Vertex operators

Open-string operator products

For later calculations, we'll need a short list of operator products. But first we need to emphasize some differences and similarities for open and closed strings. For the closed string we have left- and right-handed modes $\widehat{X}_L$ and $\widehat{X}_R$, while for the open string $\widehat{X}_L = \widehat{X}_R$:

$$X(z,\bar{z}) = \begin{cases} \sqrt{\frac{\alpha'}{2}}[\widehat{X}_L(z) + \widehat{X}_R(\bar{z})] & \text{for closed} \\ \sqrt{\frac{\alpha'}{2}}[\widehat{X}(z) + \widehat{X}(\bar{z})] & \text{for open} \end{cases}$$

All these $\widehat{X}$'s have conveniently normalized propagators

$$\langle \widehat{X}(z)\ \widehat{X}(z') \rangle = \langle \widehat{X}_L(z)\ \widehat{X}_L(z') \rangle = \langle \widehat{X}_R(z)\ \widehat{X}_R(z') \rangle = -ln(z - z')$$

from which follows directly those for X itself:

$$\frac{2}{\alpha'}\langle X(z,\bar{z})\ X(z',\bar{z}') \rangle = \begin{cases} -ln(|z-z'|^2) & \text{for closed} \\ -ln(|z-z'|^2) - ln(|z-\bar{z}'|^2) & \text{for open} \end{cases}$$

where the open string has extra contributions from crossterms, now involving the same $\widehat{X}$.

For open-string amplitudes involving only open-string external states, all the vertex operators will be on the boundary,

$$z = \bar{z} \quad \Rightarrow \quad X(z,\bar{z}) = \sqrt{2\alpha'}\widehat{X}(z)$$

Therefore, when Fourier transforming wave functions we use the exponentials

$$e^{ik\cdot X(z,\bar{z})} = \begin{cases} e^{i\hat{k}\cdot\widehat{X}(z)} & \text{for open} \\ e^{i\hat{k}\cdot\widehat{X}_L(z)}e^{i\hat{k}\cdot\widehat{X}_R(\bar{z})} & \text{for closed} \end{cases} \quad \Rightarrow \quad \hat{k} = k \times \begin{cases} \sqrt{2\alpha'} & \text{for open} \\ \sqrt{\frac{\alpha'}{2}} & \text{for closed} \end{cases}$$

Closed-string vertex operators are the product of left- and right-handed ones, which are functions of z and $\bar{z}$, respectively, and thus take the form of the product of 2 independent open-string vertex operators.

Working directly in terms of $\widehat{X}$, we then have the "operator products"

$$(i\partial\widehat{X})(z')\ e^{i\hat{k}\cdot\widehat{X}(z)} \approx \hat{k}\ \frac{1}{z'-z}\ e^{i\hat{k}\cdot\widehat{X}(z)}$$

$$\textit{or} \quad (\partial\widehat{X})(z')\ f(X(z,\bar{z})) \approx -\ \frac{1}{z'-z}\ (\partial f)(X(z,\bar{z})) \times \begin{cases} \sqrt{2\alpha'} & \text{for open} \\ \sqrt{\frac{\alpha'}{2}} & \text{for closed} \end{cases}$$

$$(i\partial\widehat{X})(z')\ (i\partial\widehat{X})(z) \approx \frac{1}{(z'-z)^2}$$

(Note the context: $\partial\widehat{X}$ is a z derivative, ∂f is an x derivative. The "i" associated with $\partial\widehat{X}$ is from Wick rotation.)

For example, we can use these results to determine the proper normalization of massless vertex operators, by comparison with that of tachyons: For the tachyon,

$$W_{\hat{k}}(z) = e^{i\hat{k}\cdot\widehat{X}(z)}, \quad \hat{k}^2 = 2 \quad \Rightarrow \quad W_{\hat{k}}(z')\ W_{-\hat{k}}(z) \approx \frac{1}{(z'-z)^2} e^{i\hat{k}\cdot[\widehat{X}(z')-\widehat{X}(z)]}$$

(For the closed string, we have the product of left and right versions of the above. Note this correctly gives $k^2 = 1/\alpha'$ for the open string tachyon and $4/\alpha'$ for the closed, where α' is the slope of the open-string Regge trajectory, and the parameter that appears in the action that describes both open- and closed-string states.) The z factors are canceled in string field theory by considering the gauge-fixed kinetic term $\langle 0|V(c_0\Box)V|0\rangle$, where $V = cW$.

Gauge-independent vertex operators

When ghosts are included, vertex operators can be generalized to arbitrary gauges for the external gauge fields. (This result follows from the same method applied to relate integrated and unintegrated vertices in subsection XIIB8 of *Fields*. We'll do a better job of that here.) The main point is the existence of integrated and unintegrated vertex operators: Integrated ones are natural from adding backgrounds to the gauge-invariant action; unintegrated ones from adding backgrounds to the BRST operator. We'll relate the two by going in both directions. The following discussion will be for general quantum mechanics (except in the relativistic case we use τ in place of t), but we'll add some special comments for open strings at the end.

The action can be written as

$$S \sim \int d\tau\, H_I$$

plus the usual terms for converting Hamiltonian to (first-order) Lagrangian, where the interacting Hamiltonian consists of the free part plus linearized vertex

$$H_I = H_0 + W$$

BRST invariance with respect to the free BRST operator then implies

$$\begin{aligned} & [Q_0, S] \approx 0 \\ \Rightarrow \quad & [Q_0, \textstyle\int d\tau\, W] \approx 0 \\ \Rightarrow \quad & [Q_0, W] \approx \partial_\tau V \\ \Rightarrow \quad & \{Q_0, V\} \approx 0 \end{aligned}$$

for some V, where "$\approx$" means "at the linearized level". The BRST invariants $\int W$ and V are thus our integrated and unintegrated vertex operators, respectively.

Going in the other direction, we start with interacting BRST

$$Q_I = Q_0 + V$$

where fully interacting BRST invariance implies at the linearized level

$$Q_I^2 = 0 \quad \Rightarrow \quad \{Q_0, V\} \approx 0$$

The full gauge-fixed action is then defined (in relativistic quantum mechanics, or otherwise in the ZJBV formalism) by

$$H_I = \{Q_I, b\} \approx H_0 + W$$

It then follows that

$$0 = [Q_I, H_I] \approx [Q_0, H_0] + ([Q_0, W] + [V, H_0])$$

which agrees with the above, since H_0 gives the (free) time development:

$$[H_0, V] = \partial_\tau V$$

The only modifications for the open string are eliminating σ dependence:

$$Q_0 = \int \frac{d\sigma}{2\pi} J, \qquad H_0 = \int \frac{d\sigma}{2\pi} T, \qquad b \to \int \frac{d\sigma}{2\pi} b \quad (0 - mode)$$

$$V \to V|_{\sigma=0}, \qquad W \to W|_{\sigma=0}$$

After combining the left and right-handed modes into functions of just z over the whole plane, as usual, we can then replace σ and τ with z in our definitions in an appropriate way.

Vector vertex

The simplest case is the massless vector. The choice for integrated vertex was obvious from the gauge transformation of the external field:

$$W = \dot{X} \cdot A(X), \quad \delta A(x) = -\partial\lambda(x)$$

$$\Rightarrow \quad \int d\tau \; W = \int dX \cdot A(X), \quad \delta \int d\tau \; W = -\int d\lambda(X) = 0$$

As usual, the τ integral gets converted into a z integral over the boundary (real axis).

Besides this "background" gauge invariance, we also need the "quantum" BRST invariance. The unintegrated vertex V and the BRST invariance of $\int W$ then follow from the same calculation:

$$[Q,W] = \partial V \quad \Rightarrow \quad Q\int W = QV = 0$$

We use the BRST operator

$$Q = \int \frac{dz}{2\pi i}\; J, \quad J = cT + c(\partial c)b, \quad T = \tfrac{1}{2}(i\partial\widehat{X})^2, \quad [Q,W(z)] = \oint_z \frac{dz'}{2\pi i}\; J(z')\; W(z)$$

(For the open string, this is all of Q; the closed string has $Q = Q_L + Q_R$, with Q_L and Q_R given by the above, with "L" or "R" subscripts on everything. For now, we stick to the open string. There is a sign convention change from *Fields* for Q and T.)

For $T(z')W(z)$, we get "single-contraction" (tree/classical) terms from the singular part of either ∂X with W (one propagator), and nonsingular (ordinary) product of the other ∂X (no propagator). So we evaluate

$$(\partial\widehat{X}^a)(z')\;[(\partial\widehat{X})\cdot A(X)](z) \approx -\frac{1}{(z'-z)^2}\; A^a(X(z)) - \sqrt{2\alpha'}\;\frac{1}{z'-z}\;[(\partial\widehat{X})^b\partial^a A_b(X)](z)$$

We also get "double-contraction" (1-loop) terms from the singular part of the product of the second ∂X with the above:

$$(\partial\widehat{X})(z')\cdot(\text{right-hand side of above}) \approx$$
$$2\sqrt{2\alpha'}\frac{1}{(z'-z)^3}\;\partial\cdot A(X(z)) + 2\alpha'\frac{1}{(z'-z)^2}(\partial\widehat{X}\cdot\Box A)(z)$$

We then need to integrate, using

$$\oint_z \frac{dz'}{2\pi i}\;\frac{1}{(z'-z)^{n+1}}\; f(z') = \frac{1}{n!}\partial^n f(z)$$

Putting it all together,

$$W = (i\partial\widehat{X})\cdot A \quad \Rightarrow \quad [Q,W] = \partial V - \alpha'(i\partial c)(\partial\widehat{X}^a)\partial^b F_{ba}$$
$$V = c(i\partial\widehat{X})\cdot A - \sqrt{\frac{\alpha'}{2}}(i\partial c)\partial\cdot A$$

(We have repeatedly used the identity $\partial_z f(X) = (\partial X)\cdot\partial f = \sqrt{2\alpha'}(\partial\widehat{X})\cdot\partial f$.)

Thus BRST invariance of $\int W$ and V requires the background satisfy only the (free) gauge-covariant field equations $\partial^b F_{ba} = 0$. This was to be expected, since quantum BRST invariance of Yang-Mills in a Yang-Mills background requires the same in field theory. We also find an order α' correction to the vertex operator V: This can be explained by noting that, while $c\partial X$ creates a Yang-Mills state from the vacuum, ∂c creates its Nakanishi-Lautrup field plus $\partial\cdot A$, in a combination that vanishes by that field's equation of motion.