Week 3: Lectures 5, 6

Renormalization: introduction by ϕ^3_4

The ϕ^3 Lagrangian in D dimensions is taken as

$$\mathcal{L}_D = \frac{1}{2}\left((\partial_\mu\phi)^2 - m^2\phi^2\right) - \frac{g\,\mu^\varepsilon}{3!}\,\phi^3\,, \tag{41}$$

where for applications to $D = 4$ we choose $\varepsilon = 2 - D/2$, which gives g units of mass for $D = 4$. For $\mathcal{L}_{6-2\varepsilon}$ we take $\varepsilon = 3 - D/2$, which makes g dimensionless for $D = 6$.

The mass counterterm, scheme and scale

In the self energy, Eq. (32), we see an additive, momentum-independent ultraviolet divergence, represented by a simple pole in $\varepsilon = (1/2)(4 - D)$. Now the Fourier transform of a constant is a delta function, so in coordinate space the divergence is *local*. As a result, it can be cancelled by a new vertex. Such a vertex can cancel the p^2-independent pole, but will not affect the momentum dependence otherwise. This is the procedure we're after – to modify the theory in such a way as to make its ill-defined predictions for constant terms finite, while preserving its sensible predictions for momentum dependence.

We implement this procedure by introducing a new vertex in the theory, modifying the Lagrange density. Specifically, we add a new term (and also change the notation for the mass and coupling slightly, attaching a subscript 'R', for "renormalized"),

$$\mathcal{L}_{\rm ren} \;=\; \frac{1}{2}\left((\partial_\mu\phi)^2 - m_R^2\phi^2\right) - \frac{g_R\mu^\varepsilon}{3!}\,\phi^3 - \frac{1}{2}\,\delta m^2\phi^2\mathcal{L}_{\rm class} + \mathcal{L}_{\rm ctr} \tag{42}$$

where $\mathcal{L}_{\rm ctr}$ is a *mass counterterm*,

$$\mathcal{L}_{\rm ctr} = -\frac{1}{2}\,\delta m^2\phi^2\,, \tag{43}$$

with

$$\delta m^2 \;= -\frac{g^2}{2(4\pi)^2}\left(-\frac{1}{\varepsilon} + c_m\right) m_R^2\,. \tag{44}$$

At this point, the constant c_m and the renormalization mass μ are still arbitrary.

- In (44), the constant c_m expresses the fact that the theory makes no prediction on the Σ at any value of p^2, only on differences. A choice of c_m, then simply corresponds to the baseline value we are going to choose for the self-energy. The choice of c_m is called a *renormalization scheme.*

- δm^2 is called the *mass shift.*

- m_R is the *renormalized mass.* It is not necessarily the same as the *physical mass*, at which $\Gamma_2 = 0$.

- g_R is the *renormalized coupling*, which in this case is just a way of denoting the coupling with integer units of mass, independent of D.

The renormalized two-point function, is now found from the full one-loop pertubation expansion, as the sum of the original self-energy diagram, Eq. (33) plus the counterterm, which gives

$$\Sigma_{ren} = \frac{g^2}{2(4\pi)^2}\left\{c_m - \gamma_E + 2 + \ln\left(\frac{4\pi\mu^2}{m_R^2}\right) - \sqrt{1-\frac{4m^2}{p^2}}\,\ln\left[\frac{\sqrt{1-\frac{4m^2}{p^2}}-1}{\sqrt{1-\frac{4m^2}{p^2}}+1}\right]\right\} + \mathcal{O}(\varepsilon)\,. \tag{45}$$

Schemes

Perhaps the most common choice is the $\overline{\text{MS}}$ or 'modified minimal subtraction' scheme, specified by,

$$c_{\overline{\text{MS}}} = -\ln 4\pi + \gamma_E\,,$$

which cancels terms, which result from the Gamma function and the one-loop momentum integrals. If $c_m = 0$, the scheme is called 'MS' or simply 'minimal subtraction'. These 'minimal' schemes specify the value of Σ only indirectly. Such terms are, in fact, ubiquitous in one-loop calculations using dimensional regularization, so this is a very natural choice to make.

Other conventional choices are called 'momentum subtraction' schemes, where we specify the value of $\Sigma(p^2, m_R^2)$ directly, for example, we can specify that $\Sigma(p_0^2, m_R^2) = 0$, so that $\Gamma_2(p_0^2 - m_R^2) = p_0^2 - m_R^2$. If we choose in addition $p_0^2 = m_P^2$ with m_P the physical mass, then the renormalized mass is fixed to be the physical mass and we have what is known as an 'on-shell' momentum subtraction scheme.

The tadpole diagram in ϕ^3

There are only three diagrams in ϕ_4^3 with superficial degree of divergence ≥ 0, the self energy, and the tadpole diagrams at order g and g^3. For the latter we have (with a conventional $-i$ for the 1PI function),

$$\begin{aligned} \gamma_1 &= \frac{g}{2(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\Gamma(\varepsilon-1)m^2 \\ &\equiv \tau\,. \end{aligned}$$

which is again p^2-independent. The integral, proportional to $\int d^Dk\,(k^2-m^2+i\varepsilon)^{-1}$, is finite only for $\varepsilon > 1$, that is $D < 2$, but as usual can be extended to all D by using the expression we get in dimensional continuation. Note that the delta function continues to give only a simple pole for $D = 4$, and that the continued integral is actually finite for $D = 3$.

All order-g tadpole diagrams are automatically cancelled by a counterterm $-\tau\phi(x)$, with τ chosen as above. The g^3 tadpole can be cancelled in the same way. Note that τ is real. Of course, this is a specific choice of renormalization scheme, which we make for simplicity.

Summary for ϕ_D^3

$$\mathcal{L}_{\text{ren}} = \mathcal{L}_{\text{KG}} - \frac{g\mu^{\varepsilon}}{3!}\,\phi^3 - \delta m^2(c_m)\,\phi^2 - \tau\phi\,.$$

With the choices for δm^2 and τ given above, perturbation theory generated from this Lagrange density is completely finite, order by order in g. Every divergent subdiagram will be matched with a counterterm, and the result will be free of poles in ε, although of course if it will still depend on our choices of μ and the scheme (that is, c_m).

The key to renormalization: bare and renormalized Lagrangians

What are we to do with this finite perturbation theory? The method for ϕ_4^3 illustrates the general procedure.

- We begin by choosing a scheme (c_m) and a renormalization mass (μ).

- From the explicit expression for Σ, Eq. (33), we find the position of the physical mass for this c_m and μ, by solving

$$\Gamma_2(p^2 = m_P^2) = m_P^2 - m_R^2 - \Sigma(m_P^2, m_R^2, g_R) = 0\,, \tag{46}$$

which ensures that to this order in perturbation theory, $G_2(p^2) = i/\Gamma_2(p^2)$ has a pole at $p^2 = m_P^2$.

- We then "measure" (more realistically choose) a physical value of m_P, something like 1 GeV or 1 MeV.

- We then solve Eq. (46) for $m_R(\mu)$ as a function of g_R (which in this superrenormalizable case we can also just specify).

- With this value of $m_R(\mu)$ and g_R we can now go ahead and compute *every other physical quantity and correlation function* in the theory to the same order in g_R. In this example, one calculation and one "measurement" is all we need to get a completely finite perturbative expansion.

We can further interpret the procedure by the following reorganization of the Lagrange density,

$$\begin{aligned}
\mathcal{L}_{\rm ren}(m_R, g_R\mu^\varepsilon, c_m) &= \frac{1}{2}\left((\partial_\mu\phi)^2 - m_R^2\phi^2\right) - \frac{g_R\mu^\varepsilon}{3!}\,\phi^3 - \delta m^2(c_m)\,\phi^2 \\
&= \frac{1}{2}\left((\partial_\mu\phi)^2 - (m_R^2 + \delta m^2(c_m))\phi^2\right) - \frac{g_R\mu^\varepsilon}{3!}\,\phi^3 \\
&= \frac{1}{2}\left((\partial_\mu\phi)^2 - m_0^2\phi^2\right) - \frac{g_0}{3!}\,\phi^3 \\
&= \mathcal{L}_{\rm class}(m_0, g_0)\,,
\end{aligned} \tag{47}$$

where in the third line we define the *bare mass*, $m_0^2 = m_R^2 + \delta m^2$ and the *bare coupline*, $g_0 = g_R\mu^\varepsilon$. The final line is once again the classical Lagrangian, now in terms of m_0 and g_0, which we refer to as the *bare Langranian* (density). We summarize this result as:

$$\textit{The renormalized Lagrangian} \;=\; \textit{The bare Lagrangian}\,.$$

We can summarize all of our calculations above as: From the renormalized Lagrangian, we can generate a finite perturbation series; its equality to the bare Lagrangian ensures that this series depends upon only two parameters, and therefore only requires two inputs. In this case, the inputs are a "measurement" of the physical mass and a specification of the coupling.

Renormalization for ϕ_6^3

The classical Lagrange density for ϕ_6^3 is the same as for ϕ_4^3, except that we let g_R be dimensionless, so that it now appears as $g_R\mu^\varepsilon$, with $\varepsilon = 3 - D/2$.

Renormalization will follow the same pattern, but now, because $[g] = 0$, the superficial degree of divergence of each Γ_E depends only on E, not on the order, V (ee Eq. (40)). Only for $E = 2$ and $E = 3$ will we need counterterms; the tadpoles, $E = 1$, are treated as in four dimensions.

The self energy ($E = 2$) is now

$$\begin{aligned}\Sigma(p^2, m^2) &= -\frac{g^2}{2(4\pi)^3}\left\{\left(-\frac{p^2}{6}\frac{1}{\varepsilon} + m^2\frac{1}{\varepsilon}\right)\right. \\ &\quad + \left(-\frac{p^2}{6} + m_R^2\right)\left(1 + \ln\frac{4\pi\mu^2 e^{-\gamma_E}}{m_R^2}\right) \\ &\quad \left. -m_R^2 \int_0^1 dx\, F(x)\ln F(x) + \mathcal{O})(\varepsilon)\right\} \end{aligned} \tag{48}$$

where

$$F(x) = 1 - x(1-x)p^2/m_R^2\,.$$

The poles appear only times p^2 and m_R^2.

For the vertex correction (triangle diagram, with $E = 3$), the finite, ε^0 are more complex because they depend on two independent external momenta, but here we need only the pole terms, given by

$$\Gamma_3 = (g_R\mu^\varepsilon)\frac{g^2}{2(4\pi)^3}\left(\frac{1}{\varepsilon} + \ln\left(4\pi e^{-\gamma_E}\right)\right) + \mathcal{O}(\varepsilon^0)\,, \tag{49}$$

which is independent of the external momenta.

The UV poles (divergences for $D = 6$) in Eqs. (48) and (49) are cancelled by three new counterterms, which we add to the classical Lagrange density in the following rather unintuitive notation as

$$\begin{aligned}\mathcal{L}_{\rm ren}(m_R, g_R\mu^\varepsilon) &= \frac{1}{2}\left((\partial_\mu\phi_R)^2 - m_R^2\phi_R^2\right) - \frac{g_R\mu^\varepsilon}{3!}\,\phi_R^3 \\ &\quad + \frac{1}{2}\left(Z_\phi - 1\right)(\partial_\mu\phi_R)^2 - \delta m^2\,\phi_R^2 - \frac{\delta g}{3!}\,\mu^\varepsilon\phi_R^3 \end{aligned} \tag{50}$$

The δm^2 and $Z_\phi - 1$ counterterms automatically cancel the p^2/ε and m^2/ε poles of the self energy, respectively. The δg counterterm cancels the pole in the vertex function. One-loop calculations in the theory are then all finite, although still functions of c_m and μ. Notice that now the field as well as the mass and coupling has subscript 'R', denoting renormalization. Unlike ϕ_4^3, two- and higher-loop renormalization is now necessary, beginning with all the two-loop diagrams that we get from the classical Lagrange density plus counterterms computed to one loop. Although the calculations are more elaborate, the steps in the method are the same. Only the renormalization constants $Z_{\phi,g,m}$ are necessary, and are themselves expansions in the (square of) the coupling,

$$Z_i = 1 + \sum_{j\geq 1} z_i^{(j)}(\varepsilon) g_R^{2j} \tag{51}$$

where the coefficients $z_i^{(j)}$ are constructed to cancel poles that appear in j-loop diagrams (i.e., at order g_R^{2j}).

Reprise of the method

We summarize the renormalization of ϕ_6^3 following what we did for ϕ_4^3.

- Pick a scheme.
- Calculate two (at least) physical quantities, $\sigma_i^{(calc)}(m_R, g_R)$. One of these could be the position of the physical mass, m_P, as we did for ϕ_4^3, but this is not the only choice (both here and for ϕ_4^3.)
- Measure the $\sigma_i^{(exp)}$ (or postulate values).
- Set $\sigma_i^{(exp)} = \sigma_i^{(calc)}$.
- Solve for m_R and g_R as functions of μ in this scheme.
- Can now calculate all other quantities in the theory to the accuracy in which m_R and g_R have been determined (that is, the original order of $\sigma_i^{(calc)}$.

Multiplicative renormalization

We get further insight by relating bare to renormalized quantities, including now the field and coupling, as well as the mass.

$$\begin{aligned}
\phi_R Z_\phi^{1/2} &= \phi_0 \\
m_R^2 Z_m &= m_0^2 \\
g_R \mu^\varepsilon Z_g &= g_0 \\
\delta m^2 &= m_R^2 \left(Z_m Z_\phi - 1\right) \\
\delta g &= g_R \mu^\varepsilon \left(Z_g Z_\phi^{3/2} - 1\right)
\end{aligned} \tag{52}$$

$$\begin{aligned}
\mathcal{L}_{\rm ren}(m_R, g_R\mu^\varepsilon) &= \frac{1}{2}\left((\partial_\mu \phi_R)^2 - m_R^2 \phi_R^2\right) - \frac{g_R \mu^\varepsilon}{3!}\,\phi_R^3 \\
&\quad + \frac{1}{2}\left(Z_\phi - 1\right)(\partial_\mu \phi_R)^2 - \left(Z_m Z_\phi - 1\right)\, m_R^2 \phi_R^2 - \left(Z_g Z_\phi^{3/2} - 1\right)\, \frac{g_R}{3!}\,\mu^\varepsilon \phi_R^3 \\
&\equiv \frac{1}{2}\left((\partial_\mu \phi_0)^2 - m_0^2 \phi_0^2\right) - \frac{g_0}{3!}\,\phi_R^3
\end{aligned} \tag{53}$$

And once again, we summarize this as

$$\mathcal{L}_{\rm ren}(g_R, M_R, c_i, \mu) = \mathcal{L}_{\rm class}(g_0, m_0) \tag{54}$$

where the left hand side generates a finite perturbation expansion in g_R and m_R (which depends on c_m, c_ϕ, c_g and μ), while from the right hand side we learn that this expansion still depends on only two independent parameters.

Summary of one-loop counterterms

Defining $a \equiv \frac{g_R^2}{2(2\pi)^3}$, we have from the one-loop results quoted above in Eqs. (48) and (49),

$$\begin{aligned}
\delta m^2 &= -\frac{a}{2}\left(\frac{1}{\varepsilon} + c_{\delta m}\right) \\
Z_\phi - 1 &= -\frac{a}{12}\left(\frac{1}{\varepsilon} + c_\phi\right) \\
\delta g &= g\mu^\varepsilon \left[-\frac{a}{2}\left(\frac{1}{\varepsilon} + c_{\delta g}\right)\right]
\end{aligned} \tag{55}$$

from which we derive

$$
\begin{aligned}
Z_m - 1 &= -\frac{5a}{12}\left(\frac{1}{\varepsilon} + c_m\right) \\
Z_g - 1 &= -\frac{3a}{8}\left(\frac{1}{\varepsilon} + c_g\right)
\end{aligned}
\tag{56}
$$

where we can readily determine $c_{\delta m}$ and $c_{\delta g}$ from $c_{m,g,\phi}$ that appear in the Z_i.