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Week 7: Lectures 13, 14

March 15, 2012

Majorana spinors

- So far, we have only considered massless, two-component spinors, the mass having been introduced with the Dirac four-component spinor. Before moving on to vectors fields, we introduce a massive field with only two-components, the "Majorana spinor". We'll denote this spinor by χ. The equation of motion for this Majorana field can be written as

$$(\underline{\partial})_{a\dot{b}}\,\chi^a + m(\chi*)_{\dot{b}} = 0\,. \tag{8}$$

This is like one of the two 2×2 equations that led to the Dirac equation, but with the dotted and undotted spinors related by complex conjugation.

- The Majorana field can be "promoted" to a Dirac field by the choice

$$\psi = \begin{pmatrix} (\chi^*)_{\dot{b}} \\ \chi^a \end{pmatrix}$$

- To really interpret this equation, we will need to use a feature of spinor *fields*, as opposed to conventional solutions to this equation of motion. As we'll see later, the fields *anticommute* with each other, $\chi_b\chi_a = -\chi_a\chi_b$. With this in mind, we can write a Lagrange density for χ,

$$\mathcal{L} = (\chi^*)^{\dot{b}}\,(\underline{\partial})_{a\dot{b}}\,\chi^a + \frac{m}{2}\left(\chi_a\chi^a + \chi_{\dot{a}}(\chi*)^{\dot{a}}\right)\,, \tag{9}$$

from which we can derive the Majorana equation of motion using $\partial\mathcal{L}/\partial(\chi^*)^{\dot{b}}$. (Note that if the fields commute, the mass term vanishes identically.)

- A striking consequence of Eq. (8) is a lack of $U(1)$ or phase invariance for the Majorana field. This means that there is no conserved change associated with the field, and when we quantize the field, it will equal its own antiparticle.

Vector fields

- Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{10}$$

- The Maxwell Lagrangian is invariant under 'gauge transformations',

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x) \tag{11}$$

- Reduction to two "physical" degrees of freedom: if we solve

$$\partial_\mu \partial^\mu \alpha(x) = \partial_\mu A^\mu\,,$$

then A'^μ satisfies

$$\partial_\mu A'^\mu = 0\,. \tag{12}$$

That is, for any given A, we can find an α so that (12) holds. Put differently, if, among all solutions to the equations of motion, we only keep those that solve (12), we will leave out no physically distinct solutions.

- We can formalize this observation by introducing a gauge fixed Lagrange density ,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \lambda \left(\partial_\mu A^\mu\right)^2$$

whose equation of motion is

$$\partial_\mu \partial^\mu A_\nu - (1-\lambda) \partial_\nu \partial_\mu A^\mu = 0\,.$$

Acting from the left with ∂^ν gives

$$\lambda\, \partial_\nu \partial^\nu \left(\partial \cdot A\right) = 0\,.$$

Then if we set $\partial \cdot A(x_0, \vec{x}) = 0$ at any time x_0 for all $\vec{x}$, properties of the wave function ensure that it will remain zero for all times. Our equation of motion then reduces to the wave equation,

$$\partial_\mu \partial^\mu A_\nu = 0\,.$$

with solutions

$$A^{\mu}_{\bar{k}}(x) = a^{\mu}\, e^{-i\bar{k}\cdot d}\,. \tag{13}$$

where $\bar{k}_0 = |\vec{k}|$, and where a^{μ} is a constant vector. This "amplitude" may be of arbitrary size, but the Lorentz condition (12) requires that it be orthogonal to the wave vector $\bar{k}^{\mu}$:

$$a_{\mu}k^{\mu} = 0\,. \tag{14}$$

- The transversality condition (14) reduces the number of physical degrees of freedom in the Maxwell field from four to three. In fact, there are only two. We can do this because the Lorentz condition does not fix all freedom to do gauge transformations. Here's how it goes. We start by writing the most general a_{μ} satisfying (12) as

  $$a^{\mu} = c\bar{k}^{\mu} + \epsilon^{\mu}\,,$$

  where c is any constant, (recall that $\bar{k}^2 = 0$) and where $\epsilon \cdot \bar{k} = 0$. Now, for any choice of a^{μ} we can adjust the constant c so that the first term gives the entire zeroth component of a: $c\bar{k}^0 = a^0$. Equivalently, without any loss of generality, we can choose

  $$\begin{aligned} \epsilon^0 &= 0 \\ \vec{\epsilon}\cdot\vec{k} &= 0\,, \end{aligned} \tag{15}$$

  which means that ϵ^{μ} has only two independent degrees of freedom. We will refer to ϵ as a physical polarization (think of the transversality of electromagnetic waves.) At this point, we can do another gauge transformation (11) while remaining in the Lorentz gauge. For our given a^{μ}, we assume that we've adjusted the constant c so that Eq. (15) holds. Now we choose the gauge-changing function α as

  $$\alpha(x) = ic\, e^{-i\bar{k}^{\mu}x_{\mu}}\,.$$

  After the transformation (11), the k term in a^{μ} is cancelled, and we find simply,

  $$a^{\mu} = \epsilon^{\mu}\,,$$

  that is, the amplitude for the Maxwell field can be reduced by gauge transformations to a physical polarization, transverse to the spatial wave vector $\vec{k}$.

- An alternative to the Maxwell field is the "Proca field", for which we add a mass to the Lagrange density (10)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A^2 \tag{16}$$

  The resulting equation of motion, "Proca's equation", is

$$\partial_\mu \partial^\mu A_\nu - \partial_\nu \partial_\mu A^\mu + m^2 A_\nu = 0\,.$$

  Acting with ∂^ν as above, we can readily show that the Lorentz condition, (12) is automatic, and the actual equation of motion is a separate Klein-Gordon equation for each of the components of A,

$$\left(\partial_\mu \partial^\mu + m^2\right) A_\nu = 0\,.$$

  The Lorentz condition reduces the number of degrees of freedom of the Proca field from four to three. The Proca Lagrangian, however, does not enjoy the gauge invariance of the original Maxwell Lagrangian, and we cannot reduce it to two.

- The solutions of the Klein Gordon equation are again plane waves, (13), now with $\bar{k}^2 = m^2$. The third degree of freedom has the interpretation of a longitudinal polarization, familiar from electromagnetic waves in media.

Local gauge invariance

- The "abelian" local gauge transformations are a combination of phase rotations on a fermion field, typically a Dirac spinor, combined with a gauge transformation on a Maxwell vector field, as in (11) above,

$$\psi'(x) = \mathrm{e}^{-ie\alpha(x)}\,\psi, \quad \bar{\psi}'(x) = \bar{\psi}(x)\,\mathrm{e}^{ie\alpha(x)} \quad A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x)\,. \tag{17}$$

  Notice the factor of e conventionally incorporated in the phase. For the spinor field, the difference from a global phase transformation is to make the phase an arbitrary function of position. By itself, the Dirac Lagrange density (3) is not form invariant because of the action of the derivative on the phase. In effect, an x-dependent phase transformation changes the kinetic energy. (GS Note: in my book $\alpha \to -\alpha$ relative to the convention here.)

- Quantum electrodynamics, or QED, is a theory that is form invariant under the *combination* of abelian local gauge transformations, (17). It is built from the Maxwell Lagrange density (10) simply added to the Dirac density (3), but with the derivative ∂^μ in the latter modified to the *covariant derivative*, by adding a term ieA^μ, with e the same constant as in (17) above, that relates the transformations of spinor and vector fields. This combination is often referred to as "minimal coupling". The notation for the covariant derivative and the resulting, full QED density are then,

$$\begin{aligned} D_\mu[A] &\equiv \partial_\mu + ieA_\mu \\ \mathcal{L}_{\mathrm{QED}} &= \bar{\psi}\,[\,i\gamma\cdot D[A] - m\,]\,\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \end{aligned} \tag{18}$$

  With the covariant derivative, the change in kinetic energy due to the position dependence of the spinor phase is precisely cancelled by the gauge transformation of the vector field in the term $\bar{\psi}\gamma\cdot eA\psi$, which itself would not be gauge invariant.

- Recall that gauge invariance is the key to reducing the number of degrees of freedom in the massless vector field from four to two for each wave number. With L_{QED}, (18), we maintain this invariance – and hence the correct counting of degrees of freedom – in a theory that couples the massless vector field to a spinor field. The same can be done with a complex scalar field, again by simply replacing derivatives by covariant derivatives, and generalizing the global phase invariance of the complex field to a local invariance. Notice, it cannot be done for a real scalar field, because this field does not have a global phase invariance that we can turn local.

Local Nonabelian Gauge Invariance

- We start with the *global* form invariance $U(N) = U(1)\times SU(N)$ of a Dirac density when there are N identical Dirac fields,

$$\begin{aligned} \mathcal{L}^{(N)}_{\mathrm{Dirac}} &= \sum_{i=1}^{N}\bar{\psi}_i(\,i\gamma\cdot\partial - m\,)\psi_i \\ \psi_i' &= U_{ij}\psi_j\,. \end{aligned}$$

For the case $N = 1$, we've seen how to couple a Dirac field to a Maxwell field, combining a local phase transformation with the vector gauge transformation.

- The $U(1)$ part of $U(N)$ is already "taken care of" by a Maxwell field. Let's assume this is done, and concentrate on the $SU(N)$ part of $U(N)$ in the following. The procedure can be carried out for any nonabelian group.

- To generalize the $U(1)$ treatment to $SU(N)$, we introduce N^2-1 gauge fields, A^μ_a, with "group index" $a = 1 \ldots N^2 - 1$ (more generally to the number of generators in the group.) By contracting the gauge field with the generators, the gauge field can be written in matrix form, in *any* representation R of the group, where $(i, j = 1 \ldots \dim(R).)$

$$(A^\mu)^{(R)}_{ij} = \sum_{a=1}^{N^2-1} A^\mu_a \, (T^{(R)}_a)_{ij} \, .$$

  For our discussion, we will specialize to $N \times N$ representations ("defining", or if there is more than one, "fundamental" representations of $SU(N)$) and we will generally suppress the superscript R for the representaiton.

- The conventional normalization of the generators (common, not universal in physics applications) is

$$\mathrm{Tr}\,(T_a\, T_b) = \frac{1}{2}\, \delta_{ab} \, . \tag{19}$$

- Local $SU(N)$ transformations of the spinor and gauge fields are defined as

$$\begin{aligned} \psi'_j &= U_{ji}(x)\psi_i(x), \\ \bar{\psi}'_j &= \bar{\psi}_i(x)\, U^\dagger_{ij}(x), \\ A'^\mu(x) &= U(x)A^\mu U(x) + \frac{i}{g}\, (\partial^\mu U)U^{-1} \, . \end{aligned}$$

- A general transformation $U(x)$ is a matrix determined by N^2-1 x-dependent group parameters. The parameters are real, and the generators are hermitian, so U is unitary.

$$\begin{aligned}
[U(x)]_{ij} &= \exp\left[ig\sum_{a=1}^{N^2-1}\Lambda_a(x)T_a\right]_{ij} \\
&\to \left[1+ig\sum_{a=1}^{N^2-1}\delta\Lambda_a(x)T_a\right]_{ij} + \ldots\,,
\end{aligned}$$

  where the second equality gives the infinitesimal form.

- The infinitesimal gauge transformations of the vector field in matrix and component forms are

$$\begin{aligned}
A'^{\mu}(x) &= A^{\mu}(x) - \partial^{\mu}\delta\Lambda(x) + ig[\delta\Lambda(x), A^{\mu}(x)] \\
A'^{\mu}_{a}(x) &= A^{\mu}_{a} - \partial^{\mu}\delta\Lambda_a - gC_{abc}\delta\Lambda_b(x)A^{\mu}_{c}\,. \qquad (20)
\end{aligned}$$

  The derivative term is the analog of the abelian transformation (11); the commutator term is the specifically nonabelian part.

- The covariant derivative and field strength are defined in matrix form as

$$\begin{aligned}
D_{\mu}[A] &= \partial_{\mu} + igA_{\mu} \\
F^{\mu\nu} &= \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} + ig[A^{\mu}, A^{\nu}] \\
F^{\mu\nu} &= \sum_{a=1}^{N^2-1} F^{\mu\nu}_{a}T_a\,. \qquad (21)
\end{aligned}$$

  The covariant derivative should always be thought of as a matrix. The component forms of the field strengths are found by taking traces with T_a and using the orthonormality property (19) of the generators,

$$F^{\mu\nu}_{a} = \partial^{\mu}A^{\nu}_{a} - \partial^{\nu}A^{\mu}_{a} - gC_{abc}A^{\mu}_{b}A^{\nu}_{c}\,. \qquad (22)$$

- The term "covariant" applies to $D(A)$ because

$$D^{\mu}(A')_{ij}\psi'_{i} = U(x)_{ij}\, D^{\mu}(A)_{jk}\psi_k\,, \qquad (23)$$

  when the spinor and vector fields transform as above. That is, the covariant derivative of the spinor field transforms just like the spinor field itself.

- Putting it all together, the gauge invariant $SU(N)$ density for spinor fields is

$$\mathcal{L}_{\rm Dirac}^{(N)} \;\; = \;\; \sum_{i,j=1}^{N} \bar{\psi}_i(\, i\gamma \cdot D[A] - m\,)_{ij}\psi_j + \mathcal{L}_{\rm YM}\,,$$

where the Yang-Mills Lagrange density (which determines the dynamics of the vector fields in this theory) is defined as

$$\mathcal{L}_{\rm YM} \;\; = \;\; -\frac{1}{4}\sum_{a=1}^{N^2-1} \; F_{\mu\nu\;a}F^{\mu\nu}{}_a = -\frac{1}{2}{\rm Tr}\,[F_{\mu\nu}F^{\mu\nu}]\,.$$

The invariance of the spinor part of the Lagrange density follows from the transformation of the covariant derivative of ψ, (23), and of the vector density from the "covariant" transformation of the field strengths,

$$F^{\mu\nu}(A') = U(x)F^{\mu\nu}(A)U^{-1}(x)\,.$$

- A scalar SU(N) gauge theory is built in the same way,

$$\mathcal{L} = \sum_{i=1}^{N} \; (D_\mu[A]\Phi)_i^\dagger \, (D^\mu[A]\Phi)_i - m^2 \sum_{i=1}^{N} |\Phi_i|^2 - \frac{1}{4}\sum_{a=1}^{N^2-1} \; F_{\mu\nu\;a}F^{\mu\nu}{}_a\,.$$

- Interactions that have local phase invariance can always be added to the spinor and scalar nonabelian densities above. Using a vector symbol to represent the group indices of the fields, any form like $V(|\vec{\Phi}|^2)$ or $V(\vec{\bar{\psi}} \cdot \vec{\psi})$ will be fine, although renormalizability will generally restrict us to scalar potentials only in four dimensions.