array}{lll} \mathrm{D}=3: G_{AB}=\tfrac{1}{2}[\zeta_A,\zeta_B\}, & \{\zeta_A,\zeta_B]=\eta_{AB} & \\ \mathrm{D}=4: G_A{}^B=\tfrac{1}{2}[\bar\zeta_A,\zeta^B\}, & \{\bar\zeta_A,\zeta^B]=\delta_A^B, & h=\tfrac{1}{2}[\zeta^A,\bar\zeta_A\} \\ \mathrm{D}=6: G_{AB}=\tfrac{1}{2}\{\zeta^a{}_A,\zeta_{aB}], & [\zeta_{aA},\zeta_{bB}\}=C_{ab}\eta_{AB}, & h_{ab}=\tfrac{1}{2}\{\zeta_a{}^A,\zeta_{bA}] \end{array}$$

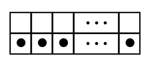
with indices A, B in the defining representation (and defining Sp(2) = SU(2) indices a, b for D = 6) and η the OSp metric (and C the Sp(2) metric), where h is the superhelicity (generating the little group U(1) for D = 4, or SU(2) for chiral D = 6). For D = 4 we have given the U(N|2,2) generators; in coordinate representations, only (P)SU(N|2,2) need be defined. Note that twistors are essentially γ-matrices for OSp, or creation/annihilation operators for U, satisfying graded *anti*commutation relations; thus the bosonic ones anticommute with the fermionic ones. Since each supertwistor carries one index, it's easy to order them (ζ and $\bar\zeta$ separately for D = 4) in the field equations so that the graded symmetrization defined by the Young tableaux reduces them by their commutation relations to δ terms (which are also generated by ordering them), which are subtracted by the definition of the Young tableaux. We'll return to supertwistors later.

Note that for D = 4, even for representations other than supertwistors,

$$superhelicity \equiv str\ G = str\ D$$

where $str\ G \equiv G^A{}_A = (-1)^A G_A{}^A$ (making the signs explicit for the more common index ordering). The equivalence for G and D is because $str\ \hat{G}$ is abelian, and so commutes with $g(\alpha)$. The superhelicity is thus part of the "spin", and becomes nontrivial when relaxing the "S" constraint of the superconformal group (P)SU. It's related to the Abelian gauge invariance $\delta G_A{}^B = \delta_A^B \mathcal{O}$, except in the case N = 4, where $str\ I = 0$, and the latter gauge invariance is the definition of the "P" in "PSU(4|2,2)". In the twistor representation, it counts the number of $\bar{\zeta}$'s minus ζ's.

Again, all higher-derivative constraints follow this pattern: E.g., for D = 4, all of them are consistent subsets of the parts of the form



(Note that for N = 2 and no fermionic indices this agrees with our earlier result for SU(2).) Explicitly, we have consistent subsets of

$$D_{(A_1}{}^{(B_1} ... D_{A_{n+1}]}{}^{B_{n+1}]} - \delta\ terms = 0$$

4D superspaces

Supercosets

An element of the group (or algebra) GL(N|4) is just an arbitrary real matrix (with appropriate grading). Thus, no consideration of exponentiation or constraints on the coordinates is necessary. This construction is simpler than using (the defining representation of) SO(D,2) for arbitrary D (for N = 0), or the superconformal groups OSp(N|4) for D = 3 or OSp*(8|2N) for D = 6, since the latter require a quadratic (orthosymplectic) constraint on the matrices. The fact that the relevant cosets are projective spaces is a significant further simplification. Then we can relate to (P)SU(N|2,2) by (1) implementing (P)S as gauge invariances (or constraints), and (2) finding the modification of the reality conditions corresponding to Wick rotating to GL(N|4).

We therefore use as coordinates the unconstrained matrix $Z_M{}^A$, or its inverse $Z_A{}^M$, where "A" and "M" are "local" (gauge) and "global" (symmetry) GL(N|4) indices, respectively. The symmetry generators and covariant derivatives are then very simple:

$$G_M{}^N = Z_M{}^A \partial_A{}^N, \qquad D_A{}^B = (\partial_A{}^M) Z_M{}^B$$

where $\partial_A{}^M = \partial/\partial Z_M{}^A$, and we ordered the derivatives to the left in D to allow use of matrix notation (later), and keep grading signs trivial. (The derivatives are meant to act only to the right of the Z. It's a kind of "normal ordering".) This corresponds to the identification of the symmetry generators G acting to the left of the group coordinates $Z_M{}^A$ and the covariant derivatives D to the right, or the reverse for the inverse $Z_A{}^M$. This is seen from the index structure, but easily checked by performing an infinitesimal transformation on Z generated by $\epsilon_M{}^N G_N{}^M$ or $\epsilon_A{}^B D_B{}^A$. In fact, Z is simply $g(\alpha)$ in the defining representation: i.e., $g(Z) = Z$.

The usual full superspace is then obtained by gauging away the diagonal blocks, as well as the lower-left triangle ("lowering operators"), leaving only the coordinates for translations and supersymmetry:

$$Z_M{}^A \to \begin{array}{c} \\ \mu \\ \underline{m} \\ \dot\mu \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \alpha & \underline{a} & \dot\alpha \end{array} \\ \begin{pmatrix} I & \theta_\mu{}^{\underline{a}} & x_\mu{}^{\dot\alpha} \\ 0 & I & \bar\theta_{\underline{m}}{}^{\dot\alpha} \\ 0 & 0 & I \end{pmatrix} \end{array}$$

More choices can be obtained by also subdividing the N-valued internal indices, perhaps not equally, into n and N−n: Again gauging away diagonal blocks and the lower-left triangle, we are left with an additional n(N−n) internal (R-symmetry) coordinates,

$$Z_M{}^A \to \begin{array}{c} \\ \mu \\ m \\ m' \\ \dot\mu \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & a & a' & \dot\alpha \end{array} \\ \begin{pmatrix} I & \theta_\mu{}^a & \theta_\mu{}^{a'} & x_\mu{}^{\dot\alpha} \\ 0 & I & y_m{}^{a'} & \bar\theta_m{}^{\dot\alpha} \\ 0 & 0 & I & \bar\theta_{m'}{}^{\dot\alpha} \\ 0 & 0 & 0 & I \end{pmatrix} \end{array}$$

For N = 1 (“simple” superspace) this is identical to the previous case, but for N > 1 it allows for generalizations that have proven necessary for most practical applications. However, so far only N = 2 superspace (“hyperspace”) has been developed to a point approaching the usefulness of N = 1.

Projective spaces are obtained by gauging away parts of GL groups in the same manner as above (diagonal blocks + lower triangle), but dividing up the indices into only 2 parts. So we reassemble the previous 4 parts, but differently than the 2 original blocks (bosonic + fermionic) as indicated by our reordering. We then do a second reordering, as the standard bosonic + fermionic within each block:

$$Z_M{}^A \to \begin{array}{c} \\ \underline{M} \\ \underline{M}' \end{array} \begin{array}{c} \begin{array}{cc} \underline{A} & \underline{A}' \end{array} \\ \begin{pmatrix} I & w_{\underline{M}}{}^{\underline{A}'} \\ 0 & I \end{pmatrix} \end{array} = \begin{array}{c} \\ m \\ \mu \\ m' \\ \dot\mu \end{array} \begin{array}{c} \begin{array}{cccc} a & \alpha & a' & \dot\alpha \end{array} \\ \begin{pmatrix} I & 0 & y_m{}^{a'} & \bar\theta_m{}^{\dot\alpha} \\ 0 & I & \theta_\mu{}^{a'} & x_\mu{}^{\dot\alpha} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{array}$$

where $\underline{M} = (m, \mu)$, $\underline{M}' = (m', \dot\mu)$, etc. We also have a simple expression for the inverse matrix, in this gauge:

$$Z_A{}^M \to \begin{array}{c} \\ \underline{A} \\ \underline{A}' \end{array} \begin{array}{c} \begin{array}{cc} \underline{M} & \underline{M}' \end{array} \\ \begin{pmatrix} I & -w_{\underline{A}}{}^{\underline{M}'} \\ 0 & I \end{pmatrix} \end{array}$$

where this matrix w is the same as the previous. (Symmetry and gauge indices lose their distinction after gauge fixing.)

This case has the same bosonic coordinates but half the anticommuting coordinates of the previous. Superspaces with half the fermionic coordinates of the full superspace are called “chiral” (if all θ’s), “antichiral” (if all $\bar\theta$’s), or “twisted chiral” (if mixed). (Frequently this is called “BPS” after papers by Bogolmon’yi and by Prasad and Sommerfeld about nonsupersymmetric monopoles, for obscure reasons.) This is the smallest number of fermions we can get, since the gauge algebra must close, and we can’t kill both a supersymmetry and its complex conjugate without killing translations. This is useful for constructing actions, since $\int d\theta = \partial/\partial\theta$ has positive mass dimension, so more θ's would require a Lagrangian lower in dimension. For N = 1, this is either chiral superspace (N = 0, no $\bar{\theta}$'s), in the chiral representation, or antichiral (n = 1, no θ's), in the antichiral representation. For general N, n = 0 is again chiral, and n = N antichiral, both with no y's. Other important special cases are N = 2n, where w is a square matrix; this applies to N = 4 (Yang-Mills), N = 2, and of course N = 0. (The last example should be kept in mind in all of the following.)

Note that the supertwistor representation is also a projective space. Besides dividing up the N-valued $\underline{a}$ index as n + (N−n) for arbitrary n, we could have done the same for 4-valued index $\underline{\alpha}$. (This would do the same kind of thing for the x coordinates as we have done for the y's.) The 0+4 case is trivial (it gives no spacetime coordinates), the 2+2 case gives normal 4D spacetime as discussed above, while the 1+3 case gives supertwistors. However, this would give a complex space, so we need to include the complex-conjugate twistor to define real fields. Identifying the complex conjugate with the canonical conjugate (as for creation and annihilation operators) then prevents doubling the dimension of the space.

The super Penrose transform gives the solution to the equations of motion in projective superspace by identifying G's for "translations" in w (obvious by generalization from the SU(2) and conformal cases, but we'll derive it later) to the corresponding ones for supertwistors: For scalars,

$$-i\partial_{\underline{M}'}{}^{\underline{M}} = \pm\bar{\zeta}_{\underline{M}'}\zeta^{\underline{M}} \quad \Rightarrow \quad \mathit{\Phi}(w) = \sum_{\pm} \int d\zeta\, d\bar{\zeta}\; e^{\pm i\zeta w\bar{\zeta}} \chi_{\pm}(\zeta, \bar{\zeta})$$

(restoring the "i" for hermiticity), relating the projective superfield $\mathit{\Phi}(w_{\underline{M}}{}^{\underline{M}'})$ with the twistor superfields $\chi_{\pm}(\zeta^{\underline{M}}, \bar{\zeta}_{\underline{M}'})$ for positive and negative-energy solutions. The choice of n determines how the fermionic twistor coordinates are distributed between ζ and $\bar{\zeta}$. Note that, unlike ζ^M or $\bar{\zeta}_M$, these coordinates are not a representation (but only a nonlinear realization) of the superconformal group: For example, the conformal boosts are represented as quadratic in their "momenta". In comparison to the usual projective superspaces given above, supertwistors are like lightcone superspace, keeping: (1) just 3 out of 4 x's (since they're on shell), (2) 1/2 the θ's (say $\theta^{m'+}$ and $\bar{\theta}_m{}^{\dot{+}}$), and (3) none of the y's.

As usual, the twistor superfields can be Fourier transformed to functions of just ζ^M (or just $\bar{\zeta}_M$, or something in-between): Integrating over just $\bar{\zeta}_{\underline{M}'}$,

$$\Phi(w) = \sum_{\pm} \int d\zeta^{\underline{M}}\ \tilde{\chi}_{\pm}(\zeta^{\underline{M}}, -\zeta^{\underline{N}} w_{\underline{N}}{}^{\underline{M}'}) = \sum_{\pm} \int d\zeta^{M}\ \delta(\zeta^{\underline{M}'} + \zeta^{\underline{N}} w_{\underline{N}}{}^{\underline{M}'})\ \tilde{\chi}_{\pm}(\zeta^{M})$$

$$= \sum_{\pm} \int d\zeta^{M}\ \delta(\zeta^{M} Z_{M}{}^{\underline{A}'})\ \tilde{\chi}_{\pm}(\zeta^{M})$$

(There is a similar relation with $\delta(Z_{\underline{A}}{}^{M}\bar{\zeta}_{M})$.)

As we'll discuss in detail later, only the case n = N/2 (and thus even N) allows real superfields, since only it makes w a square matrix, with equal range for the $\underline{A}$ index and its "charge conjugate" $\underline{A}'$. This is especially clear if we note that it's the only case where there are equal numbers of θ's and $\bar{\theta}$'s. (Of course, the full superspaces also allow real superfields.) Since this makes them the most useful, we'll often use the term "projective" to refer to them specifically.

Projective approach

The interesting properties of projective spaces follow from the fact that the coset coordinates fit into a rectangle. Furthermore, although the full, "left" index is required for manifest symmetry, the gauge group necessarily breaks the "right" index into 2 pieces. We can therefore begin, before choosing a gauge, with a rectangle that keeps the full left index, but only the part of the right index that contains the coset:

$$Z_{M}{}^{A} \to \bar{z}_{M}{}^{\underline{A}'} = \begin{array}{c} \\ \underline{M} \\ \underline{M}' \end{array} \!\!\begin{array}{c} \underline{A}' \\ \begin{pmatrix} z_{\underline{M}}{}^{\underline{A}'} \\ z_{\underline{M}'}{}^{\underline{A}'} \end{pmatrix} \end{array}$$

And we can do the analogous for the inverse group element:

$$Z_{A}{}^{M} \to z_{\underline{A}}{}^{M} = \begin{array}{c} \\ \underline{A} \end{array} \!\!\begin{array}{cc} \quad\underline{M} & \quad\underline{M}' \\ \multicolumn{2}{c}{\begin{pmatrix} z_{\underline{A}}{}^{\underline{M}} & z_{\underline{A}}{}^{\underline{M}'} \end{pmatrix}} \end{array}$$

Then all that's left of the relation between the group element and its inverse is the orthogonality relation

$$z_{\underline{A}}{}^{M}\bar{z}_{M}{}^{\underline{A}'} = 0$$

(Similar orthogonality relations appeared for supertwistors above.) Furthermore, all that's left of the original gauge invariance is the block diagonal pieces, one of which acts only on z (GL(n|2) for the superconformal group), and the other only on $\bar{z}$ (GL(N−n|2)). (But all the symmetry remains linear, since that index hasn't been restricted.) Note that neither z nor $\bar{z}$ contains the coordinates for conformal boosts.

Some simple but interesting cases of projective spaces are RP(n) (which is related to S^n) and CP(n), which are described by (n+1)-vectors (i.e., (n+1)×1 rectangles), real or complex, for which the gauge parameter is a single number, real or complex. Using a vector for a representation of SO(n) or SU(n) (which becomes a nonlinear realization after fixing the gauge) is much simpler than using a coset space. The chiral case of the above superconformal is HP($\frac{1}{2}$N|1).

The surviving coordinates w can be defined in a gauge-invariant way, which is a simpler way to see their symmetry transformations. An easy way to do this is by solving the orthogonality condition, as

$$\bar{z}_M{}^{\underline{A}'} = (w_{\underline{M}}{}^{\underline{N}'}, \delta_{\underline{M}'}^{\underline{N}'})\bar{u}_{\underline{N}'}{}^{\underline{A}'}, \qquad z_{\underline{A}}{}^M = u_{\underline{A}}{}^{\underline{N}}(\delta_{\underline{N}}^{\underline{M}}, -w_{\underline{N}}{}^{\underline{M}'})$$

or in matrix notation

$$\bar{z} = \begin{pmatrix} w \\ I \end{pmatrix} \bar{u}^{-1}, \qquad z = u^{-1} \begin{pmatrix} I & -w \end{pmatrix}$$

Only u and $\bar{u}$ transform under their respective gauge transformations. This defines w as the "ratio" of the 2 blocks of either z or $\bar{z}$:

$$w_{\underline{M}}{}^{\underline{M}'} = \bar{z}_{\underline{M}}{}^{\underline{A}'}(\bar{z}_{\underline{M}'}{}^{\underline{A}'})^{-1} = -(z_{\underline{A}}{}^{\underline{M}})^{-1} z_{\underline{A}}{}^{\underline{M}'}$$

where the inverses are matrix inverses of those blocks.

It's actually easier to derive the explicit form of the symmetry generators on w from the form of finite transformations, rather than using $G = g\partial_g$ and $D = (\partial_g)g$. On the original full group element $Z_M{}^A$, the transformation was $Z' = g_0 z$ and thus $Z'^{-1} = Z^{-1} g_0^{-1}$, so in terms of

$$g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g_0^{-1} = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix}$$

the symmetry transformation of w follows as a "fractional linear" ("projective") transformation: From the rectangles,

$$\bar{z}' = g_0 \bar{z}, \quad z' = z g_0^{-1} \quad \Rightarrow \quad w' = (aw+b)(cw+d)^{-1} = (w\tilde{c}+\tilde{d})^{-1}(w\tilde{a}+\tilde{b})$$

(Of course, $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ can be expressed in terms of a, b, c, d, and vice versa: See *Fields*, subsection IIC3 if you want to see explicit expressions.) A special case is ordinary conformal symmetry (N = 0), where all the above are 2×2 matrices: This takes a simpler form than in the usual vector notation, just as for the case of SO(3,1) on 2D Euclidean space. Here the simplification arises from using quaternions instead of 4D vectors, while in the 2D case it was complex numbers in place of 2-vectors.

From the same derivation we also have the transformations of the u's:

$$\bar{u}' = \bar{u}(cw+d)^{-1}, \qquad u' = (w\tilde{c}+\tilde{d})^{-1}u$$

Eventually we need to apply the "S" gauge condition (or constraint):

$$sdet(g_0) = 1 \quad \Rightarrow \quad sdet(cw+d) = sdet(w\tilde{c}+\tilde{d})$$

$$sdet\ Z = 1 \quad \Rightarrow \quad sdet\ u = sdet\ \bar{u} \equiv 1/x^-$$

We can also construct symmety invariants in a similar way to (and implied by) the coset construction (consider $g^{-1}dg$ and $g_2^{-1}g_1$), as differentials or finite differences:

$$z_{\underline{A}}{}^M d\bar{z}_M{}^{\underline{A}'} = u_{\underline{A}}{}^{\underline{M}}(dw_{\underline{M}}{}^{\underline{M}'})\bar{u}_{\underline{M}'}{}^{\underline{A}'}, \qquad z_{2\underline{A}}{}^M \bar{z}_{1M}{}^{\underline{A}'} = u_{2\underline{A}}{}^{\underline{M}}(w_1 - w_2)_{\underline{M}}{}^{\underline{M}'}\bar{u}_{1\underline{M}'}{}^{\underline{A}'}$$

The u's are pure gauge; symmetry- and gauge-invariant quantities depend only on differentials or differences of w, according to the translation ("b") part of the symmetry. (An analog in field theory is the dilaton, or local Weyl scale compensator of general relativity, which cancels in all locally scale invariant actions.) These translations include the usual spacetime ones, some of the internal symmetry, and half the supersymmetries (as in the special case of chiral superspace).

The transformation law for the projective integration measure dw can be found from $d\bar{z}$ (or dz), which is invariant because $sdet\, g = 1$. (The relation of the superdeterminant to Jacobians, as the generalization of the bosonic case, follows from its definition in terms of a Gaussian integral.) This is true already for the part of the measure $d\bar{z}$ coming from any one particular value of $\underline{A}'$ in $\bar{z}_{\mathcal{M}}{}^{A'}$. We then separate out dw and $d\bar{u}$ in $\bar{z} = (w, I)\bar{u}^{-1}$:

$$d\bar{z}_M{}^{\underline{A}'} = (dw_{\underline{M}}{}^{\underline{N}'}, 0)\bar{u}_{\underline{N}'}{}^{\underline{A}'} + (w_{\underline{M}}{}^{\underline{N}'}, \delta_{\underline{M}'}^{\underline{N}'})d\bar{u}_{\underline{N}'}{}^{\underline{A}'}$$

$$\Rightarrow \quad d\bar{z} = dw\,(sdet\ \bar{u})^{-str\, I_u} \times d(\bar{u}^{-1}), \qquad str\ I_u = \mathrm{n} - 2$$

where the exponent comes from multiplying the contributions from each particular value of $\underline{M}$. The superconformal transformation of $d(\bar{u}^{-1})$ then follows from that of $\bar{u}^{-1}$ by a similar manipulation:

$$d(\bar{u}^{-1})' = d(\bar{u}^{-1})[sdet(cw+d)]^{str\, I_{\bar{u}}}, \qquad str\ I_{\bar{u}} = (\mathrm{N}-\mathrm{n}) - 2$$

$$\Rightarrow \quad dw' = dw\,[sdet(cw+d)]^{-str\, I}, \qquad str\, I = \mathrm{N} - 4$$

So we have invariant measures

$$dw(x^-)^{N-4}\ , \qquad dw\, dx^-(x^-)^{N-5}$$

We have already looked at SU(2) in such a treatment; it follows as a special case of the above:

$$\text{SU(2)}: \quad w \to y \ , \quad x^- \to 1/\sqrt{y_0}$$

(and drop the "4"'s in the above measure expressions: N$-4 \to 2 = str\, I$). Another important and simple example is SU(2,2) as the N = 0 case of SU(N|2,2) (only fermionic indices):

$$\text{SU(2,2)}: \quad w \to x \ , \quad x^- \to x_0$$

Then we have a 2×2 matrix x, with

$$x' = (ax+b)(cx+d)^{-1}$$

as a simple matrix generalization of the SU(2) case. We can easily recognize b as translations, and a and d as SL(2,C) Lorentz on the 2 spinor indices, and scale. This leaves c to give conformal boosts.

Correlators

As an example of (super)conformal invariants, we consider 2- and 3-point correlators. For scalars, these can be constructed directly from the symmetry invariants considered above. (The more complicated case of operators with spin can be treated by slight generalization with the methods given in the next section.) To illustrate, we'll look at the simplest case, N = 0. The symmetry (but not gauge) invariant is then

$$z_{2\alpha}{}^{\underline{\mu}} \bar{z}_{1\underline{\mu}}{}^{\dot{\alpha}} = u_{2\alpha}{}^{\mu} x_{12\mu}{}^{\dot{\mu}} \bar{u}_{1\dot{\mu}}{}^{\dot{\alpha}} = u_2^{-1} x_{12} \bar{u}_1^{-1}$$

for any two points 1,2 (including switching 1↔2), where $x_{12} = x_1 - x_2$. The idea is then to cancel all the "Lorentz" pieces of all the u's and $\bar{u}$'s, to get invariants under not only the symmetry group (conformal) but also the gauge group (SL(2,C)). In particular, we have the invariant

$$\det(u_2^{-1} x_{12} \bar{u}_1^{-1}) = \frac{det(x_{12})}{det(u_2) det(\bar{u}_1)}$$

where $det\, x_{12} = x_{12}^2$. Furthermore, the S of SU(2,2) says

$$\det u = \det \bar{u} \equiv x_0$$

which is the N = 0 case of x^-, while the power of $det\, u$ at any point gives the scale weight of the operator there. The 2-point correlator is then easily found to be

$$\langle \mathring{\mathcal{O}}_1(1)\ \mathring{\mathcal{O}}_2(2) \rangle \sim \left(\frac{x_{01} x_{02}}{x_{12}^2} \right)^{\triangleleft}$$

in terms of the 4D (composite) scalar fields $\mathring{\mathcal{O}}$, where $\lhd$ is the scale weight of both, attributed to

$$\mathring{\mathcal{O}}(x, x_0) = x_0^{\lhd}\mathcal{O}(x)$$

by dimensional analysis, or by the coset constraint on the covariant derivative for dilatations. (The $x_0^{\lhd}$ is one of the "vielbein"-type factors $e(\gamma)$ for coset spaces.) Clearly both $\mathring{\mathcal{O}}$'s must have the same weight (although they might still be different operators, if we haven't diagonalized our basis.)

For higher-point functions, we could try to contract indices on the $u^{-1}x\bar{u}^{-1}$'s, but there is no nontrivial way to do this till 4-point. (Consider, in Dirac notation, $tr(\not{a}\not{b}...\not{c})$.) But this means the 3-point function is till given by just det's, namely

$$\langle \mathring{\mathcal{O}}_1(1)\ \mathring{\mathcal{O}}_2(2)\ \mathring{\mathcal{O}}_3(3)\rangle \sim \left(\frac{x_{01}x_{02}}{x_{12}^2}\right)^a \left(\frac{x_{02}x_{03}}{x_{23}^2}\right)^b \left(\frac{x_{03}x_{01}}{x_{31}^2}\right)^c$$

The x_0 dependence then tells us

$$\lhd_1 = a + c\ , \quad \lhd_2 = a + b\ , \quad \lhd_3 = b + c$$

$$\Rightarrow \quad a = \tfrac{1}{2}(\lhd_1 + \lhd_2 - \lhd_3)\ , \quad b = \tfrac{1}{2}(\lhd_2 + \lhd_3 - \lhd_1)\ , \quad c = \tfrac{1}{2}(\lhd_1 + \lhd_3 - \lhd_2)$$

4-point and higher are where things start to get interesting: They give "amplitudes", not just weights and "couplings". Unfortunately, the result is not unique, even up to a constant, since there are ratios of conformal invariants that are also gauge invariant.

Generalization to spin is easy: Just as the scale weight of a field is carried by x_0, the spin is carried by (the unit determinant part of) u and $\bar{u}$. (I.e., all of u and $\bar{u}$, subject to $det\, u = det\, \bar{u} = 1$, now make up the vielbein for arbitrary spins.) Matching u's and $\bar{u}$'s with the fields then gives factors of x_{ij} (with free indices) in the correlators. For example, for the 2-point, after removing the u's and $\bar{u}$'s, including the x_0's, (i.e., going to a "unitary gauge")

$$\langle \mathcal{O}_{\dot{\mu}...\dot{\nu}}{}^{\sigma...\tau}(1)\ \mathcal{O}_{\dot{\mu}'...\dot{\nu}'}{}^{\sigma'...\tau'}(2)\rangle \sim \frac{x^{(\sigma}{}_{\dot{\mu}'}...x^{\tau)}{}_{\dot{\nu}'}x^{(\sigma'}{}_{\dot{\mu}}...x^{\tau')}{}_{\dot{\nu}}}{(x^2)^{\widetilde{\lhd}}}$$

where here $x \equiv x_{12}$ and $\widetilde{\lhd}$ is $\lhd$ + 1/2 the number of spinor indices on $\mathcal{O}$. The result looks much worse in vector notation, especially for half-integer spin. It's also much simpler when written with these local indices instead of global ones (the "embedding" formalism), a general truth for anything in coset spaces. Similar generalizations are possible for superfields, particularly those that live in projective superspaces.

4D superfields

Spin

We start with the parametrization of a group element of PSU(2,2|4) as for projective superspace, but keeping all elements:

$$Z_M{}^A = \begin{pmatrix} I & w \\ 0 & I \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \bar{u}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -v & I \end{pmatrix} = \begin{pmatrix} u - w\bar{u}^{-1}v & w\bar{u}^{-1} \\ -\bar{u}^{-1}v & \bar{u}^{-1} \end{pmatrix}$$

We separate the derivative form of the symmetry generators and covariant derivatives as

$$G = Z\partial_Z \equiv \begin{pmatrix} G_u & -G_v \\ G_w & -G_{\bar{u}} \end{pmatrix} , \qquad D = \partial_Z Z \equiv \begin{pmatrix} D_u & -D_v \\ D_w & -D_{\bar{u}} \end{pmatrix}$$

For spin, as for the bosonic case (and in analogy to SU(2)), we keep $D_v = 0$ unmodified from the spinless case, and gauge $v = 0$. (Then w, u, and $\bar{u}$ appear as previously.) But we introduce spin to replace D_u and $D_{\bar{u}}$, since they include Lorentz. (Of course, D_w is not in the gauge group.) Then (as easily seen from the finite gauge transformations)

$$D_u \equiv \partial_u u = s_u \equiv u^{-1}\hat{s}_u u, \qquad D_{\bar{u}} \equiv \bar{u}\partial_{\bar{u}} = s_{\bar{u}} \equiv \bar{u}\hat{s}_{\bar{u}}\bar{u}^{-1}$$

where the $\hat{s}$'s are defined to act on "curved" indices M, M' rather than "flat" indices A, A'.

Our flat/curved terminology is by analogy to general relativity, where "flat" indices carry the Lorentz gauge symmetry, and are how spin is introduced, while "curved" indices, and the coordinates that carry them, are acted on by any global symmetry of the space under consideration. In fact, in the bosonic case our gauge group GL(2)⊗GL(2) is just the Lorentz group, scale transformations (for which the "spin" part is the scale weight), and the purely gauge GL(1) that reduces GL(4) to the (Wick-rotated) conformal group SL(4).

Since our gauge group is GL(n|2)⊗GL(N-n|2), it's clear how this works: The gauge group covariant derivatives D_u and $D_{\bar{u}}$ carry flat indices; their irreducible matrix representations carry arbitrary mixtures of these defining indices, up and down, with arbitrary graded (anti)symmetrizations (but with arbitrary values of the Abelian GL(1) charges, and maybe some supertrace conditions). Thus our original fields $\Phi(w, u, \bar{u})$ carry these flat indices, are scalars with respect to the symmetry group, and satisfy the constraints $D_u - s_u = D_{\bar{u}} - s_{\bar{u}} = 0$. But we can explicitly solve these constraints in terms of fields Φ that carry only curved indices, by using u and $\bar{u}$ as "vielbeins" to convert flat indices into curved. The fields with curved indices then depend only on w, and are gauge invariant, but are no longer scalars: The $\hat{s}$'s in G act the same way on the curved indices as the s's acted on the flat (and themselves carry curved indices).

It's sufficient to consider an example with one of each type of index, primed and unprimed (up vs. down indices should be obvious):

$$s_{\underline{A}}{}^{\underline{C}}\Phi_{\underline{B}'}{}^{\underline{D}} = \delta_{\underline{A}}^{\underline{D}}\Phi_{\underline{B}'}{}^{\underline{C}} - r\delta_{\underline{A}}^{\underline{C}}\Phi_{\underline{B}'}{}^{\underline{D}}, \qquad s_{\underline{A}'}{}^{\underline{C}'}\Phi_{\underline{B}'}{}^{\underline{D}} = \delta_{\underline{B}'}^{\underline{C}'}\Phi_{\underline{A}'}{}^{\underline{D}} - \bar{r}\delta_{\underline{A}'}{}^{\underline{C}'}\Phi_{\underline{B}'}{}^{\underline{D}}$$

(with extra signs from index reordering implicit) where $r+\bar{r}$ is the superscale weight (see below) and $str\ s - str\ \bar{s}$ (the "$-$" comes from the definition of $D_{\bar{u}}$ and $G_{\bar{u}}$) is related to the super-(internal-)U(1) charge (or superhelicity). The solution to the constraints is

$$\Phi_{\underline{A}'}{}^{\underline{A}}(w,u,\bar{u}) = (x^-)^{r+\bar{r}}\bar{u}_{\underline{A}'}{}^{\underline{M}'}\Phi_{\underline{M}'}{}^{\underline{M}}(w)u_{\underline{M}}{}^{\underline{A}}, \qquad \Phi'_{\underline{A}'}{}^{\underline{A}}(w,u,\bar{u}) = \Phi_{\underline{A}'}{}^{\underline{A}}(w',u',\bar{u}')$$

$$\Rightarrow \quad \Phi'_{\underline{M}'}{}^{\underline{M}}(w) = [sdet(cw+d)]^{r+\bar{r}}(cw+d)^{-1}{}_{\underline{M}'}{}^{\underline{N}'}\Phi_{\underline{N}'}{}^{\underline{N}}(w')(w\tilde{c}+\tilde{d})^{-1}{}_{\underline{N}}{}^{\underline{M}}$$

where r and $\bar{r}$ appear only in the combination $r+\bar{r}$ because we have used the "S" constraint.

In the physically most interesting cases (e.g., the N = 2 scalar multiplet or the N = 4 vector multiplet) the field strength $\Phi(w,u,\bar{u})$ is a scalar (and $r+\bar{r}$ is nonzero). It then depends on only w and the "extra coordinate" x^-. This is well known from the nonsupersymmetric case, where this extra coordinate is the x_0 of the projective lightcone.

A linear form of transformation on indices can be obtained by using z and $\bar{z}$ to convert flat indices into full GL(N|4) curved indices; e.g.,

$$\Phi_M{}^N \sim \bar{z}_M{}^{\underline{A}'}\Phi_{\underline{A}'}{}^{\underline{A}}z_{\underline{A}}{}^N$$

But such fields are constrained,

$$z_{\underline{A}}{}^M\Phi_M{}^N = \Phi_M{}^N\bar{z}_N{}^{\underline{A}'} = 0$$

Solving the constraints leads back to the above fields and yields their nonlinear transformations.

Note that the fermionic part of the spin is usually assumed to vanish, in agreement with known physical examples. This implies that their superpartners do also, so in those cases s vanishes except for the chiral case, where only s_u (consisting of just $s_\alpha{}^\beta$) is nonvanishing, or the antichiral case, where only $s_{\bar{u}}$ is. In some of the most interesting cases (like N = 4 Yang-Mills), the only nonvanishing parts of the spin are the r and $\bar{r}$ pieces. (Some of the restrictions come from the quadratic field equations, which we consider below.)

The form of the symmetry generators in terms of w and u can again easily be derived from the finite forms of the transformations (taking the infinitesimal limit). We thus find the basis

$$G_w = \partial_w, \qquad G_u = w\partial_w + u\partial_u, \qquad G_{\bar{u}} = \partial_w w + \partial_{\bar{u}}\bar{u}, \qquad G_v = w\partial_w w + u\partial_u w + w\partial_{\bar{u}}\bar{u}$$

If we apply the spin constraints, the symmetries then become

$$G_w = \partial_w, \qquad G_u = w\partial_w + \hat{s}_u, \qquad G_{\bar{u}} = \partial_w w + \hat{s}_{\bar{u}}, \qquad G_v = w\partial_w w + \hat{s}_u w + w\hat{s}_{\bar{u}}$$

(For the special case of N $= 0$, we can replace w with x, and $\hat{s}$ with Lorentz and scale, to find the usual in spinor notation.) One can also check that for N $=$ 2n these operators are permuted by the "inversion" (a particular case of the above finite transformations)

$$g_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}: \qquad w \to -w^{-1}, \qquad u \to w^{-1}u, \qquad \bar{u} \to \bar{u}w^{-1}$$

Although the covariant derivatives D_u and $D_{\bar{u}}$ for the gauge group are obvious from the way they act on the group indices, the remaining derivatives D_w can't be found that commute with the symmetry generators G_v. However, we can define

$$D_w = \bar{u}\partial_w u$$

that commute with all but G_v. This is the usual procedure for ordinary conformal symmetry, where coordinates are not introduced for conformal boosts, so ("covariant") translational derivatives don't commute with them. (We would have found the same result if we had derived truly invariant derivatives before fixing the gauge $v = 0$.)

Field equations

We saw before the 4D superconformal (free) field equations

$$D_{(A}{}^{(C} D_{B]}{}^{D]} = 0 \; mod \; \delta \; terms$$

that includes the massless Klein-Gordon equation $p^2 = 0$. The equation is determined only up to Kronecker δ terms, which don't contribute to the Klein-Gordon equation, and has this ambiguity because of the gauge invariance

$$D_A{}^B \to D_A{}^B + \delta_A^B A$$

for arbitrary operator A. (Because it's Abelian, this is the same gauge symmetry as in the gauge group generated by G's; "Abelian" means it can be considered as either left or right.)

The supersymmetric equation of motion also includes the general (massless) supersymmetry free field equation $\not{p}d = 0$, the Pauli-Lubański equation, several dd equations often seen in supersymmetry, equations involving the internal symmetry generators, and various redundant equations.

The number of field equations we wrote above is reduced by the gauge constraints. The net result is that the equations on projective space reduce to (all mod δ terms for the s's)

$$\partial_{(\underline{M}'}{}^{(\underline{P}}\partial_{\underline{N}']}{}^{\underline{Q}]} = s_{\underline{M}}{}^{(\underline{P}}\partial_{\underline{N}'}{}^{\underline{Q}]} = s_{(\underline{M}'}{}^{\underline{P}'}\partial_{\underline{N}']}{}^{\underline{Q}} = 0$$

$$s_{\underline{M}}{}^{\underline{P}}s_{\underline{N}'}{}^{\underline{Q}'} = s_{(\underline{M}}{}^{(\underline{P}}s_{\underline{N}]}{}^{\underline{Q}]} = s_{(\underline{M}'}{}^{(\underline{P}'}s_{\underline{N}']}{}^{\underline{Q}']} = 0$$

The first set of equations is for arbitrary massless representations of supersymmetry, the second set restricts the index structure for specialization to conformal supersymmetry. (A similar separation can be made for the bosonic case in arbitrary dimensions.) Specifically, the second set places the restriction that superconformal representations have only primed or only unprimed indices, and fixes the value of the superscale weight.

In supertwistor space, we find that already the equations linear in spin and in derivatives restrict the supertwistor space solutions to the analog of those for the bosonic case:

$$\Phi_{\underline{M}'\ldots\underline{N}'}{}^{\underline{M}\ldots\underline{N}}(w) = \sum_{\pm}\int d\zeta\, d\bar{\zeta}\; e^{\pm i\zeta w\bar{\zeta}}\; \bar{\zeta}_{\underline{M}'}\cdots\bar{\zeta}_{\underline{N}'}\zeta^{\underline{M}}\cdots\zeta^{\underline{N}}\chi_{\pm}(\zeta,\bar{\zeta})$$

Since a ζ and $\bar{\zeta}$ are produced by a w derivative, this effectively reduces Φ to have only unprimed or only primed indices, graded antisymmetric in all of them, as implied by the second (spin-only) set of superconformal field equations. (However, fields that are total derivatives on shell need not be so off; but such field strengths are generally not conformal.) In the purely ζ^M or $\bar{\zeta}_M$ form, the full indices can be used, but because of the constraint enforced by the δ function, the fields will satisfy the analogous constraints on the indices, as described in the previous section. The superhelicity is now given by the number of unprimed minus primed indices.

Moving back to coordinate space, the spin-free part of these equations decomposes as:

$$\partial_x\partial_x = \partial_x\partial_\theta = \partial_\theta\partial_\theta = \partial_\theta\partial_{\bar{\theta}} + \partial_x\partial_y = \partial_y\partial_\theta = \partial_y\partial_y = 0$$

(and complex conjugates). Internal indices are symmetrized, while Weyl spinor indices are contracted (antisymmetrized). The ∂_y-free equations should be familiar from N = 1 chiral scalars: They include the Klein-Gordon, Weyl spinor, and auxiliary field equations, respectively. The equation with all types of derivatives (and thus 2 different types of terms, each with only 1 of each kind of index, and thus no symmetrization possible) shows that any y-dependent term shows up without y at higher order in θ and $\bar\theta$ with x-derivatives, and that all terms with both θ and $\bar\theta$ are of this form.

Taylor expansion is sufficient for the y's, since setting both primed indices equal and both unprimed indices equal in the $\partial_y\partial_y$ equation says the field is linear in each y. (Of course we can always Taylor expand in the θ's.) Then the non-∂_x equations say that all component fields in this Taylor expansion in y's and θ's are totally antisymmetric in unprimed internal indices and separately also in primed.

We now examine the component expansion for N = 4, n = 2. (So only the $SU(2)^2 \otimes U(1)$ subgroup of SU(4) is realized linearly.) The result is straightforward:

$$\begin{aligned} \Phi(w) = (\phi + y_m{}^{m'}\phi_{m'}{}^m + \tfrac{1}{2}y^2\bar\phi) + \theta_\mu{}^{m'}(\lambda_{m'}{}^\mu + y_{m'}{}^m\lambda_m{}^\mu) + \bar\theta_m{}^{\dot\mu}(\bar\lambda_{\dot\mu}{}^m + y_{m'}{}^m\bar\lambda_{\dot\mu}{}^{m'}) \\ +(\theta^2_{\mu\nu}f^{\mu\nu} + \bar\theta^{2\dot\mu\dot\nu}\bar f_{\dot\mu\dot\nu}) - i\theta_\mu{}^{m'}\bar\theta_m{}^{\dot\mu}\partial_{\dot\mu}{}^\mu(\phi_{m'}{}^m + y_{m'}{}^m\bar\phi) \\ -i\theta^2_{\mu\nu}\bar\theta_m{}^{\dot\mu}\partial_{\dot\mu}{}^\mu\lambda^{m\nu} - i\bar\theta^{2\dot\mu\dot\nu}\theta_\mu{}^{m'}\partial_{\dot\mu}{}^\mu\bar\lambda_{\dot\nu m'} - \theta^2_{\mu\nu}\bar\theta^{2\dot\mu\dot\nu}\partial_{\dot\mu}{}^\mu\partial_{\dot\nu}{}^\nu\bar\phi \end{aligned}$$

where we have used the internal $SL(2)^2$ metrics to raise, lower, and contract indices. Each component field, as a function of x, satisfies the Klein-Gordon equation, and each non-scalar satisifies a Weyl equation (which for f is the combination of the usual field equation and Bianchi identity for the Yang-Mills field strength). Note that all component fields appear at $y = 0$, but some only with x derivatives; as stated above, this is a general feature, following from the equation $\partial_\theta\partial_{\bar\theta} + \partial_x\partial_y = 0$; the same is not true off shell.

The usual component (bosonic-)twistor fields are obtained by evaluating the expansion of χ over the fermionic ζ's. (As mentioned previously, this is like expanding in only 1/2 the θ's.) The expansion of the Penrose transform in y gives new component fields, but the expansion terminates because of the anticommutativity of the corresponding ζ's. The expansion in θ (and $\bar\theta$) also gives new component fields, but with spinor indices from bosonic ζ, which then satisfy the usual Weyl equation (as in the nonsupersymmetric twistor formalism), and faster termination because there are fewer fermionic ζ's than θ's, and because y dependence may give extra fermionic ζ's. Also note that expansion in both θ and $\bar\theta$ will give both ζ^μ and $\bar\zeta_{\dot\mu}$, which is equivalent to an x derivative. We also see that all fields with y dependence also occur without y, but with x derivatives, because fermionic ζ's can come from either θ or y (but y's give only equal numbers of ζ^μ and $\bar{\zeta}_{\dot{\mu}}$). All of this agrees with our previous evaluation in terms of the field equations directly.

Powers of the projective superfield Φ, multiplied at the same point in projective superspace, are still projective. Of particular interest are their Yang-Mills traces, which are gauge-independent: The operators

$$tr(\Phi^n)$$

for arbitrary nonnegative integer n (though $n = 0$ is just a constant, and $n = 1$ is nontrivial only in the abelian case) satisfy nonrenormalization theorems for similar reasons to those for chiral operators in N = 1 supersymmetry. In the AdS/CFT correspondence, they correspond to 10D IIB supergravity on the AdS side (and not the rest of the string). They satisfy the "field equations"

$$D_{(A_1}{}^{(B_1} ... D_{A_{n+1}]}{}^{B_{n+1}]} tr(\Phi^n) = 0 \; mod \; \delta \; terms$$

since at least one of the Φ's will have (at least) 2 D's hitting it in the same way as the Φ field equations. (Actually, the Φ field equations generally involve gauge-covariant derivatives and terms nonlinear in field strengths. But the lowest-dimension, y-derivative ones don't, and the rest then follow from repeated application of the coset constraints.) Thus, for $n = 1$ we get the free field equations satisfied by an abelian Yang-Mills multiplet. Another interesting case is $n = 2$ (corresponding to 5D maximal supergravity on AdS), which is the multiplet including the energy-momentum tensor; the cubic constraint then includes the conservation laws for this tensor and other conserved currents. A less exciting case is $n = 3$, where the quartic constraint (as for lower n) restricts how far in the θ expansion independent operators appear. (The $\theta^4\bar{\theta}^4$ term is independent only for n $\geq$ 4.) For all n, the lowest dimension, y-derivative equations (together with the U(1) weight) determine the SU(4) representation of the scalar composite at $\theta = 0$: It's a rank-n totally symmetric, traceless SO(6) tensor.

General composite operators constructed from this field strength will no longer be projective, because the covariant derivatives will introduce u dependence. (Compare chiral superfields for N = 1.) But if only certain spinor derivatives are used, only certain fermionic coordinates will be introduced, so the operators will depend on some number of θ's intermediate between those of the full superspace (16) and those of projective superspace (8). Sometimes these go by the name of "semi-chiral", or "1/4-BPS" (12), "1/8-BPS" (14), etc. ("BPS" is then called "1/2-BPS".) Similar remarks apply to other supersymmetric theories.

One might wonder why projective superspace was enough to describe N = 4 Yang-Mills: An internal space of 4 coordinates is insufficient to describe all representations of SU(4), which would require 6 coordinates (from constraining just the Cartan subalgebra and corresponding lowering operators). The reason is that there is a correspondence between the internal SU(4) and spacetime SU(2,2) (or just SL(2,C)) representations: Any field (or composite) in the theory (and any other physical theory for that matter) is a "spinor" representation of each or a "tensor" representation of each. So the y coordinates need provide only a subset of those required (specifically, the traceless totally symmetric tensors of SO(6)), while the rest are provided by the SU(4) transformations of the θ coordinates. (On the AdS side, the 4 coordinates produce the representations found from the Kaluza-Klein expansion for S^5.)

Charge conjugation

As explained previously, only the cases N = 2n, where w is square, allow the existence of real superfields. Because of the Wick rotation used to conveniently describe the superconformal group, fields will satisfy nontrivial reality conditions. We really don't need to Wick rotate: If you ignore reality, it doesn't make a difference; just treat any variable and its complex conjugate as algebraically independent. (However, there can be some topological complications, which we'll ignore, at least for now.) Reality for the superconformal group is expressed as a pseudo-unitarity condition (the "U" in (P)SU(N|2,2)),

$$g^\dagger \Upsilon g = g \Upsilon g^\dagger = \Upsilon, \qquad \Upsilon^2 = I, \qquad \Upsilon^\dagger = \Upsilon; \qquad \Upsilon^{\dot{M}N} = \begin{array}{c} \\ \dot{m} \\ \dot{\mu} \\ \dot{m}' \\ \mu \end{array} \!\!\begin{array}{c} \begin{array}{cccc} n & \nu & n' & \dot{\nu} \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & -iC \\ 0 & 0 & I & 0 \\ 0 & iC & 0 & 0 \end{pmatrix} \end{array}$$

in terms of the SL(2) and U(2) metrics, e.g.,

$$C^{\mu\nu} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad I^{\dot{m}n} = \delta_m^n$$

(The dots on indices refer to complex conjugation.)

It isn't useful to solve for the reality conditions on the components of g because of the nonlinearity, and because some of the complex conjugates of components of w have been gauged away. (So we have chosen a complex gauge by eliminating them.) Instead, we use this unitarity condition to define "charge conjugates" of elements of

g that transform in the same way under the symmetry group, although differently under the gauge group. Specifically, we need this only for the coset:

$$\mathcal{C}(w') = (\mathcal{C}w)'$$

where $\mathcal{C}$ acts on w' as if it were w, and $'$ acts on $\mathcal{C}w$ as if it were w; thus superconformal transformations and charge conjugation commute. We therefore need to use only the fact that the symmetry transformation g_0 used in $Z' = g_0 Z$ satisfies the same unitarity condition as g above. This fact can then be applied as well to the transformations on the projective space, $\bar{z}' = g_0\bar{z}$ and $z' = zg_0^{-1}$. The goal will be to define a charge conjugation $\mathcal{C}$ of fields that involves their complex (hermitian) conjugation, but still gives fields that depend on w (and not $w^\dagger$, whatever that is). Thus

$$(\mathcal{C}\Phi)(w, u, \bar{u}) \equiv [\Phi(\mathcal{C}w, \mathcal{C}u, \mathcal{C}\bar{u})]^\dagger$$

where "$\mathcal{C}w$" is some function of $w^\dagger$ (so $\Phi^\dagger$ gives back w) that transforms the same as w under superconformal transformations. The relation for curved superfields then follows. For real fields (when they can be defined), $\mathcal{C}\Phi$ is identified with Φ.

We thus define the action of charge conjugation $\mathcal{C}$ on the full coordinates $Z_M{}^A$ by

$$\mathcal{C}Z \equiv Z\Upsilon\widehat{\Upsilon} = \Upsilon(Z^{-1})^\dagger\widehat{\Upsilon}, \qquad \widehat{\Upsilon}^{\dot{A}B} = \begin{array}{c} \\ \underline{\dot{A}} \\ \underline{\dot{A}}' \end{array} \!\!\begin{array}{c} \begin{array}{cc} \underline{B} & \underline{B}' \end{array} \\ \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \end{array}$$

In the former form the symmetry transformation is obvious, while in the latter form Υ mixes only the symmetry indices, with $\widehat{\Upsilon}$ chosen to mix the gauge indices to relate the pieces appearing in the projective approach:

$$(\mathcal{C}\bar{z}_M{}^{\underline{A}'})^\dagger = -z_{\underline{A}}{}^N \Upsilon_{N\dot{M}}, \qquad (\mathcal{C}z_{\underline{A}}{}^M)^\dagger = \Upsilon^{\dot{M}N}\bar{z}_N{}^{\underline{A}'}$$

relating z to the complex conjugate of $\bar{z}$. (The "$-$" sign, from $\widehat{\Upsilon}$, preserves $sdet\, g = 1$.) The gauge indices don't match because charge conjugation switches primed and unprimed indices; but w is gauge invariant.

Independent of coordinate choice, we find as a result

$$(\mathcal{C}G)^\dagger = -\Upsilon^{-1}G\Upsilon, \qquad (\mathcal{C}D)^\dagger = -\widehat{\Upsilon}^\dagger D\widehat{\Upsilon}^{\dagger -1}$$

(but we have chosen $\Upsilon^{-1} = \Upsilon$, $\widehat{\Upsilon}^{-1} = \widehat{\Upsilon}^\dagger = -\widehat{\Upsilon}$). More explicitly, and taking into account (i.e., undoing) that the above hermitian conjugation includes matrix transposition,

$$\mathcal{C}: \qquad D_w \to -D_w, \quad D_v \to -D_v, \quad D_u \leftrightarrow -D_{\bar{u}}$$

We then find the conjugation of w, which we can write as

$$(\mathcal{C}w)^{\dagger \underline{\dot{M}'}}{}_{\underline{\dot{N}}} = \begin{matrix} \\ \dot{m}' \\ \mu \end{matrix} \begin{matrix} \dot{n} & \dot{\nu} \\ \left(\begin{matrix} -y^{-1}{}_{m'}{}^{n} \\ -iC^{\mu\nu}\theta_\nu{}^{n'}y^{-1}{}_{n'}{}^{n} \end{matrix} \right. & \left. \begin{matrix} -iy^{-1}{}_{m'}{}^{n}\bar{\theta}_n{}^{\dot{\mu}}C_{\dot{\mu}\dot{\nu}} \\ -C^{\mu\nu}(x_\nu{}^{\dot{\mu}} - \theta_\nu{}^{n'}y^{-1}{}_{n'}{}^{n}\bar{\theta}_n{}^{\dot{\mu}})C_{\dot{\mu}\dot{\nu}} \end{matrix} \right) \end{matrix}$$

For N = 0, $\mathcal{C}x$ is just x. (The factors of C in its hermitian conjugation are because it's $x^{\mu\dot{\mu}}$ that's hermitian. Note that $C_{\mu\nu} = -C^{\mu\nu}$.) But $\mathcal{C}y$ is $-y^{-1}$. Thus $\mathcal{C}$ treats x normally, while on y it acts like an inversion for SU(N) (as we have already seen for SU(2)).

For considering spin, we also have

$$(\mathcal{C}u)^\dagger = \bar{u}\bar{\mathcal{A}}^{-1}(w), \qquad (\mathcal{C}\bar{u})^\dagger = \mathcal{A}^{-1}(w)u$$

$$\mathcal{A}_{\underline{M\dot{N}'}} = \begin{matrix} \\ m \\ \mu \end{matrix} \begin{matrix} \dot{n}' \quad\ \nu \\ \begin{pmatrix} y_m{}^{n'} & 0 \\ \theta_\mu{}^{n'} & -iC_{\mu\nu} \end{pmatrix} \end{matrix}, \qquad \bar{\mathcal{A}}^{\underline{\dot{M}N'}} = \begin{matrix} \\ \dot{m} \\ \dot{\mu} \end{matrix} \begin{matrix} n' \quad\ \dot{\nu} \\ \begin{pmatrix} y_m{}^{n'} & \bar{\theta}_m{}^{\dot{\nu}} \\ 0 & -iC^{\dot{\mu}\dot{\nu}} \end{pmatrix} \end{matrix}$$

$$\Rightarrow \quad sdet\, \mathcal{A} = sdet\, \bar{\mathcal{A}} = det\ y$$

For our previous spin example, we then have

$$(\mathcal{C}\varPhi_{\underline{M'}}{}^{\underline{M}})(w) = (det\ y)^{r+\bar{r}}\bar{\mathcal{A}}^{-1}_{\underline{M'\dot{N}}}[\varPhi(\mathcal{C}w)]^{\dagger\underline{\dot{N}}}{}_{\underline{\dot{P}'}}\mathcal{A}^{-1\underline{\dot{P}'M}}$$

The field equations for N = 4 Yang-Mills are also implied by the combination of Taylor expandability in y with the "reality" condition,

$$\mathcal{C}\varPhi = \varPhi, \qquad r + \bar{r} = 1$$

where the latter equation implies a factor of $det\ y = y^2$ (4-vector square of the 4 y's) under charge conugation. (If we keep the factor x^- in the "flat" version of $\varPhi$, it comes from there.) The fact that $\varPhi$ is Taylor expandable in y implies that $\mathcal{C}\varPhi$ is no higher than y^2 (indices contracted), which is equivalent to the $\partial_y\partial_y = 0$ field equation (indices symmetrized); we then know the other equations must also be satisfied by superconformal invariance.

(Similar remarks apply for the N = 2 scalar multiplet, also described by a scalar with $r + \bar{r} = 1$. In that case, since $\mathcal{C}^2 = (-1)^{(N/2)(r+\bar{r})}$, the field is pseudoreal, so the field is doubled and satisfies $(\mathcal{C}\varPhi)^i = C^{ij}\varPhi_j$. In the interacting case, the reality condition for the scalar hypermultiplet involves the prepotential for the vector hypermultiplet. That prepotential is also real, but it has superscale weight 0, so no field equations are implied for it.)

AdS

Maximally symmetric spaces

We begin with a discussion of fields on anti-de Sitter space and how they represent the AdS symmetry. We'll restrict our discussion of general massive fields to just their spectrum: Later we'll go into more detail on 10D IIB supergravity (which is 10D massless) on $AdS_5 \times S^5$, and how Kaluza-Klein reduction over S^5 gives specific massive AdS_5 fields.

First we define AdS, as a maximally symmetric space. (This discussion is taken from *Fields* subsection IXC2.) This can be done in either an "analytic" or "algebraic" way. The analytic way is to look at a curvature tensor that has only a scalar component, which is a constant. The curvature can be determined from the torsion-free covariant derivative with a "flat" index:

$$[\nabla_A, \nabla_B] = \tfrac{1}{2} R_{AB}{}^{CD} M_{DC}$$

To be general, we start with a space $\mathcal{X}^A$ with p space and q time coordinates. Here M_{AB}, which gauges SO(p,q), is a "second-quantized" spin operator, meaning it's defined to act only on fields; thus its action on covariant derivatives themselves is determined by acting before an explicit spacetime solution is chosen:

$$M_{AB} \nabla_C = \nabla_{[A} \eta_{B]C} \quad \Rightarrow \quad [M_{AB}, M^{CD}] = -\delta_{[A}^C \delta_{B]}^D$$

We then constrain the curvature to be

$$R_{AB}{}^{CD} = k \delta_{[A}^C \delta_{B]}^D \quad \Rightarrow \quad [\nabla_A, \nabla_B] = -k M_{AB} \ , \quad |k| = \frac{1}{\mathcal{R}^2}$$

In convenient coordinates, we can choose the radius of curvature $\mathcal{R} = 1$, unless $k = 0$ (flat space). Then the usual definition of the curvature scalar, by tracing, is

$$R = \mathrm{D}(\mathrm{D}-1)k \ , \quad \mathrm{D} = p + q$$

Since the rest of the curvature tensor vanishes, this space can be expressed in appropriate coordinates as a Weyl scale transform of flat space:

$$\nabla_A = \Phi \partial_A + (\partial^B \Phi) M_{AB} \quad \Rightarrow \quad R_{AB}{}^{CD} = \Phi \delta_{[A}^{[B} \partial_{C]} \partial^{D]} \Phi - \delta_{[A}^C \delta_{B]}^D (\partial \Phi)^2$$

where we started with only flat local indices, but the right-hand sides have only flat global indices (in terms of raising/lowering and contracting). For $\mathrm{D} > 2$ the condition on the curvature separates into

$$2\Phi \partial^2 \Phi - \mathrm{D}(\partial \Phi)^2 = \mathrm{D}k \ , \quad \mathrm{D} \partial_A \partial_B \Phi = \eta_{AB} \partial^2 \Phi$$

for flat d'Alembertians ∂^2. (For D = 2 the latter equation isn't implied.) The latter equation for $A \neq B$ says that Φ is the sum of functions of a single coordinate; for $A = B$ it says the functions are quadratic and have the same quadratic coefficient. (This is of the same form as the $\partial_y \partial_y$ equation we encountered for the N = 4 superconformal field strength previously; there it appeared in spinor notation. This is related to the fact that the y coordinates on the AdS side describe S^5.) Then the former equation gives k:

$$\Phi = A + B_A \mathcal{X}^A + C \tfrac{1}{2} \mathcal{X}^A \mathcal{X}_A \ , \quad k = 2AC - B^2$$

Common coordinate choices are

$$\Phi = \begin{cases} 1 + \frac{1}{4} k \mathcal{X}^2 & \text{(Cartesian) stereographic} \\ B \cdot \mathcal{X} & \text{“Poincaré” (due to Liouville and Beltrami)} \end{cases}$$

where B is chosen to give k the appropriate sign and magnitude. (We can even choose it complex if we don't mind complex "gauges".) Such coordinates may not cover the whole space, but are sufficient when spacetime has Euclidean signature, as for Feynman diagrams.

The algebraic approach is to consider a D-dimensional hyperboloid, embedded in D+2 flat dimensions $\mathcal{Y}^I$ (with 1 extra space and 1 extra time dimension), as the intersection of a (hyper)cone

$$\mathcal{Y}^2 = 0 \quad \Rightarrow \quad \mathcal{Y}^I = e \mathcal{W}^I \ , \quad (\mathcal{W}^+, \mathcal{W}^A, \mathcal{W}^-) = (1, \mathcal{X}^A, \tfrac{1}{2}\mathcal{X}^2) \ , \quad d\mathcal{Y}^2 = e^2 d\mathcal{X}^2$$

(since $\mathcal{W}^2 = 0 \Rightarrow \mathcal{W} \cdot d\mathcal{W} = 0$; e is a variable related to the worldline metric) with a plane

$$n_I \mathcal{Y}^I = 1 \quad \Rightarrow \quad e^{-1} = n \cdot \mathcal{W} = n_+ + n_A \mathcal{X}^A + n_- \tfrac{1}{2} \mathcal{X}^2$$

(Without this second constraint we would just have a general conformally flat metric.) We then see the relation to the previous as

$$e^{-1} = \Phi \ , \quad k = -n^2$$

$$n_I = \begin{cases} (1, 0, \frac{1}{2}k) & \text{stereographic} \\ (0, n_A, 0) & \text{Poincaré} \end{cases}$$

Besides the metric, we can also look at invariants for finite differences in position as

$$(\mathcal{Y}_1 - \mathcal{Y}_2)^2 = -2 \mathcal{Y}_1 \cdot \mathcal{Y}_2 = e_1 e_2 (\mathcal{X}_1 - \mathcal{X}_2)^2$$

(This is the chord length in the embedding space, which is a function of, but not the same as, the arc length through the symmetric space.)

From this approach we can easily see the cosets in terms of the unbroken symmetry: Noting that the symmetry group leaves $n \cdot y$ invariant, while the "vacuum" has symmetry SO(p,q), the coset is

$$\left.\begin{array}{ll} k>0: & \text{SO(p+1,q)} \\ k=0: & \text{ISO(p,q)} \\ k<0: & \text{SO(p,q+1)} \end{array}\right\} /\text{SO(p,q)}$$

(The SO(p,q) indicates both the number of dimensions of the space and its signature.) Note the relation to general relativity, where SO(p,q) is identified with the local Lorentz symmetry on the tangent space, as appears in the covariant derivatives, while the global symmetry is described by Killing vectors that commute with the covariant derivatives. Some frequent examples are

	$k > 0$	$= 0$	< 0
q $= 0$	sphere	Euclidean	hyperbolic
1	de Sitter	Minkowski	anti-de Sitter

(For D $= 1$ you can easily picture this construction as giving an ellipse, parabola, and hyperbola, respectively, though you need higher D to get actual geometry.)

Another way to look at it for $n^2 \neq 0$ is to pick a coordinate in $\mathcal{Y}$ in the n direction: Then

$$\mathcal{Y}^I = (\mathcal{Y}_0, \mathcal{Y}^{\mathcal{A}}) = (\mathcal{R}, X^{\mathcal{A}}) \quad \Rightarrow \quad X^2 + n^{-2} = 0 \ , \quad d\mathcal{Y}^2 = dX^2$$

from which we recognize the hyperboloid in constrained coordinates $X^{\mathcal{A}}$ where the symmetry group is manifest, with $\mathcal{R} = |n^2|^{-1/2}$. In comparison with Poincaré coordinates of the previous discussion

$$\mathcal{X}^A = (x_0, x^a) \ , \qquad n_A \mathcal{X}^A = x_0/\mathcal{R}$$

$$-ds^2 = \mathcal{R}^2 \frac{dx^2 \pm dx_0^2}{x_0^2} \ , \quad (\mathcal{Y}_1 - \mathcal{Y}_2)^2 = \mathcal{R}^2 \frac{(x_1 - x_2)^2 \pm (x_{01} - x_{02})^2}{x_{01} x_{02}}$$

where "x" now refers to just x^a, and $-1/k = n^{-2} = \pm\mathcal{R}^2$, we now have

$$I = (+, 0, a, -) \ , \quad \mathcal{A} = (+, a, -) \ , \quad A = (0, a)$$

$$X^{\mathcal{A}} = (X^+, X^a, X^-) = \frac{\mathcal{R}}{x_0}(1, x^a, \tfrac{1}{2}(x^2 \pm x_0^2))$$

What is often called the "horizon" ($x_0 = \infty$) in AdS, because of its derivation from the near-horizon limit of D-brane solutions to supergravity, is actually part of the *boundary* ($x_0 = 0$) of AdS, as can be seen if one uses coordinates that are nonsingular there: AdS$_D$ is a maximally symmetric space, and has a boundary and the bulk, nothing else. In fact, in the Euclidean version of AdS ($x^2 \geq 0$), the so-called "horizon" is actually the point (since the $1/x_0^2$ kills the dx^2) at infinity on the boundary, which completes it topologically from R^{D-1} to S^{D-1}. (This is the same as what we do in 2D electrostatics or on the worldsheet, where the complex plane is then treated as a sphere topologically.) This is convenient for conformal transformations on the boundary: For example, inversions $X^+ \leftrightarrow X^-$ switch infinity with the origin. In terms of the embedding coordinates $X^{\mathcal{A}}$, these are the 2 related points $(0,0,\infty)$ and $(\infty,0,0)$. So, statements often found in the literature about the "horizon limit" of AdS should properly be stated as "long-distance" limits on the boundary. (Note that in CFT, as on the boundary, geometry is poorly defined, as we really have only conformal geometry: The metric is defined only up to a scale, which is here $1/x_0^2$.)

Fields on AdS

We next look at the Klein-Gordon equation on AdS_{D+1}. (So in the previous construction, $\mathcal{Y}$ would be for SO(D+1,2) in D+3 dimensions, and X would be for SO(D,2) in D+2 dimensions, before constraining.) Rather than find the general solution, we examine how the mass relates to the asymptotic behavior of wave functions toward the AdS boundary. (This will prove useful later for checking the AdS/CFT correspondence.) A convenient coordinate system for this is Poincaré coordinates: From the above analysis we see for AdS $k < 0 \Rightarrow n^2 > 0$, so we must pick a spatial coordinate x_0 for which

$$-ds^2 = \mathcal{R}^2 \frac{dx^2 + dx_0^2}{x_0^2}$$

Since for $dx = 0$, $-ds^2 = \mathcal{R}^2 (d\,ln\,x_0)^2$, the space has $ln\,x_0 \in [-\infty, \infty] \Rightarrow x_0 \in [0, \infty]$, so the boundary is $x_0 = 0$. (For quantum purposes, we'll stick mostly to Euclidean signature, i.e., hyperbolic space: Then $x_0 = \infty$ is just a point. In Lorentzian signature, these coordinates don't cover the whole space.)

The mass^2 is defined by the d'Alembertian, e.g., as a Casimir of the spacetime symmetry group; this definition is independent of the representation of this group; it is the relation between energy and spatial momentum. Using the above expression for the covariant derivative

$$\varPhi = \frac{x_0}{\mathcal{R}} \quad \Rightarrow \quad \mathcal{R}\nabla_0 = x_0 \partial_0 \ , \quad \mathcal{R}\nabla_a = x_0 \partial_a + M_{a0}$$

$$\Rightarrow \quad \mathcal{R}^2 m^2 = \mathcal{R}^2 \Box \equiv \mathcal{R}^2 \nabla^A \nabla_A = [(x_0\partial_0)^2 - \mathrm{D} x_0 \partial_0 + (M_{a0})^2] + [2x_0 \partial^a M_{a0}] + [x_0^2 \partial_a^2]$$

where we have used $[M_{a0}, \nabla^a] = -\mathrm{D}\nabla_0$ for D values of a.

We now want to evaluate the boundary limit $x_0 \to 0$. Noting that the metric in the x_0 direction goes like dx_0^2/x_0^2, it's clear that $ln\ x_0$ should be interpreted as the usual coordinate. So we're looking for the usual exponential fall-off in a spatial coordinate at ∞ (compare, e.g., a massive flat-space propagator). Here that means $x_0 = \infty$, which is a point in hyperbolic space (Euclidean, Wick-rotated signature). Another way to think of it is that we are looking for solutions generated by sources near the boundary, so they should have exponential decay in $ln\, x_0$ away from $x_0 = 0$.

For the AdS/CFT correspondence, we'll have a source term $\int dx\, \overset{\circ}{\phi}\overset{\circ}{\mathcal{O}}$: If the CFT operator $\mathcal{O}$ has some scale weight $\triangleleft$, then the AdS field ϕ must have weight D$-\triangleleft$ for scale invariance. ($\triangleleft$ will always be nonnegative, and 0 only for the identity operator.) Thus

$$\lim_{x_0 \to 0} \overset{\circ}{\phi}(x, x_0) = (x_0)^{D-\triangleleft}\phi(x)$$

Similarly

$$\overset{\circ}{\mathcal{O}} = (x_0)^{\triangleleft}\mathcal{O}$$

The conformally invariant boundary integration measure is actually

$$dx \equiv d^D x/(x_0)^D$$

so x_0's cancel, a manifestation of scale invariance.

The leading power of x_0 in the above Klein-Gordon equation then gives ($x_0\partial_0 =$ D$-\triangleleft$ on ϕ)

$$\mathcal{R}^2 m^2 = \triangleleft(\triangleleft - \mathrm{D}) + M_{a0}^2$$

(Note that there is no ordering ambiguity in M_{a0}^2, since for each a we have a square.) We can also think of M_{a0}^2 as the difference of the 2 quadratic Casimirs of SO(D,1) and SO(D$-$1,1),

$$M_{a0}^2 = \tfrac{1}{2}(M_{AB}^2 - M_{ab}^2)$$

i.e., it's the Casimir of the coset SO(D,1)/SO(D$-$1,1), dS$_D$.

For example, for scalars $M_{a0} = 0$. From the reality of $\triangleleft$, we then have the Breitenlohner-Freedman bound

$$\mathcal{R}^2 m^2 = \triangleleft(\triangleleft - D) = (\triangleleft - \tfrac{1}{2}D)^2 - (\tfrac{1}{2}D)^2 \geq -(\tfrac{1}{2}D)^2$$

We next evaluate M_{a0}^2 for some of the more interesting cases. After the trivial case of the scalar comes the Dirac spinor, for which in our conventions

$$S_{AB} = -\tfrac{1}{2}\gamma_{[A}\gamma_{B]}\ , \quad \{\gamma_A, \gamma_B\} = -\eta_{AB} \quad \Rightarrow \quad M_{a0}^2 = -\tfrac{1}{4}\mathrm{D}$$

where we have used the relation between first- and second-quantized operators that $S_{AB} = -M_{AB}$ acting on a field. (Remember that x_0 is spacelike. Also note that in our conventions $(\gamma_0)^2 = -1/2$.) However, the mass in the Klein-Gordon equation (which is more useful for supersymmetric comparison to the scalar), is not the same as the mass parameter in the Dirac equation; the only way we know to identify it as mass is by squaring the Dirac equation (even if only for purposes of normalization). The relation is

$$0 = \left(\nabla\!\!\!/ + \frac{i}{\sqrt{2}}M\right)\left(\nabla\!\!\!/ - \frac{i}{\sqrt{2}}M\right) = \gamma^A\gamma^B\nabla_A\nabla_B + \tfrac{1}{2}M^2$$

$$= \tfrac{1}{2}\{\gamma^A,\gamma^B\}\nabla_A\nabla_B + \tfrac{1}{2}[\gamma^A,\gamma^B]\nabla_A\nabla_B + \tfrac{1}{2}M^2 = \tfrac{1}{2}(-\Box + M^2) + \tfrac{1}{2}\gamma^A\gamma^B\mathcal{R}^{-2}\tfrac{1}{2}\gamma_{[A}\gamma_{B]}$$

where we have used the AdS curvature in $[\nabla_A, \nabla_B]$. We thus find

$$\mathcal{R}^2M^2 = \mathcal{R}^2m^2 + \tfrac{1}{4}(\mathrm{D}^2 + \mathrm{D}) = \lhd(\lhd - \mathrm{D}) + \tfrac{1}{4}\mathrm{D}^2 = (\lhd - \tfrac{1}{2}\mathrm{D})^2$$

$$\Rightarrow \quad \mathcal{R}M = |\lhd - \tfrac{1}{2}\mathrm{D}|$$

The next case we consider is p-forms. Looking only at forms for which the "longitudinal" (0) components vanish towards the boundary (after evaluating the action of M_{AB}), and noting that M_{AB} acts the same on a vector as on ∇_A, we have

$$M_{a0}^2 = -p$$

However, again the mass parameter M^2 appearing in the field equation that follows from the action is not the same as the mass in the Klein-Gordon equation derived from it by gauge fixing (in the Stückelberg formalism when massive) or applying the divergence of the field equation as a constraint. We can make that comparison in a way similar to the Dirac spinor, by treating p-forms as Dirac matrices, i.e., writing an arbitrary spinor$\otimes$spinor as a sum over antisymmetric products of γ-matrices. The field equation then looks like the Dirac equation, with the γ in $\nabla\!\!\!/$ multiplying from one particular side of this field matrix. (This is not the usual massive field equation for p-forms, which instead involves the sum of multiplying on either side of the field matrix by $\nabla\!\!\!/$, with similar results: See exercise IIB4.1 in *Fields*.) But M_{AB} yields a spin operator S_{AB} that's the sum of terms multiplying from either side. Using the identity

$$\gamma^A\gamma_{B_1\ldots B_p}\gamma_A = -\tfrac{1}{2}(-1)^p[2p - (\mathrm{D}+1)]\gamma_{B_1\ldots B_p}$$

(e.g., using $B_n = n$ for the totally antisymmetrized $\gamma_{B_1\ldots B_p}$) we find

$$\mathcal{R}^2 M^2 = \mathcal{R}^2 m^2 - \tfrac{1}{2}\gamma^A\gamma^B(\gamma_{[A}\gamma_{B]} + \gamma'_{[A}\gamma'_{B]}) = \mathcal{R}^2 m^2 + \tfrac{1}{4}(\mathrm{D}^2+\mathrm{D}) - \tfrac{1}{2}\gamma^A\gamma^B(...)\gamma_{[B}\gamma_{A]}$$
$$= \lhd(\lhd - \mathrm{D}) + p(\mathrm{D}-p)$$

Fortunately, for the comparison we're interested in we'll only need to look at the true mass m^2 that appears in the Klein-Gordon equation. Also, for the simple case we consider we'll only need to look at the scalar, as the rest will be determined by supersymmetry.

Boundary limit

The basic idea is that AdS_{D+1} is described by the coset SO(D,2)/SO(D,1), in the same way that S^{D+1} is SO(D+2)/SO(D+1). (In particular, for D = 4 we have SU(2,2)/USp(2,2) and SU(4)/USp(4).) The boundary of AdS can then be defined by taking a group contraction where SO(D,1) becomes the Poincaré group ISO(D-1,1), while preserving the symmetry group. This corresponds to a limit where the radius $\mathcal{R}$ becomes small in a particular way. If we Wick rotate the sphere so it gains a boundary, a similar procedure can be applied there.

The (singular) boundary limit $x_0 \to 0$ is equivalent to the limit $\mathcal{R} \to 0$ (if $\mathcal{R}$ dependence is defined appropriately), describing flat space of one less dimension. This can be interpreted as the relation between active and passive approaches: Instead of moving to the boundary, we shrink the distance scale, effectively moving the boundary closer. (It is a type of long-distance limit, in contrast to the short-distance limit $\mathcal{R} \to \infty$ related to flat space.) In terms of the above metric, we first rescale

$$x_0 \to \mathcal{R}x_0 \quad \Rightarrow \quad -ds^2 \to \frac{dx^2 + \mathcal{R}^2 dx_0^2}{x_0^2}$$

(Alternatively, we can scale $x \to x/\mathcal{R}$ instead.) The limit $\mathcal{R} \to 0$ pinches AdS into a lightcone, reducing the conformal analysis to that of the projective lightcone: If we go back to our treatment of maximally symmetric spaces and use the same construction, but drop the condition $n \cdot \mathcal{Y} = 1$, we have a hypercone with metric $d\mathcal{Y}^2 = e^2 d\mathcal{X}^2$, with e arbitrary. (This is the same as we did for SU(2), after Wick rotation.) In this limit for the hyperboloid, x takes the place of $\mathcal{X}$ (and e was already $1/x_0$).

This limit contracts the gauge group SO(D,1) of the coset to ISO(D−1,1), while leaving the symmetry group SO(D,2) intact. In this limit x_0 survives in a trivial way: It makes dx^2/x_0^2 conformally invariant, and gives a simple way of seeing it. For purposes of describing just this flat space, it can be removed by introducing a "projective" scale invariance $\delta\mathcal{X} = \lambda(\mathcal{X})\mathcal{X}$ (as exists for the hypercone coordinates $\mathcal{Y}$ before applying $n \cdot \mathcal{Y} = 1$). The only symmetry generators with dependence on x_0 are the dilatation and conformal boosts.

As a simple, well-known example we consider a free, massless scalar field on AdS_{D+1}. Explicitly, we have for the *symmetry generators* for dilatations and conformal boosts

$$\Delta = x \cdot \partial_x + x_0 \partial_0$$

$$K_a = x_a(x \cdot \partial_x + x_0 \partial_0) - \tfrac{1}{2}(x^2 + \mathcal{R}^2 x_0^2)\partial_a$$

(Here "Δ" has nothing to do with propagators.) In the limit $\mathcal{R} \to 0$, these take the D-dimensional flat-space form, with $x_0 \partial_0$ acting as the scale weight.

A simple example of the AdS/CFT correspondence is the classical kinetic term for a scalar: Using the free field equation, this term becomes a total derivative, and thus a boundary term:

$$\int d^D x\, dx_0 \sqrt{-g}\,[(\nabla \mathring{\phi})^2 + m^2 \mathring{\phi}^2] = \int d^D x\, dx_0\, \partial_0(\mathring{\phi}\sqrt{-g} g^{00} \partial_0 \mathring{\phi}) = x_0^{1-D} \int d^D x\, \mathring{\phi} \partial_0 \mathring{\phi}$$

But the leading power in x^0 of $\partial_0 \mathring{\phi}$ in the boundary limit is related nonlocally in x to the leading power in $\mathring{\phi}$, since they come from different powers in x^0 of $\mathring{\phi}$. However, we know that the propagator $\langle \mathring{\phi}\ \mathring{\phi} \rangle$ in the bulk must be a function of the invariant $(x_{12}^2 + x_{012}^2)/x_{01}x_{02}$, so that in a boundary limit

$$\Delta(1,2) \equiv \lim_{x_{01}\ or\ x_{02} \to 0} \langle \mathring{\phi}(1)\ \mathring{\phi}(2) \rangle_{AdS} \sim \left(\frac{x_{01}x_{02}}{x_{12}^2 + x_{012}^2} \right)^a$$

for some a. The propagator is used in general as

$$\mathring{\phi}(1) \sim \int d\Sigma_2^m\, \Delta(1,2) \overleftrightarrow{\partial}_{2m} \mathring{\phi}(2)$$

where in our case, as from integration by parts above,

$$d\Sigma^m = \delta_0^m x_0^{1-D} d^D x$$

Taking $x_{02} \to 0$, using also the asymptotic dependence of $\mathring{\phi}$ on x_0,

$$\lim_{x_0 \to 0} \mathring{\phi} = x_0^{D-\lhd} \phi(x) \quad \Rightarrow \quad \mathring{\phi}(1) \sim x_{02}^{1-D} \int d^D x_2 \left(\frac{x_{01}x_{02}}{x_{12}^2 + x_{012}^2} \right)^a x_{02}^{D-\lhd-1} \phi(x_2)$$

We see that we can now take $x_{02} = 0$ consistently if

$$a = \lhd$$

so

$$\overset{\circ}{\phi}(1) \sim \int d^D x_2 \left(\frac{x_{01}}{x_{12}^2 + x_{01}^2} \right)^{\lhd} \phi(x_2)$$

In particular, since

$$\lim_{x_{01}\to 0} \left(\frac{x_{01}}{x_{12}^2 + x_{01}^2} \right)^{\lhd} \sim x_{01}^{D-\lhd} \delta^D(x_{12})$$

this is consistent with that limit. We can now neglect the x_0's in $x_{12}^2 + x_{01}^2$ in the limit $x_{01} \to 0$ when taking the derivative,

$$(\partial_0 \overset{\circ}{\phi})(1) \sim x_0^{\lhd-1} \int d^D x_2 \left(\frac{1}{x_{12}^2 + x_{01}^2} \right)^{\lhd} \phi(x_2)$$

(One might expect again a δ^D, times a different power of x_{01}, but this is spoiled by the term we dropped, higher order in x_{01}.) Finally, the term in (the exponent of) Z_{string} is then

$$Z_{string} \sim \int d^D x_1\, d^D x_2 \, \frac{\phi(x_1)\phi(x_2)}{(x_{12}^2)^{\lhd}} \quad \Rightarrow \quad \langle \mathcal{O}(x_1)\ \mathcal{O}(x_2) \rangle_{CFT} \sim \frac{1}{(x_{12}^2)^{\lhd}}$$

which is the expected result.

Note that $\langle \mathcal{O}(x_1)\ \mathcal{O}(x_2) \rangle_{CFT}$ is similar to $\langle \phi(1)\ \phi(2) \rangle_{AdS}$ (at least near the boundary), in spite of the fact that they are "dual" to each other, in the sense that $\overset{\circ}{\phi} \sim x_0^{D-\lhd}$ while $\overset{\circ}{\mathcal{O}} \sim x_0^{\lhd}$. (See the δ function definition above.)

$\mathrm{AdS}_5{\times}\mathrm{S}^5$

IIB supergravity

The field strengths of 10D IIB supergravity have a simple description on chiral superspace, where the chiral 16-component θ is the complex combination of the left- and right-handed θ's, as $\theta = \theta_1 + i\theta_2$, which we can do for IIB because both θ's have the same Lorentz-chirality. There is a U(1) symmetry that mixes θ_1 and θ_2 (as the usual SO(2)), so the complex θ and its complex conjugate $\bar{\theta}$ have opposite U(1) eigenvalues. (This is a continuous symmetry for supergravity, but broken to a discrete subgroup by the massive string states.) The superfield strength is a scalar, with expansion

$$\chi(\theta) = \phi + \theta^\alpha\lambda_\alpha + \tfrac{1}{2}\theta^\alpha\theta^\beta H_{\alpha\beta} + \tfrac{1}{6}\theta^\alpha\theta^\beta\theta^\gamma\mathcal{R}_{\alpha\beta\gamma} + \tfrac{1}{24}\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta R_{\alpha\beta\gamma\delta} + ...$$

where H is the field strength of the complex 2-form, $\mathcal{R}$ is the γ-traceless one of the complex gravitino, and R is the Weyl tensor + the covariant derivative of the selfdual field strength (less curl and divergence) of the real 4-form. (Note that the constant vacuum values of the Ricci tensor and selfdual field strength for $\mathrm{AdS}_5{\times}\mathrm{S}^5$ don't contribute to χ.) The counting for spinor notation vs. mixed vector-spinor is (from the Young tableaux for SO(16) vs. SO(10))

$$H : 16\cdot 15/2 = 120 = 10\cdot 9\cdot 8/6\ , \qquad \mathcal{R} : 16\cdot 15\cdot 14/6 = 560 = (10\cdot 9/2 - 10)\cdot 16\ ,$$

$$R\ (+\nabla F) : 16\cdot 15\cdot 14\cdot 13/24 = 1820 = 770 + 1050$$

$$= (11\cdot 10\cdot 10\cdot 9/12 - 11\cdot 10/2) + \tfrac{1}{2}(11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6/144 - 10\cdot 9\cdot 8\cdot 7/24)$$

Since R and F are real, the U(1) weight of θ implies increasing (or decreasing, depending on convention) U(1) weights for the fields as one moves lower in orders in θ.

The reality condition on R, and the determination of the unlisted higher-θ components in terms of spacetime derivatives of the complex conjugates of the lower ones, as well as the field equations, are all contained in the condition

$$d^4_{\alpha\beta\gamma\delta}\chi = \bar{d}^4_{\alpha\beta\gamma\delta}\bar{\chi}$$

where d^4 and $\bar{d}^4$ refer to chiral and antichiral covariant spinor derivatives, totally antisymmetric in spinor indices, corresponding to the θ expansion. When comparing to the 4D boundary later, we'll see this θ includes both 4D supersymmetry θ's and 4D S-supersymmetry θ's (as the full superspace has 32 θ's, not 16), so expansion in the 10D θ produces components that can have either higher or lower 4D conformal weight.

This chiral superspace has a nice coset on $AdS_5\times S^5$: Starting with PSU(2,2|4), we just separate the bosonic and fermionic indices:

$$Z_M{}^A = \begin{array}{c} \\ \underline{m} \\ \underline{\mu} \end{array} \!\!\begin{array}{c} \underline{a} \quad \underline{\alpha} \\ \begin{pmatrix} y & \theta \\ \bar\theta & x \end{pmatrix} \end{array}$$

The gauge group then has the bosonic part USp(2,2)⊗USp(4), which gives $AdS_5\times S^5 =$ (SU(2,2)/USp(2,2))⊗(SU(4)/USp(4)), while the fermionic is just 1 of the 2 fermionic blocks of the matrix, leaving the other block as the chiral θ. This makes it more convenient to use rectangles than the full square, like the 4D projective spaces:

$$z_{\underline{a}}{}^M \bar z_M{}^{\underline{\alpha}} = 0$$

$$\bar z = \bar z_M{}^{\underline{\alpha}} = (\theta_{\underline{m}}{}^{\underline{\nu}}, \delta_{\underline{\mu}}^{\underline{\nu}}) x_{\underline{\nu}}{}^{\underline{\alpha}} = \begin{pmatrix} \theta \\ I \end{pmatrix} x^{-1}, \qquad z = z_{\underline{a}}{}^M = y_{\underline{a}}{}^{\underline{n}} (\delta_{\underline{n}}^{\underline{m}}, -\theta_{\underline{n}}{}^{\underline{\mu}}) = y^{-1} \begin{pmatrix} I & -\theta \end{pmatrix}$$

The covariant derivatives thus take a similar form to those of the projective spaces considered earlier: Noting the correspondence

$$(y, x, \theta) \leftrightarrow (u, \bar u, w)$$

we then have

$$D_y = \partial_y y \ , \quad D_x = x\partial_x \ , \quad D_\theta = x\partial_\theta y$$

This x and y are the "square roots" of the usual 6D unit-vectors X and Y for AdS_5 and S^5: The latter are the gauge-group invariants (using the antisymmetric USp metrics Ω)

$$X = -X^T = X^{\underline{\mu\nu}} = x_{\underline{\alpha}}{}^{\underline{\mu}} \Omega^{\underline{\alpha\beta}} x_{\underline{\beta}}{}^{\underline{\nu}} = x^T \Omega x$$

and similarly for $Y = y\Omega y^T$. The norms of X and Y are given by their Pfaffians (which are quadratic); they are unity only after the P and S conditions have been imposed, which are

$$\det x = \det y = 1$$

In a convenient USp gauge, corresponding to Poincaré coordinates, we can choose

$$x_{\underline{\alpha}}{}^{\underline{\mu}} = \begin{matrix} & \mu & \dot{\mu} \\ \begin{matrix} \alpha \\ \dot{\alpha} \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} \delta_\alpha^\mu & x_\alpha{}^{\dot{\mu}} \\ 0 & \delta_{\dot{\alpha}}^{\dot{\mu}} x_0 \end{pmatrix}} \end{matrix} x_0^{-1/2}$$

$$\Rightarrow \quad X^{\underline{\mu\nu}} = \begin{matrix} & \nu & \dot{\nu} \\ \begin{matrix} \mu \\ \dot{\mu} \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} C^{\mu\nu} & x^{\mu\dot{\nu}} \\ -x^{\nu\dot{\mu}} & C^{\dot{\mu}\dot{\nu}}(x^2 + x_0^2) \end{pmatrix}} \end{matrix} x_0^{-1} = \begin{pmatrix} C^{\mu\nu} X^+ & X^{\mu\dot{\nu}} \\ X^{\dot{\mu}\nu} & C^{\dot{\mu}\dot{\nu}} X^- \end{pmatrix}$$

(In stereographic coordinates the square roots would be much messier.)

Note that these θ's carry 6D "curved" spinor indices and not 5D "flat" spinor indices. Of course, this does not generalize to arbitrary curved spaces, where spinors can carry only flat indices. We have converted flat into curved with x and y, the "square roots" of X and Y. Thus the θ expansion of any superfield will effectively contain extra factors of square roots of X and Y in the coefficient component fields. The result is that the chiral scalar superfield strength of 10D IIB supergravity satisfies a Klein-Gordon equation, at linear order in this field perturbed about the vacuum, that involves no θ derivatives: This modified definition of the component fields satisfies a Klein-Gordon equation that is independent of spin. (The same would not be true if we did a "covariant" expansion in θ as defined by the covariant θ derivatives, since they don't commute with the covariant x and y derivatives. This corresponds to a coordinate redefinition to flat θ's.)

Both 10D IIB supergravity and 4D N = 4 super Yang-Mills are representations of the group PSU(4|2,2). But the physical interpretation is different: For example, they satisfy different field equations, even at the free level. We saw the free field equations for (the field strengths of) 4D super Yang-Mills,

$$\text{linearized 4D N=4 super Yang-Mills:} \qquad D_{(A}{}^{(C} D_{B]}{}^{D]} = 0 \; mod \; \delta \; terms$$

and applied them in projective superspace. On the other hand, 10D supergravity satisfies different, weaker equations (since more dimensions), and with different coset gauge constraints: Its free field equations are

$$\text{linearized 10D IIB supergravity on AdS}_5{\times}\text{S}^5\text{:} \qquad D_A{}^C D_C{}^B = 0 \; mod \; \delta \; terms$$

In the boundary limit these are not the 4D Yang-Mills equations, but the equations satisfied by fields coupling to BPS color-singlet composites of the Yang-Mills superfields.

We saw the stronger equations implied $p^2 = 0$ in D = 4 by picking indices giving the highest (engineering) dimension; thus the rest of the equations followed by conformal supersymmetrization. That was easy, since all 4 indices were free in that case, whereas here some are contracted. Now we restrict to the bosonic sector of the weaker 10D equations, which is sufficient, as the supersymmetric generalization is unique. This means we truncate the symmetry group to SU(4)⊗SU(2,2), which is not the same as considering the N = 0 case. The field equations are then of the form

$$D_{\underline{a}}{}^{\underline{c}} D_{\underline{c}}{}^{\underline{b}} = \delta_{\underline{a}}^{\underline{b}} \mathcal{O}, \qquad D_{\underline{\alpha}}{}^{\underline{\gamma}} D_{\underline{\gamma}}{}^{\underline{\beta}} = \delta_{\underline{\alpha}}^{\underline{\beta}} \mathcal{O}$$

for some operator $\mathcal{O}$. These can be translated into vector notation as

$$D_{[\mathcal{AB}} D_{\mathcal{CD}]} = D_{[\mathcal{MN}} D_{\mathcal{PQ}]} = D^{\mathcal{AB}} D_{\mathcal{AB}} - D^{\mathcal{MN}} D_{\mathcal{MN}} = 0$$

which generalize to arbitrary $AdS_m \times S^n$, where only for this equation $\mathcal{A}$ and $\mathcal{M}$ are vector indices for SO(m−1,2) and SO(n+1). If we plug in the usual representations of these symmetry groups on these spaces, then the former 2 equations say that the corresponding spins vanish, while the last is the massless Klein-Gordon equation in m+n dimensions. In our supersymmetric case of $AdS_5 \times S^5$, these equations are unmodified on the chiral field strength, since the $\partial/\partial\bar{\theta}$ in the $\partial/\partial\theta\, \partial/\partial\bar{\theta}$ term vanishes.

If we had set $\mathcal{O}$ to vanish, decoupling the 2 spaces, we would instead have the massless m-dimensional Klein-Gordon equation on AdS, while on the sphere we would leave only a constant solution. In the supersymmetric case, this describes maximally supersymmetric 5D supergravity on AdS:

linearized 5D maximal supergravity on AdS_5: $\quad D_A{}^C D_C{}^B = 0 \quad$ (*including* δ *terms*)

Superlimit

We have already found the projective superspace that best describes 4D N = 4 super Yang-Mills on the CFT side of the correspondence: It has 8 bosonic coordinates and 8 fermionic, and the field strength lives in this superspace, at least on shell. (To be more precise, Yang-Mills traces of functions of it live there when the interacting field equations are satisfied. This is sufficient to describe BPS states.) On the AdS side, IIB supergravity (which also describes BPS states) has a field strength that lives in 10 bosonic dimensions and 16 fermionic (chiral superspace). This mismatch is fixed by the fact that we're actually interested only in asymptotic supergravity states. As in flat space, these are best represented (for massless fields) by lightcone superspace, which has only 9 bosonic dimensions (after solving for dependence on "time" x_0) and 8 fermionic (the usual reduction of spinors on the lightcone). As in the flat lightcone, the ninth bosonic coordinate has a distinctive role: Here its Taylor expansion gives the different 4D BPS multiplets.

Here we'll apply a slightly different procedure: In lightcone quantization the wave equation is solved for dependence on a lightlike coordinate. Furthermore, for applying twistor techniques to Feynman diagrams it's convenient to Wick rotate this idea to "spacecone" quantization, using a complex, null, spatial coordinate. We'll find it convenient to use a similar procedure here, to find the correspondence between the superspaces of AdS and CFT.

To see why such a treatment naturally arises, we work in Poincaré coordinates for S^5, after an appropriate Wick rotation. (This is the same as we did for S^2 when discussing SU(2).) Combining the two spaces (with the signs that follow from the grading),

$$ds^2 = \frac{dy^2 + \mathcal{R}^2 dy_0^2}{y_0^2} - \frac{dx^2 + \mathcal{R}^2 dx_0^2}{x_0^2} = \frac{dy^2}{y_0^2} - \frac{dx^2}{x_0^2} - \mathcal{R}^2\, d\, ln(x_0 y_0) d\, ln(x_0/y_0)$$

We can then identify $x_0 y_0$ and x_0/y_0 (or some functions of just one or just the other) as two null, spatial coordinates, to be used to define our spacecone quantization.

We then modify the usual boundary limit of AdS to

$$x^+ \equiv x_0 y_0 \to 0\ , \quad x^- \equiv x_0/y_0 \text{ fixed}$$

or (as already implemented above)

$$x_0 \to \mathcal{R} x_0\ , \quad y_0 \to \mathcal{R} y_0 \quad \text{followed by} \quad \mathcal{R} \to 0$$

in line with interpretation of $x_0 y_0$ as the spacecone "time". (Of course, x and y are also fixed.) This leaves us with 9 bosonic coordinates on the boundary, 8 of which have translation invariance, and are to be identified with the 4 x's and 4 y's of 4D N = 4 projective superspace. (There is a symmetry under translation of the ninth coordinate, but it requires also scaling of the other 8, as well as the fermions. It is associated with a combination of a dilatation with an R-symmetry U(1).) Thus the bosonic gauge group $SO(4,1)^2$ (after our Wick rotation) has contracted to $ISO(3,1)^2$. (The associated geometry is a bit funny: There is something like 8-dimensional branes corresponding to this 9D boundary. In terms of the string action, which we won't discuss here, it results from a kind of T-duality transformation on the 4 y's and some of the θ's.)

To generalize this limit to superspace, we require that this limit preserves the symmetry group PSU(4|2,2), which is a symmetry on both the AdS and CFT sides (hence the correspondence). This means that on the group coordinates $Z_M{}^A$ (or inverse $Z_A{}^M$) it should affect only the flat index A, and not the curved index M.

Thus, although the symmetry group is untouched, the gauge group is contracted, just as we saw in the pure, bosonic AdS case.

We return to the 4D parametrization of a group element of PSU(2,2|4), but keep all elements so as to include those in AdS superspace:

$$Z_M{}^A = \begin{pmatrix} I & w \\ 0 & I \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \bar{u}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -v & I \end{pmatrix}$$

In particular, it's easy to pick out x_0 and y_0 as the pieces of u and $\bar{u}$ invariant under the manifest SO(3,1) Lorentz and SO(4) internal symmetries, after killing the "PS" pieces of PSU(4|2,2):

$$u = \begin{pmatrix} \sqrt{y_0}\, I & 0 \\ 0 & \sqrt{x_0}\, I \end{pmatrix} u_0, \qquad \bar{u} = \begin{pmatrix} \sqrt{y_0}\, I & 0 \\ 0 & \sqrt{x_0}\, I \end{pmatrix} \bar{u}_0$$

$$sdet\ u_0 = sdet\ \bar{u}_0 = det\ u_0 = det\ \bar{u}_0 = 1$$

This can be seen, e.g., by considering the N = 0 case, and noting that there $det(zd\bar{z}) = dx^2/x_0^2$ is the metric of the projective lightcone. We then have

$$sdet\ u = sdet\ \bar{u} \equiv \frac{1}{x^-} = \frac{y_0}{x_0}$$

Then the limit on just x_0 and y_0, acting on just the flat indices, must be the $\mathcal{R} \to 0$ limit after the rescaling

$$z_M{}^A \to \left(\sqrt{\mathcal{R}} z_M{}^{\underline{A}}, \frac{1}{\sqrt{\mathcal{R}}} z_M{}^{\underline{A}'} \right), \qquad z_A{}^M \to \left(\frac{1}{\sqrt{\mathcal{R}}} z_{\underline{A}}{}^M, \sqrt{\mathcal{R}} z_{\underline{A}'}{}^M \right)$$

or in terms of the above variables

$$w \to w\ , \qquad u \to \sqrt{\mathcal{R}} u\ , \qquad \bar{u} \to \sqrt{\mathcal{R}} \bar{u}\ , \qquad v \to \mathcal{R} v$$

(w carries only curved indices, v has 2 flat ones.) The metric takes the form

$$ds^2 = \frac{x^- dy^2 - (x^-)^{-1} dx^2 - (x^-)^{-1} dx^+ dx^-}{x^+}$$

Then the $\mathcal{R}$ rescaling can be interpreted as defining the x^+ dependence as

$$v \sim \sqrt{x^+}\ ; \quad u, \bar{u} \sim (x^+)^{1/4}\ ; \quad w \sim 1$$

followed by scaling just $x^+ \to \mathcal{R}^2 x^+$ (or just taking $x^+ \to 0$ directly).

Spacecone gauge

The limit defined above matches up the bosonic coordinates for AdS and CFT (except for x^-), but at this point there is a mismatch between the fermions because: (1) Those for the IIB supergravity field strength are chiral, whereas the 4D N = 4 Yang-Mills superfield strength is twisted chiral; and (2) the AdS chiral superspace has 16 fermions, while the 4D projective one has 8. These problems will be fixed by first introducing a lightcone (really spacecone) superspace for AdS, and then expressing the field strength in terms of a real prepotential in a twisted chiral superspace, as suggested by the reality condition satisfied by the field strength. (In principle, there should be a 10D "supertwistor" solution to this problem that eliminates the 8 extra fermions covariantly.)

Decomposing the 10D field equations into 4D projective blocks,

$$I\mathcal{O} \sim \begin{pmatrix} D_u & -D_v \\ D_w & -D_{\bar{u}} \end{pmatrix}^2 = \begin{pmatrix} D_u^2 - D_v D_w & -D_u D_v + D_v D_{\bar{u}} \\ D_w D_u - D_{\bar{u}} D_w & -D_w D_v + D_{\bar{u}}^2 \end{pmatrix}$$

The covariant x^- derivative p^+ appears as

$$D_u \, , D_{\bar{u}} = p^+ I + ...$$

(for identity matrix "I"), which in turn defines how constraints are solved. For example, for the fermionic ("κ symmetry") constraints, in the lightcone formalism we pick the half that have p^+ and not p^-, and divide by p^+. Here we see that p^+ cancels when multiplying d_w or d_v, but not d_u nor $d_{\bar{u}}$: Therefore the latter d's are taken as "auxiliary", and we eliminate θ_u and $\theta_{\bar{u}}$ as a gauge choice. (For abbreviation we now use "x" for all bosons and "θ" for all fermions, and "p" and "d" for the corresponding covariant derivatives, with subscripts indicating from where they come.)

In such a lightcone-like gauge, expansion in lightcone θ's picks out field strengths which have a trivial relation to gauge fields. For example, for p-forms (including nonabelian Yang-Mills) and gravity we see only (in vector notation)

$$F^{+i_1 ... i_p} = p^+ A^{i_1 ... i_p} \ , \quad R^{+i+j} = (p^+)^2 h^{ij}$$

where i are the transverse indices (excluding $\pm$), as follows from the gauge choice $A^{+\cdots} = h^{+\cdots} = 0$. In our case, 10D vector indices reduce to transverse 8D vector indices that are just the 4+4 of x and y. (Remember from our coset discussion that spin is generally treated by indices from the gauge group, the bosonic part of which is SO(3,1) on x and SO(4) on y, up to some U(1)'s.) Furthermore, $p^+ = \partial/\partial(ln\, x^-)$ is just an integer (at least near the boundary), so we can essentially identify the field strengths with their gauge fields (at least for counting purposes). So we now have (distinguishing SL(2,C) dotted (barred) spinors $\bar\theta_v$ and undotted θ_w),

$$\chi(\theta_w,\bar\theta_v)=\phi+\theta\lambda+\tfrac12\theta^2p^+B+\tfrac16\theta^3p^+\psi+\tfrac1{24}\theta^4p^{+2}(h+A)+...$$

and the counting for spinor notation vs. mixed vector-spinor is, from 2 different kinds of Young tableaux for SO(8) (really SO(7,1)) that are related by triality,

$$\phi:1\ ,\qquad \lambda:8\ ,\qquad B:8\cdot7/2=28\ ,\qquad \psi:8\cdot7\cdot6/6=56=(8-1)\cdot8\ ,$$

$$h+A:8\cdot7\cdot6\cdot5/24=70=35+35=(8\cdot9/2-1)+\tfrac12 8\cdot7\cdot6\cdot5/24$$

(But remember all but h and A are complex.) This should be compared to our previous expansion for the covariant superfield in terms of field strengths.

The solution to the reality condition on the chiral field strength $\chi(\theta_w,\bar\theta_v)$ is then given in terms of the prepotential $V(\theta_w,\bar\theta_w)$ by

$$d_w^4\chi=\bar d_w^4\bar\chi\quad\Rightarrow\quad\chi=\bar d_w^4V\ ,\quad\bar\chi=d_w^4V\ ;\qquad V=\overline{V}$$

where the d^4's are now scalars from the product of all 4 components of the corresponding d's. (There are also redundant reality conditions, $\bar d_v^4\chi=d_v^4\bar\chi$ and others from switching various numbers of d_w with $\bar d_v$.) This is essentially a Fourier transform in the fermions (up to powers of p^+), replacing $\bar\theta_w$'s in V with $\bar\theta_v$'s in χ. We also have

$$d_vV=\bar d_vV=0;\qquad d_v\chi=\bar d_w\chi=0,\quad \bar d_v\bar\chi=d_w\bar\chi=0$$

using

$$\{d_v,d_w\}\sim\{\bar d_v,\bar d_w\}\sim p^+,\qquad\{d_v,\bar d_w\}=\{\bar d_v,d_w\}=0$$

$$\{d_v,d_v\}=\{d_v,\bar d_v\}=\{d_w,d_w\}=\{d_w,\bar d_w\}=0$$

So we need only look at how the covariant spinor derivatives $\bar d_w\approx\partial/\partial\bar\theta+\bar\theta p^+$ rearrange components in going from $\chi(\theta_w,\bar\theta_v)=\bar d_w^4V$ to $V(\theta_w,\bar\theta_w)$ (but only with respect to $\bar\theta$, not θ), or $\bar d_v$ for $V=(p^+)^{-4}\bar d_v^4\chi$.

Then, writing the field strengths in terms of the gauge fields as above, we have (dropping the p^+'s)

$$\begin{aligned}V=&\ (h+A)+(\theta\bar\psi+\bar\theta\psi)+(\theta^2\bar B+\bar\theta^2B)+\theta\bar\theta(h+A)+(\theta^3\bar\lambda+\bar\theta^3\lambda)+(\theta^2\bar\theta\bar\psi+\bar\theta^2\theta\psi)\\&+(\theta^4\phi+\bar\theta^4\bar\phi)+(\theta^3\bar\theta\bar B+\bar\theta^3\theta B)+\theta^2\bar\theta^2(h+A)+(\theta^4\bar\theta\bar\lambda+\bar\theta^4\theta\lambda)+(\theta^3\bar\theta^2\bar\psi+\bar\theta^3\theta^2\psi)\\&+(\theta^4\bar\theta^2\bar B+\bar\theta^4\theta^2B)+\theta^3\bar\theta^3(h+A)+(\theta^4\bar\theta^3\bar\psi+\bar\theta^4\theta^3\psi)+\theta^4\bar\theta^4(h+A)\end{aligned}$$

where unbarred fields are those that appear at lower orders in χ. But all these fields can easily be guessed just by matching U(1) weights. The different SO(4)⊗SO(3,1) components of these SO(7,1) fields are distinguished by their SO(3,1) and SO(4) spinor indices: In terms of $\theta^{a'\alpha}$ and $\bar{\theta}^{a\dot{\alpha}}$, θ^4 and $\bar{\theta}^4$ are SO(4) and SO(3,1) singlets, so we have also $(\theta^3)^{a'\alpha}$ and $(\bar{\theta}^3)^{a\dot{\alpha}}$; the other powers are $(\theta^2)^{(a'b')}$, $(\theta^2)^{(\alpha\beta)}$, $(\bar{\theta}^2)^{(ab)}$, and $(\theta^2)^{(\dot{\alpha}\dot{\beta})}$ (from symmetrizing in one type of indices and antisymmetrizing = contracting in the other, for total antisymmetry). The only components that are ambiguous are the "$h+A$": The $\theta^2\bar{\theta}^2$ break up nicely into separate h and A pieces, but the $\theta\bar{\theta}$ and $\theta^3\bar{\theta}^3$, and the 1 and $\theta^4\bar{\theta}^4$, are different linear combinations of h and A that can be fixed by comparing to χ.

Correspondence

In general in the AdS/CFT correspondence, it's the AdS gauge field that couples to the CFT composite-field "source" at the boundary. So now we have a spacecone gauge prepotential V coupling to a 4D N = 4 Yang-Mills source, both of which live in 4D N = 4 projective superspace (which is integrated over in the source term), plus the extra coordinate x^- (the x^+ in V being eliminated at the boundary). Unlike the case of bosonic AdS, the measure $\int d^4x\, d^4y\, d^4\theta\, d^4\bar{\theta}$, the 10D supergravity prepotential V, and the 4D N = 4 YM BPS operators (before extracting x^- dependence) are all dimensionless, so no powers of x^+ need be canceled.

We now investigate the significance of this ninth coordinate x_0/y_0 to the CFT. Consider expansion of the 10D theory over S^5 in terms of spherical harmonics. These can all be expressed in terms of those for the vector harmonic, which are given by a unit 6-vector; in the coordinates we've been using, these are (after scaling in an $\mathcal{R}$ to make the limit obvious)

$$Y^{\mathcal{A}} = (Y^+, Y^a, Y^-) = \frac{(1, y^a, \frac{1}{2}(y^2 + \mathcal{R}^2 y_0^2))}{y_0}\ , \qquad Y^2 = -\mathcal{R}^2$$

(Y is thus a position vector in the embedding space of the Wick rotated sphere of radius $\mathcal{R}$. Spinors can also be described by the method given above for the superspace of IIB supergravity on the AdS$_5\times$S^5 background, using the matrix square root y of Y as the spinor spherical harmonic.) In the boundary limit, this becomes a null 6-vector,

$$Y^{\mathcal{A}} \to \frac{(1, y^i, \frac{1}{2}y^2)}{y_0}\ , \qquad Y^2 \to 0$$

homogeneous in y_0. (Similarly, the spinor spherical harmonic becomes a projection operator in this limit. We saw the same behavior for arbitrary spin in the case of

SU(2), where we looked at the $\mathcal{R} \to 0$ limit of Wick-rotated S^2.) This y dependence can clearly be associated with that of the scalars of 4D N = 4 Yang-Mills, i.e., the field strength Φ at $\theta = 0$. A similar analysis can be made for the x_0 dependence of the scalars, as discussed above. (In general, interactions modify this result; but for the fundamental fields of 4D N = 4 Yang-Mills, and the BPS composite operators considered here, ultraviolet finiteness preserves conformal weights.)

Generalizing to the rest of that superfield, we note from our previous discussion of spin in projective superspace, for the case of scalar superfields, we have

$$\mathcal{O}(w, u, \bar{u}) = (x^-)^{r+\bar{r}} \mathcal{O}(w)$$

Our 10D supergravity prepotential is not required to be an eigenstate of $r + \bar{r}$. But we identified $r + \bar{r} = 1$ for the 4D N = 4 Yang-Mills superfield strength Φ.

We can easily supersymmetrize this result to identify the other fields of the supermultiplet, and see how they appear in color singlets. Returning to our analysis of general spin, noting that Φ is a scalar with $r + \bar{r} = 1$, we have

$$\Phi(w, u, \bar{u}) = x^- \Phi(w)$$

reproducing the x_0 and y_0 dependence found above for the scalars. x_0 dependence is determined by the superscale weight of the multiplet, and y_0 by the super-U(1) weight. The corresponding symmetry generators also have $\theta\partial_\theta$ terms, giving different component scale and U(1) weights to the higher spins. Thus, if we want powers of x_0 and y_0 associated with the usual component weights, we should redefine $\theta \to \sqrt{x_0 y_0}\theta$ in the θ expansion of Φ. This is automatic if we define the component expansion in a coordinate independent way by use of covariant derivatives: In these coordinates, $d_w = \bar{u}\partial_w u \sim \sqrt{x^+}$.

It then follows that the supergravity superfield source on the boundary must take the form

$$\mathcal{O}(w, x^-) = tr\{f[\Phi(w, x^-)]\}$$

for some (Taylor expandable) function f, and thus contains terms of the form

$$tr\{[x^- \Phi(w)]^n\}$$

Thus, the ninth bosonic coordinate on the boundary just counts the number of supergluons. Note that, unlike the usual $x_0 \to 0$ limit, in this limit the supergravity fields are nonvanishing, having no dependence on $x_0 y_0$ (but string excitations will have positive powers of $x_0 y_0$, corresponding to anomalous dimensions in the 4D field theory). Also for these supergravity fields on the boundary, the “momentum” p^+ conjugate to the coordinate $ln(x^-)$ is quantized.

Note that, e.g., the SU(4) representations coming from $tr(\Phi^n)$ at $\theta = 0$ will always be traceless, symmetric tensors of SO(6). This comes from the fact that the 6 scalars are contracted with the null-vector $Y^{\mathcal{A}}$. So for example,

$$(Y^{\mathcal{A}}\phi_{\mathcal{A}})^2 = Y^{\mathcal{A}}Y^{\mathcal{B}}\phi_{\mathcal{A}}\phi_{\mathcal{B}}$$

will contain only the symmetric (obviously), traceless part of $\phi \otimes \phi$ because

$$Y^{\mathcal{A}}Y^{\mathcal{B}}\eta_{\mathcal{AB}} = 0$$

(This would not be the case without the $y_0 \to 0$ limit. Thus an expansion over null vectors in R-space is more convenient than one over unit vectors.)

The 10D supergravity superfield is real. So the source superfield is also real. Thus Φ is forced to satisfy its (*interacting*) field equations.

Explictly, we can now match the component fields of the prepotential V of 10D IIB supergravity to those of the BPS CFT operators $\mathcal{O}$ (i.e., $tr(\Phi^n)$) directly (at least as V approaches the boundary), since they have the same θ coordinates, and we have identified the 10D IIB supergravity fields in V with the usual components in the chiral field strength χ. We can also associate the masses of the 10D supergravity states with the different conformal weights of the YM composites because of their x^0 dependence. But the x^0 dependence of the superfields is tied to the y^0 dependence (the masses come from Kaluza-Klein reduction over S^5), which defines the SU(4) representation. Furthermore, $p^+ = \partial/\partial(ln\, x^-)$ is just an integer, counting the power of Φ on the CFT side.

For example, consider the 10D scalars, which satisfy the 10D massless Klein-Gordon equation, in the boundary limit (by a similar analysis to our previous for just AdS),

$$\lhd(\lhd - 4) - \lhd_y(\lhd_y + 4) = 0 \quad \Rightarrow \quad \lhd = \lhd_y + 4$$

where the former term comes from the x part of the d'Alembertian, and the latter from the y. (The other solution, $\lhd = -\lhd_y$, can describe only the unit CFT operator, $\lhd = \lhd_y = 0$. Note that $\lhd_y(\lhd_y + D)$ is the quadratic Casimir for traceless, symmetric tensors of SO(D+2).) We have used the previous definition of $\lhd$, but introduced the definition of $\lhd_y$, which is always nonnegative: Acting on the CFT operator,

$$\text{CFT component:} \qquad \lhd \equiv x_0\partial_{x0} \ , \quad \lhd_y \equiv -y_0\partial_{y0}$$

This is the component defined by covariant expansion of the BPS operators $tr[(x^- \Phi)^n]$: Since these scalars appear at order θ^4, which gets an extra $(x_0 y_0)^2$, we find:

$$\triangleleft = n + 2 \ , \quad \triangleleft_y = n - 2 \ ; \qquad n \geq 2$$

which agrees with the above result on the AdS side, where the condition $n \geq 2$ follows from the component expansion of Φ (or the field equations).

A similar analysis can be made for the other components: For SU(4) representations, one needs to take into account that the global SU(4) generators have both y and θ pieces: In the language of Young tableaux, one finds representations of the form (a,b,c), labeling the number of columns of depth (1,2,3), where "b" comes from expansion in y (symmetric, traceless tensors of SO(6)), "a" comes from θ's with symmetrized R-indices, and "c" the same for $\bar{\theta}$. (Thus b is arbitrary, while a and c = 0,1,2.) There are some components missing for powers $n < 4$ of Φ because of the Φ field equations, as is clear from the explicit expansion we found for Φ.

So we have an expansion in x^- on both sides of the correspondence: On the CFT side, it is the expansion in powers n of the super Yang-Mills field strength Φ; on the AdS side it's the Kaluza-Klein expansion of massless 10D maximal supergravity in 5D AdS mass levels. $n = 0$ is trivial. $n = 1$ on the CFT side is just the Abelian part of the U(N) gauge group appearing in $tr(\Phi)$, which is free, decoupling from the interacting SU(N) part; on the AdS side we can take it as pure gauge. $n = 2$ is the supersymmetric generalization of the energy-momentum tensor ("supercurrent") on the CFT side; on the AdS side it's 5D maximal supergravity (with a cosmological constant), which can be considered a massless theory. $n > 2$ couples higher powers of Φ to massive analogs of supergravity, coming from the compactification of spin 2 and lower spins. (The spins are limited by the θ expansion of V or $tr(\Phi^n)$, which for all n is a scalar superfield living on the same projective superspace.)

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