More topology

Here are all the cases of $\chi = 2 - 2h - w - c \geq 0$:

	h	w	c	χ
sphere	0	0	0	2
disk	0	1	0	1
projective plane	0	0	1	1
torus	1	0	0	0
cylinder	0	2	0	0
Möbius strip	0	1	1	0
Klein bottle	0	0	2	0

The ones with $c \neq 0$ are a little hard to picture, and especially to understand why they correspond to insertions of crosscaps, but we can give a simple construction from the disk: Represent the disk as a square; we then consider identifying opposite sides, either with or without a twist.

First, identify just the left and right sides; the result is:

$$\text{disk} \to \begin{cases} \text{cylinder} & \text{no twist} \\ \text{Möbius strip} & \text{twist} \end{cases}$$

For the cylinder, the top edge has its ends identified, and so does the bottom, giving 2 boundaries; for the Möbius strip, the ends of the top edge have been identified with those of the bottom, leaving 1 boundary. The cylinder is easy to picture, and this definition of the Möbius strip is the usual one. The cylinder is also clearly a sphere with 2 windows. The Möbius strip has 1 boundary that can be identified with a window, but the crosscap is more subtle: Consider horizontal lines in the disk; because of the twisted identification, they are closed, with half in the top half of the disk, connected to the other half in the bottom half of the disk. Now move continuously from line to line, from the boundary (top and bottom edges) to the middle. Upon reaching the middle, the top and bottom halves of the line are identified: This is exactly a crosscap. So we started from a boundary and moved continuously away from it, as one might on a cylinder, but at the other end found a crosscap (instead of another boundary). This demonstrates the entry for the Möbius strip in the above table.

Then identify both left with right, and top with bottom, with any combination of twists; the result is a space with no boundary:

$$\text{disk} \to \begin{cases} \text{torus} & \text{no twist} \\ \text{Klein bottle} & \text{1 twist} \\ \text{projective plane} & \text{2 twists} \end{cases}$$

Note that the 2-twist case is the same as identifying opposite points on the whole boundary of the disk (easier to picture if we take the disk as round): This is clearly the same as inserting a crosscap into a sphere, just as the original disk was the same as inserting a window into a sphere, which justifies both entries in the table above for $\chi = 1$. The name "(real) projective plane", or "RP(2)", refers to the fact that we can also define this space as ordinary 3D space with all points on a ray (straight line through the origin) identified: The identification means we can first choose all points at radius 1 (the sphere), then identify opposite points; this is the same as taking half the sphere, then identifying opposite points on the boundary – a sphere with a crosscap.

The torus is easy to picture as a sphere with a handle. For the Klein bottle, start with a Möbius strip, from twisted identification of the left and right sides of the disk: Think of the Möbius strip as a cylinder with 1 boundary replaced with a crosscap, as described above. Then perform the *un*twisted identification of the top and bottom sides of the disk (as was done in the middle): This is just another crosscap at the other end of the cylinder. The usual picture of the Klein bottle, as a cylinder with one end turned inside-out before connecting to the other end, comes from performing the identifications in the opposite order, first getting a cylinder from the untwisted identification, then performing a twisted identification of its 2 boundaries.

We can also derive Euler's theorem by such constructions: χ is defined (up to some normalization) by integrating the curvature, and adding a contribution from the curvature of the boundary with respect to the surface. We can avoid considering the latter contribution by stretching the surface (which doesn't change the topology) in such a way that the boundary is always straight with the respect to the surface, never turning at some angle. For example, a disk clearly has a curved boundary, since the surface is flat, so all curvature resides in the boundary. But stretch it into a half-sphere, and the opposite is true. Thus, defining $\chi = 2$ for the sphere, $\chi = 1$ for the disk, so a window contributes -1. Alternatively, we could consider a cylinder, which has no curvature in either its surface (since the surface is made by connecting ends of a flat disk) or boundaries, and arrive at the same conclusion. The torus comes from eliminating the ends of the cylinder, so it also has $\chi = 0$, and thus a handle contributes -2. Finally, a Möbius strip comes from identifying 2 sides of a flat disk, so no curvature from the surface, while the remaining boundary is everywhere straight; thus a crosscap also contributes -1. A similar result comes from the Klein bottle, making it from a cylinder as for the torus, but with a twist.