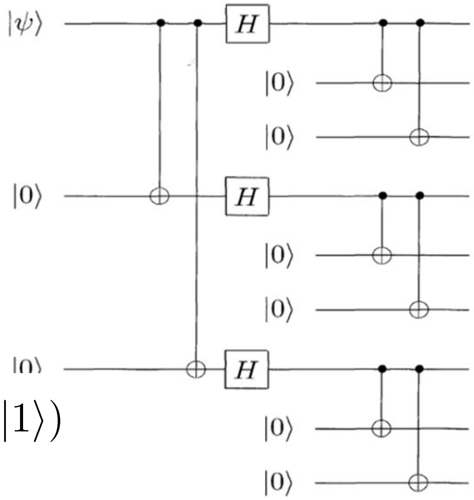
PHY682 Special Topics in Solid-State Physics: Quantum Information Science

Lecture time: 2:40-4:00PM Monday & Wednesday

Today 10/5:

1. Brief review error correction for Shor’s code
2. More of Week 6’s topic on quantum error correction

Review: Shor's 9-qubit Error Correction Code [[n,k,d]]=[[9,1,3]]



- To fight against flip error:

$$|"0"\rangle = |000\rangle \qquad |"1"\rangle = |111\rangle$$

- To fight against phase error:

$$|" + "\rangle = |+++\rangle = \frac{1}{\sqrt{8}}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$
$$|" - "\rangle = |---\rangle = \frac{1}{\sqrt{8}}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)$$

- Shor's suggestion to fight against both (and thus more):

$$|"0"\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|"1"\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

➔ Can verify the capability of correcting one-qubit errors

Review: Shor's code protects all 1-qubit errors!

Nielsen & Chuang: "the apparent continuum of errors that may occur on a single qubit can all be corrected by correcting only a discrete subset of those errors"

- Consider E_k to be a general combination

$$E_k = e_{k0} I + e_{k1} X + e_{k2} XZ + e_{k3} Z$$

Its action on a qubit ψ gives rise to superposition (of no error and three types of errors):

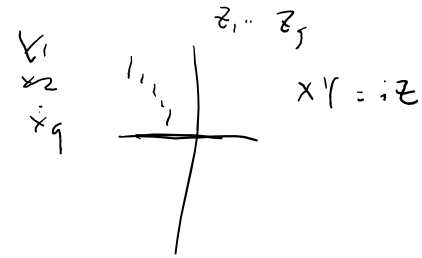
$$E_k|\psi\rangle = e_{k0}|\psi\rangle + e_{k1} X|\psi\rangle + e_{k2} XZ|\psi\rangle + e_{k3} Z|\psi\rangle$$

➔ Measuring "syndromes" collapses to either of the four components and correction can be applied to recover ψ

Review: Correctable errors---Shor's code

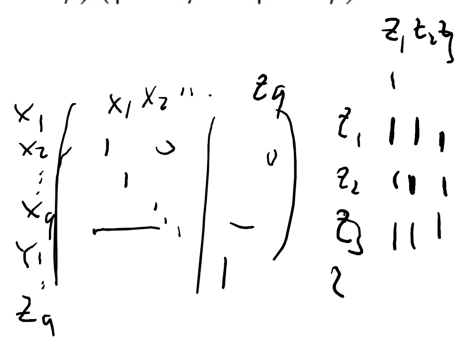
- Correctable condition: $PE_i^\dagger E_j P = \alpha_{ij} P$

Exercise 10.10 (N&C): Explicitly verify the quantum error-correction conditions for the Shor code, for the error set containing I and the error operators X_j ,Y_j ,Z_j for j = 1 through 9.



$$|"0"\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$
$$|"1"\rangle = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

$(|0\rangle\langle 0| + |1\rangle\langle 1|) X_1 X_2 (|0\rangle\langle 0| + |1\rangle\langle 1|)$



Review: Error-correction conditions for S*

- Theorem 10.8 (Nielsen & Chuang): (Error-correction conditions for stabilizer codes) Let S be the stabilizer for a stabilizer code C(S). Suppose $\{E_j\}$ is a set of operators in G_n such that $E^\dagger_j E_k$ not in N(S)−S for all j and k. Then $\{E_j\}$ is a correctable set of errors for the code C(S).

  Note: N(S) is normalizer group of S: contains elements E of G_n that preserve S, i.e. $\forall g \in S \Rightarrow E g E^\dagger \in S$ [In this case, N(S) is equal to the centralizer Z(S), the group that commutes with all elements in S]

- For Shor's code: N(S) is generated by 1 $X_1X_2X_3X_4X_5X_6$, 2 $X_4X_5X_6X_7X_8X_9$ (from phase flip) 3 Z_1Z_2, 4 Z_2Z_3, 5 Z_4Z_5, 6 Z_5Z_6, 7 Z_7Z_8, 8 Z_8Z_9 and the two logical operators: $Z = X_1X_2X_3X_4X_5X_6X_7X_8X_9$ and logical $X = Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9$
  - ✓ N(S)-S [set of elements in N(S) but not in S] contains operators of weight at least three: $X_1X_2X_3$, $X_4X_5X_6$, $X_7X_8X_9$, $Z_1Z_4Z_7$, $Z_2Z_5Z_8$, etc. $E^\dagger_j E_k$ from single-qubit errors are not in this set!

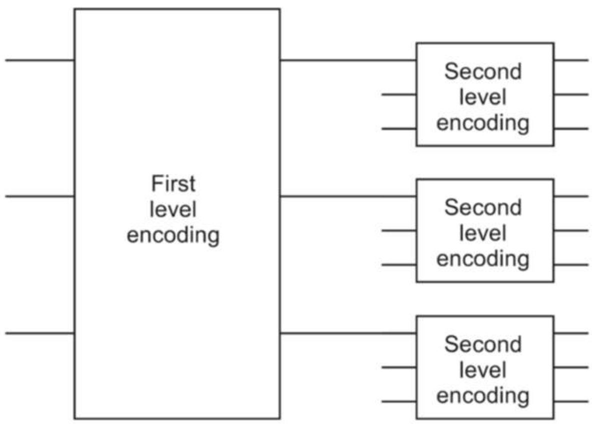
Summary: Quantum error correction

- In quantum computer: there are more errors than just bit flip and phase
  - ✓ Also due to quantum superposition, being able to correct flip and phase errors ➔ correct all one-qubit errors
- Quantum error correction has been well developed, drawing inspirations from classical coding theory;
  now also used in many fields, e.g. condensed matter physics and AdS/CFT holographic entanglement
- Quantum computers spend more effort in preventing and actively correcting errors than classical ones
  - ➔ need measurement to find errors and apply correcting operations

Fault-tolerant and threshold

[Shor, Aharonov, Ben-Or, Kitaev,…]

If the error probability p of a gate is less than some threshold p_{th} , then arbitrarily long quantum computations are possible, using noisy gates, with a reasonable overhead cost



Concatenation

- Uses concatenation (many layer)

  each layer error prob. p → cp^2

- k layers: $p \to \frac{1}{c}(cp)^{2^k}$

- Want error probability ≤ gate error rate/poly(number) of gates

  ➔ needs $p < p_{\rm th} = 1/c$

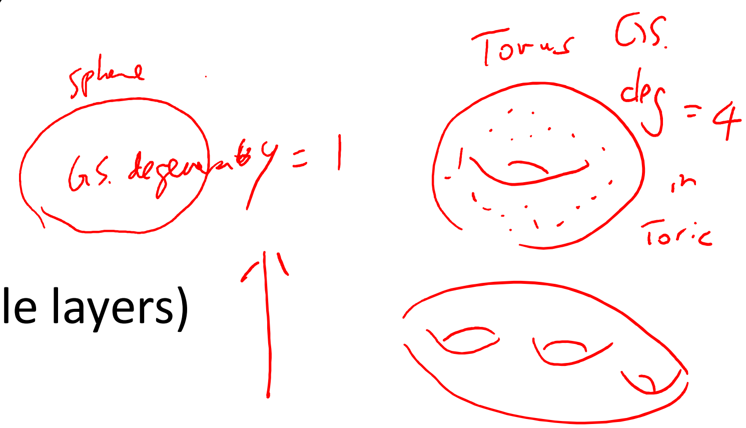
❖ p_{th} and overhead depend on error model and fault-tolerant scheme

Topological codes

Toric code, color codes, Bacon-Shor codes, subsystem codes, fracton codes, Levin-Wen string-net models, etc.

➔ Closely related to topological phases

➔ Do not need "concatenation" (i.e. multiple layers)



encode Q Info
in degenerate ground space

Kitaev's toric code

[Kitaev '03, Wen '03]

- Geometry: torus
- One qubit on each edge

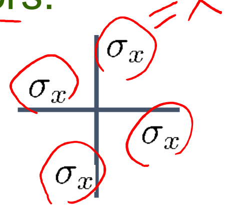
$$n_{\text{qubit}} = 2N^2$$

- Hamiltonian: $\hat{H} = -\sum_s A_s - \sum_p B_p$

This defines the stabilizer group

- Star operators:

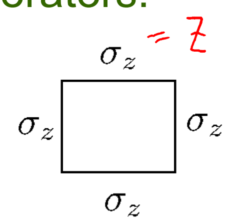
$$A_s = \prod_{j \in s} \sigma_x^{[j]}$$



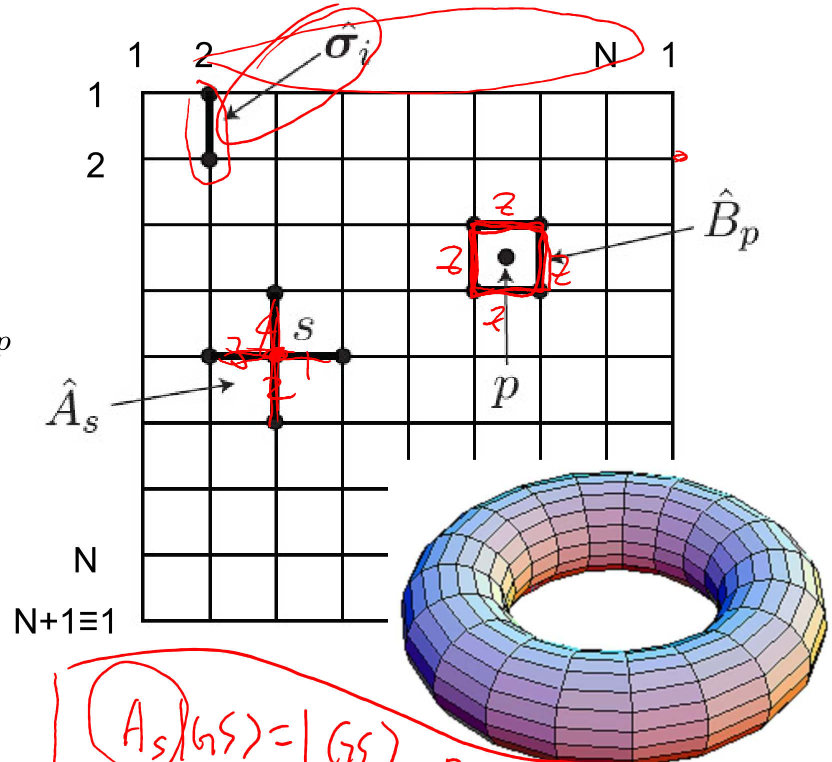
$$n_s = N^2$$

- Plaquette operators:

$$B_p = \prod_{j \in \partial(p)} \sigma_z^{[j]}$$



$$n_p = N^2$$



$A_s|GS\rangle = |GS\rangle$, $B_p|GS\rangle = |GS\rangle$

$\Rightarrow$ $[A_s, B_p] = 0$

Ground-State degeneracy

- Counting stabilizer operators

$$[A_s, B_p] = 0 \Rightarrow$$

$$A_s|\mathrm{GS}\rangle = B_p|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$$

- How many independent equations (stabilizer generators)?

$$\prod_p B_p = \prod_s A_s = I \Rightarrow 2N^2 - 2$$

star + plaq

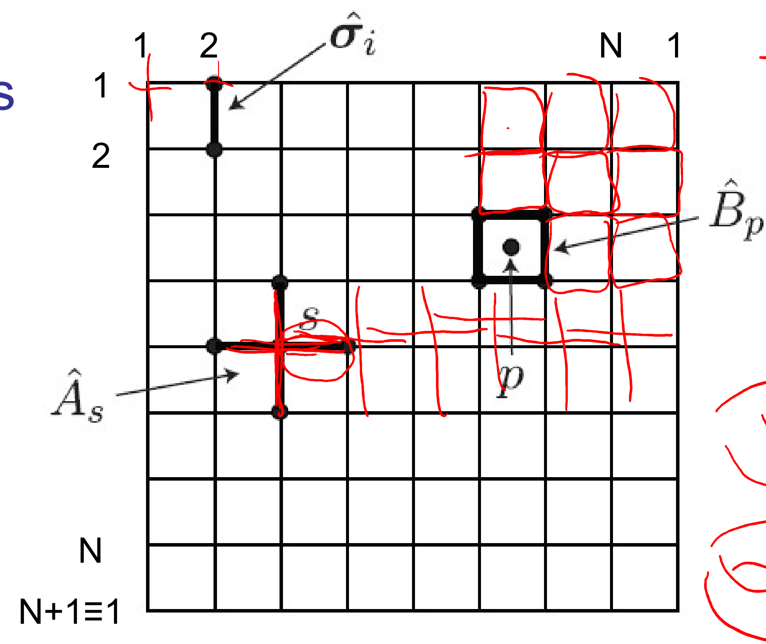
- Effective two qubits, i.e. 4-fold degeneracy

logical # of qubits = $\underbrace{2N^2}_{\text{physical}} - \underbrace{(2N^2 - 2)}_{\text{indep generator}} = 2$

- For surface with genus g: 4^g-fold degeneracy

$$E - (V + F - 2) = 2 - (V - E + F) = 2g$$

$2^{2g} = 4^g$



Δ gap

4 degenerate GS

Torus

Recall: Stabilizer group & logical code space

Proposition 10.5 (Nielsen & Chuang): Let S = <g_1 ...,g_{n-k}> be generated by n − k independent and commuting elements from G_n , and such that −I not in ∈ S. Then V_S is a 2^k -dimensional vector space (effectively k qubits):

➔ V_S is a k-qubit code space C(S) defined by the stabilizer group S

➔ Can choose two sets of k operators (logical Z's and X's)

$$\{\bar{Z}_1, \dots \bar{Z}_k\},\ \{\bar{X}_1, \dots \bar{X}_k\}$$ → commute w stabilizer S

Such that

(i) $\{g_1, g_2, \dots, g_{n-k}, \bar{Z}_1, \dots \bar{Z}_k\}$ Independent, commuting

(ii) $[g_j, \bar{X}_l] = 0,\ \boxed{[\bar{Z}_i, \bar{X}_{j\neq i}] = 0,\ \bar{Z}_i\bar{X}_i = -\bar{X}_i\bar{Z}_i}$

for same logical qubit

X's are logical X operators

➔ Can apply to Toric code: what are the logical operators?

Two qubits: effective Pauli's

- Operators on non-contractible cycles form effective qubit operators

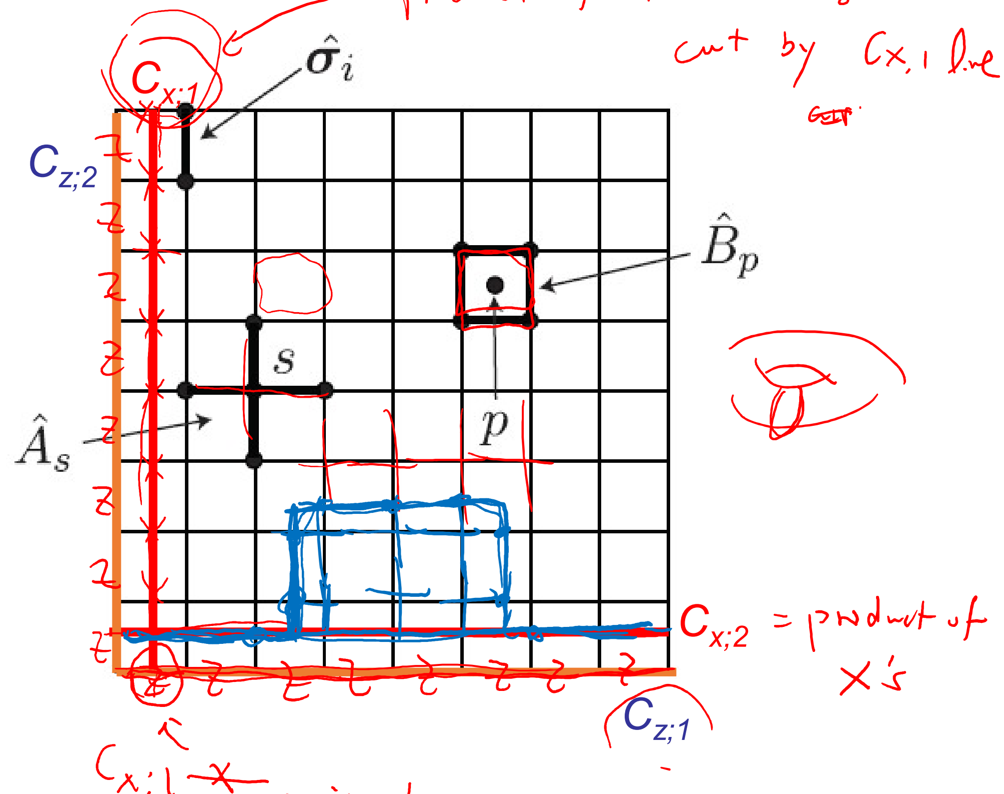
$$Z_{1/2} \equiv \prod_{j \in C_{z;1/2}} \sigma_z^{[j]}$$

1 or 2

$$X_{1/2} \equiv \prod_{j \in C_{x;1/2}} \sigma_x^{[j]}$$

➡ Commute with star and plaquette operators

➡ $\{X_1, Z_1\} = \{X_2, Z_2\} = 0$



- Action of X's and Z's on GS remains in ground space [C's can be deformed]

X_i|GS⟩ is a GS

logical L ≡ L s ← stabilizer

s|GS⟩ = |GS⟩

$$E_0 X_i|GS\rangle = X_i \hat{H}|\mathrm{GS}\rangle = \hat{H}(X_i|\mathrm{GS}\rangle) = E_0(X_i|\mathrm{GS}\rangle)$$

Excitations of toric code model

$$H = -\sum_s A_s - \sum_p B_p$$

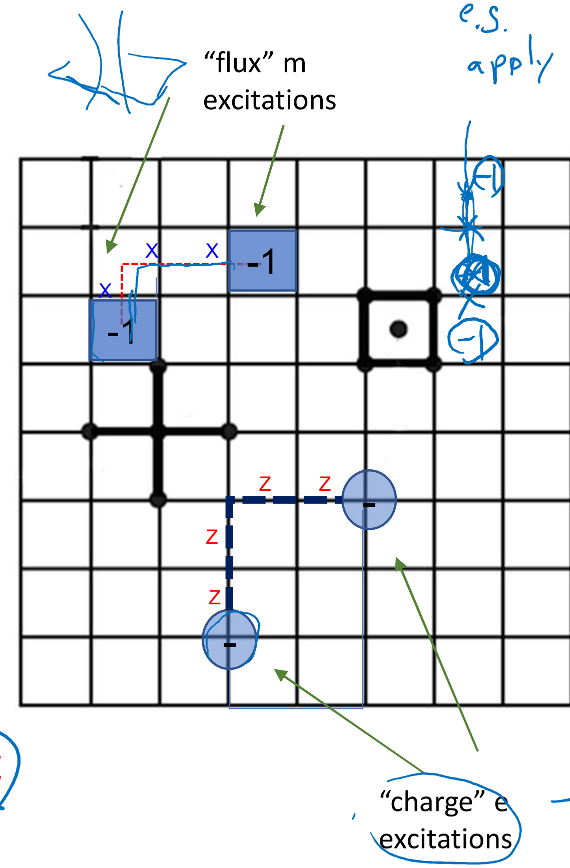
- Ground states "satisfy" all terms $A_s=B_p=1$ in the Hamiltonian
  ➔ $E_0 = -2N^2$

$+1 \to -1$) (-4)
$+1 \to -1$

- Excitations break some of them (e.g. $B_{p1}= -1$), $-1 \cdot -1 = (-2)$
  but come in pairs due to

$$\prod_p B_p = \prod_s A_s = I$$

➔ A pair of excitations has energy E_0+4

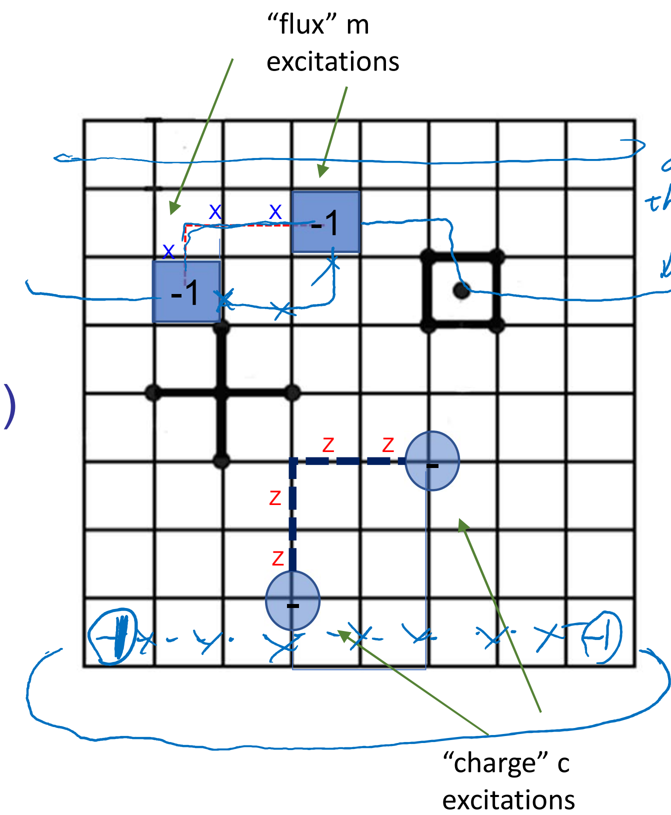


e.g. apply a single X
it commutes with A_s
but anticommutes with two plaq

Correcting local errors

$$H = -\sum_s A_s - \sum_p B_p$$

- Error syndromes are from measuring A_s=B_p (see if -1)
- Identify pairs of (e,e) and of (m,m)
- Then apply string of Z..Z or X..X to correct them



apply correction like this will not correct errors

⇒ causes logical bit flip

long distance string is unlikely

Check error correction condition

local oper

$\langle \psi_i | E_x | \psi_j \rangle$ GS GS

- Correctable condition: $PE_i^\dagger E_j P = \alpha_{ij} P$

  For toric code: P is projector to ground space.

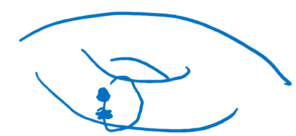
  - ✓ Can be easily checked that local operators cannot distinguish different ground states (the combined $E^\dagger_i E_j$ needs to wrap around the torus, otherwise $\alpha_{i\neq j}=0$)

    local

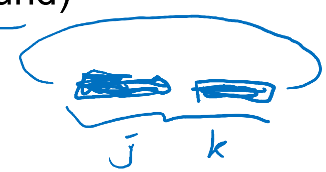
$\langle \psi_\ell | E_k | \psi_k \rangle$

- Correctable condition from stabilizer formalism:

  $E^\dagger_j E_k$ not in N(S)−S for all j and k.



  - ✓ N(S)-S is generated by logical operators (product of Pauli wrapping around)
  - ✓ Errors can be corrected as long as $E^\dagger_j E_k$ do not wrap around the torus)



Ground state: explicit construction

$B_p |0\cdots0\rangle = |0\cdots0\rangle$

- Configuration 00…0 satisfies $B_p = \prod_{j\in\partial(p)} \sigma_z^{[j]}$ $\qquad \sigma_z|0/1\rangle = \pm|0/1\rangle$

$Z|0\rangle = |0\rangle$

- *Action of A_s*: locally flipping 0000 to 1111 $\qquad A_s = \prod_{j\in s}\sigma_x^{[j]}$
  - ➔ apply all possible flipping

$A_s|GS\rangle = |GS\rangle$

- Consider group G_s generating by all A_s

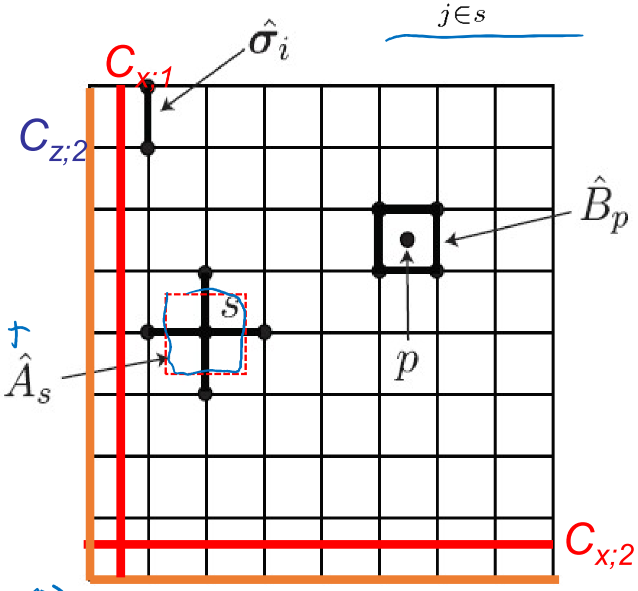
flipping by A_s or product of A_s

$$|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}}\sum_{g\in G_s} g|00\ldots0\rangle$$

  - ➔ Satisfies all A_s and B_p

g'

$A_s \sum_g g|0\cdots0\rangle = \sum_g (A_s g)|0\cdots0\rangle$



$|G_s| = 2^{n_s-1}$

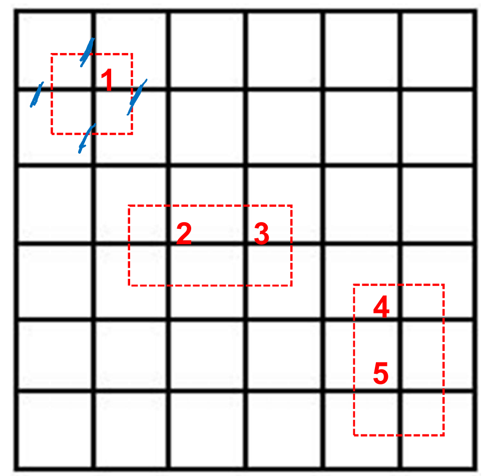
Pictorial understanding

$|0\cdots0\rangle$

$$|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}} \sum_{g\in G_s} \underline{g|00\ldots0\rangle}$$

e.g.

$$g = \underline{A_1 A_2 A_3 A_4 A_5 I} \cdots I$$



➔ Ground state G_{00} is a equal superposition of all possible (contractible) loop configurations [of 1…1]

Other ground states [4 degenerate states]

$$|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}} \sum_{g \in G_s} g|00\ldots0\rangle \qquad |G_s| = 2^{n_s - 1}$$

- Ground space: effective two qubits

  ➔ Use logical Pauli X operators to flip to get other ground states:

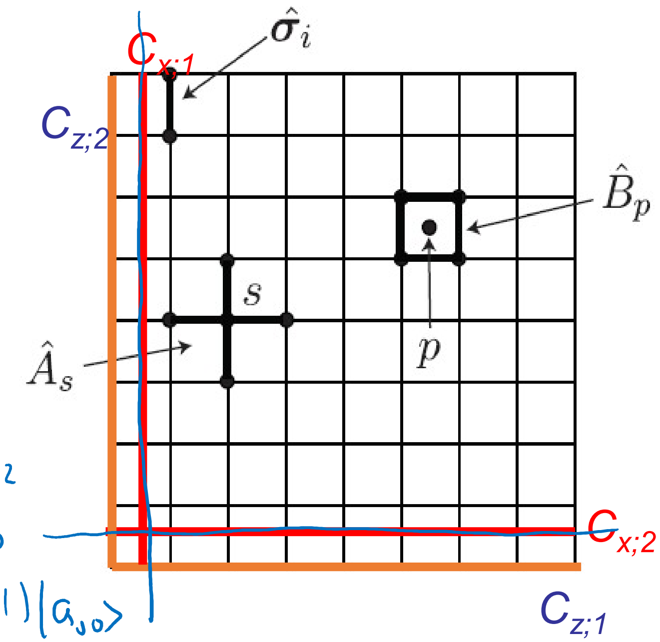
$$|G_{\alpha,\beta}\rangle \equiv (X_1)^{\alpha}(X_2)^{\beta}|G_{0,0}\rangle$$

$$X_{1/2} \equiv \prod_{j \in C_{x;1/2}} \sigma_x^{[j]}$$

Should check

- Degenerate ground states cannot be distinguished locally
  (as *X* strings can be deformed)

Z_1 or Z_2 $|G_{00}\rangle = (+1)|G_{00}\rangle$



Pictorial understanding

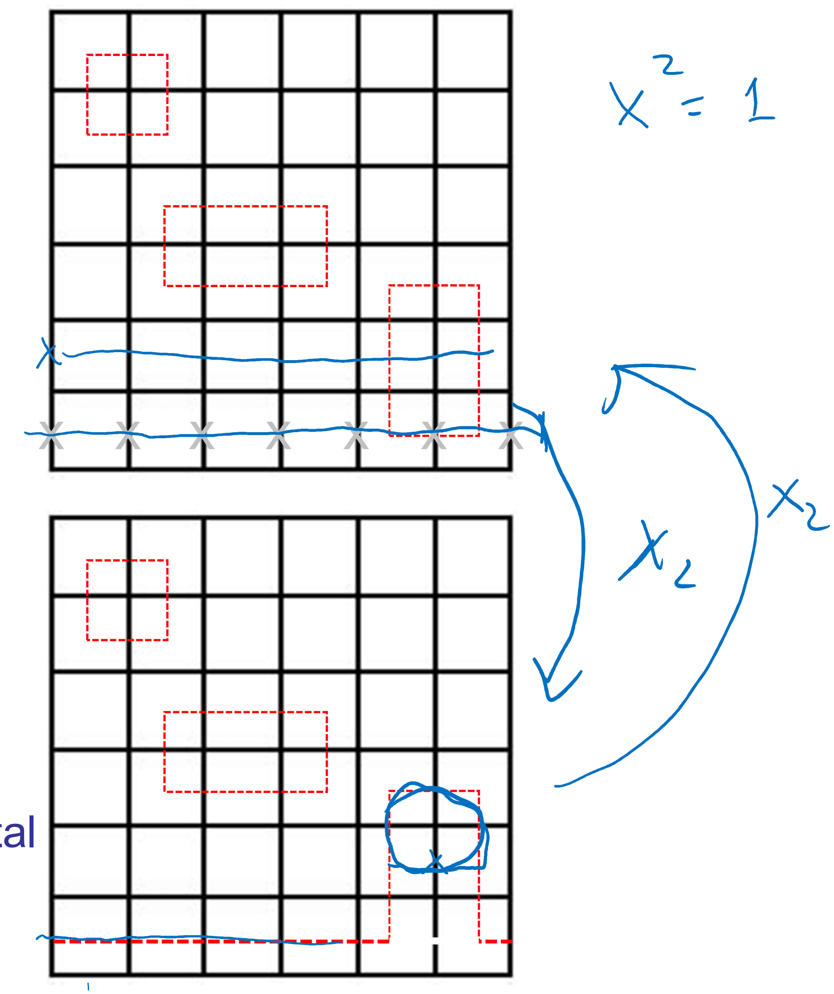
$$|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}} \sum_{g \in G_s} g|00...0\rangle$$

- Equal superposition of all possible (contractible) loop configurations

Another ground state:

$$|G_{0,1}\rangle = \Big(X_2 \equiv \prod_{j \in C_{x;2}} \sigma_x^{[j]}\Big)|G_{0,0}\rangle$$

- Non-contractible loops with winding number=1 (odd) in horizontal direction

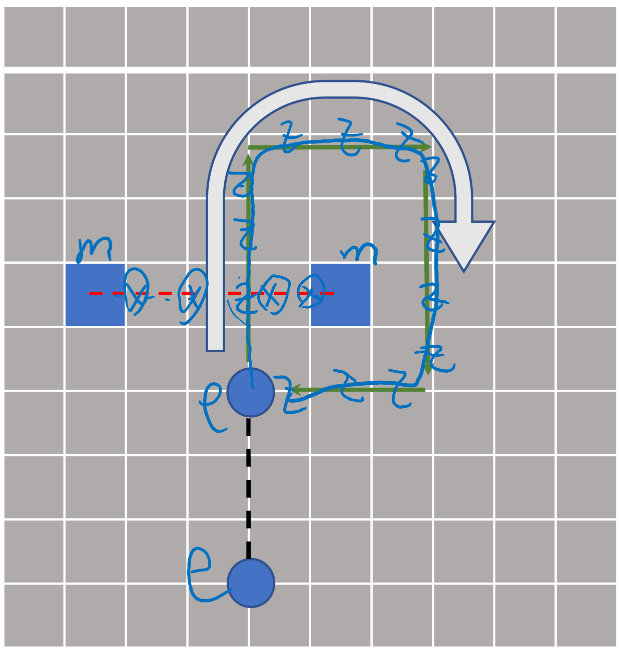


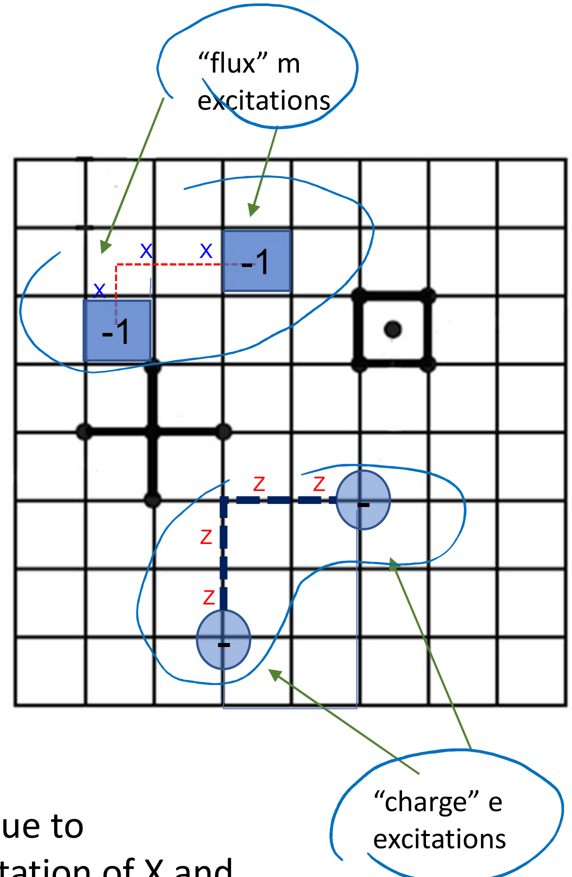
Anyons: excitations of toric code model

- We have seen ground-state vacuum (I) and charge (e): A_s =-1, flux (m) B_p= -1

➔ Exchange between e & e (or m & m) gives +1

➔ Braiding of e around m gives -1 sign





➔ -1 sign due to anticommutation of X and Z operators (crossed by red and green lines)

Anyons: I, e, m and f (fermions)

- We have now seen three anyons: vacuum (I) charge (e): A_s=-1, and flux (m) B_p= -1

➢ One more type of anyons is the usual "fermion" which is a "bound state" of e and m

f

- Fusion: *e and m fuse to f (e x m = f); e x f = m, m x f = e*

f=(e x m)

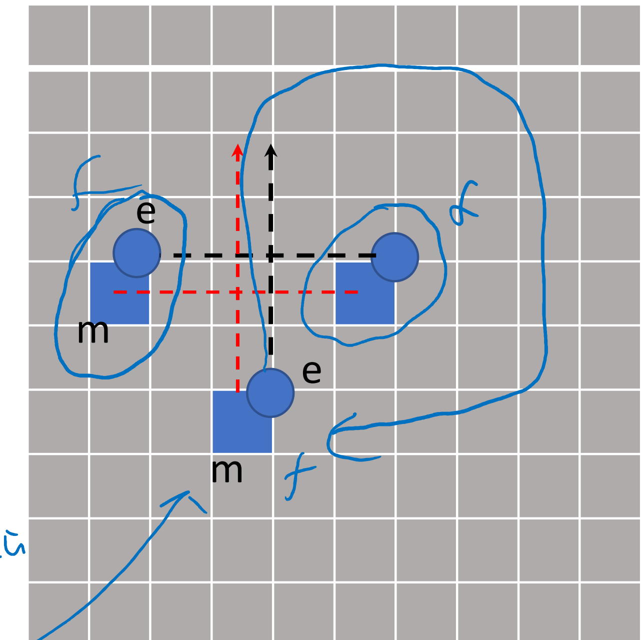
e x e = I , m x m = I

- Vacuum I is identity: *I x e = e, I x m = m, I x f = f*
- Same anyons fuse to vacuum: e x e = I = m x m = f x f

e → ← e annihilation

Note: Braiding of (e,m) around (e,m) gives +1

➔ Question: how do we reveal -1 of its fermion exchange?



f — f exchange twice (+1)

"Topological charge" basis (*)

Old logical: $|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}} \sum_{g \in G_s} g|00\ldots0\rangle$

- Four basis states in ground space (convenient basis but there is a better one)

logical : $\boxed{|G_{\alpha,\beta}\rangle \equiv (X_1)^\alpha (X_2)^\beta |G_{0,0}\rangle}$

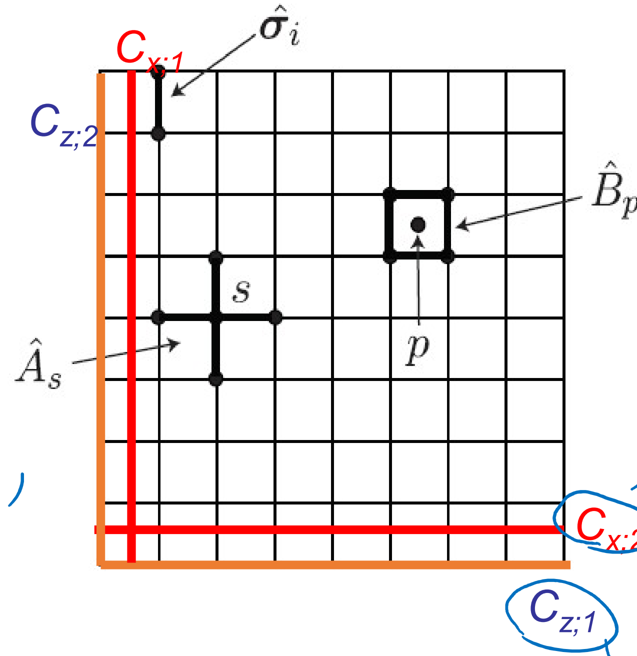
- Topological charge basis w.r.t. (e.g.) horizontal loops: (mixed basis)

$$\boxed{Z_1 \otimes X_2}|0/1,\pm\rangle = (-1)^{0/1} \cdot (\pm 1)|0/1,\pm\rangle$$

$$|0,\pm\rangle \equiv \frac{1}{\sqrt{2}}(|G_{0,0}\rangle \pm G_{0,1}\rangle)$$

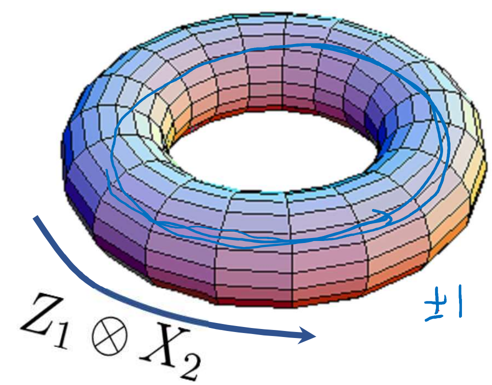
$$|1,\pm\rangle \equiv \frac{1}{\sqrt{2}}(|G_{1,0}\rangle \pm G_{1,1}\rangle)$$

usually $|0\rangle|+\rangle = (|0\rangle|0\rangle + |0\rangle|1\rangle)/\sqrt{2}$



flux

charge

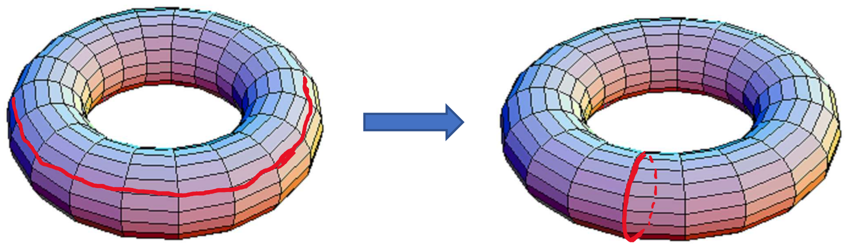


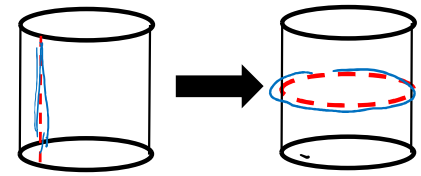
± 1

fixed topological charge

Rotate by 90° and modular S matrix

❑ Rotation by 90° of (e.g.) horizontal loops:





$$|G_{\alpha,\beta}\rangle \longrightarrow |G_{\beta,\alpha}\rangle$$

swap $\langle G'|\hat{S}|G\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

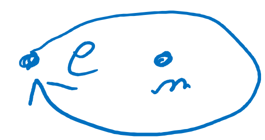
transform to 0/1, +/- basis

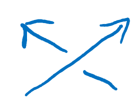
➔ Using topological charge basis:

$$\begin{pmatrix} |0,+\rangle \\ |0,-\rangle \\ |1,+\rangle \\ |1,-\rangle \end{pmatrix} \xrightarrow{\hat{S}} \underbrace{\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}}_{\text{Modular S matrix} \to \text{mutual statistics}} \begin{pmatrix} |0,+\rangle \\ |0,-\rangle \\ |1,+\rangle \\ |1,-\rangle \end{pmatrix}$$

I e m f

X is a braiding statistics



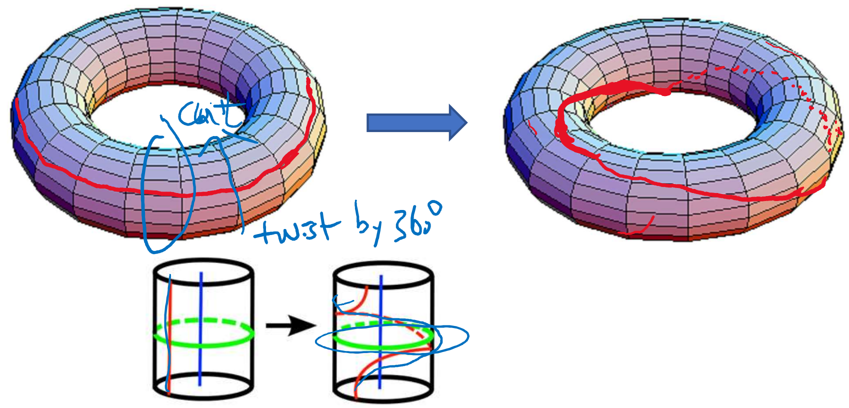


Dehn twist and modular T matrix

❑ Dehn twist: $|G_{\alpha,\beta}\rangle \longrightarrow |G_{\alpha,\beta+\alpha}\rangle$

$$\langle G'|\hat{T}|G\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CNOT



➔ Using topological charge basis:

self boson

$$\begin{pmatrix} |0,+\rangle \\ |0,-\rangle \\ |1,+\rangle \\ |1,-\rangle \end{pmatrix} \xrightarrow{\hat{T}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} |0,+\rangle \\ |0,-\rangle \\ |1,+\rangle \\ |1,-\rangle \end{pmatrix}$$

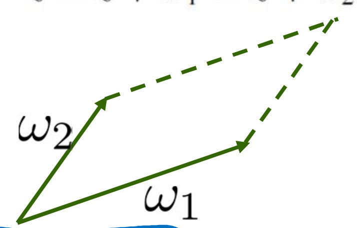
self fermion

Fermion exchange sign -1

Modular T matrix ➔ self exchange statistics

Modular transformation and statistics

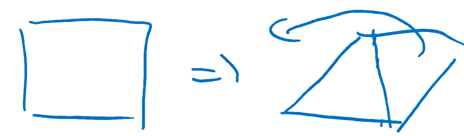
$$z \equiv z + \omega_1 \equiv z + \omega_2$$



$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \to \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

→ *SL(2,Z)*

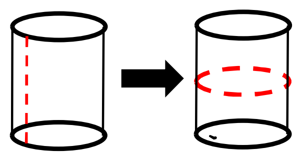
Dehn twist
= shear



- *SL(2,Z)* generated by *s* & *t*

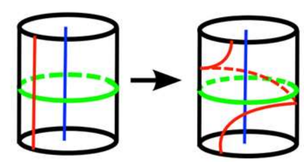
$$\hat{s} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

[90° rotation on square]



$$\hat{t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

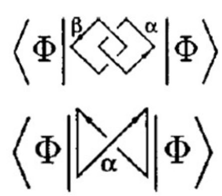
[Dehn twist]



- Modular transformation on degenerate ground states in topological basis $\{|\Xi_\alpha^{//}\rangle\}$
  ➔ modular matrices *S* (mutual statistics) & *T* (self-statistics)

$$S_{\beta\alpha} = \langle \Xi_\beta^{//} | \hat{R}(90^\circ) | \Xi_\alpha^{//} \rangle$$

$$T_{\beta\alpha} = \langle \Xi_\beta^{//} | \mathrm{Dehn\ Twist} | \Xi_\alpha^{//} \rangle$$



Check error correction condition

- Correctable condition: $PE_i^\dagger E_j P = \alpha_{ij} P$

  For toric code: P is projector to ground space.

  - ✓ From the property of topological phase that local operators cannot distinguish different ground states (the combined $E^\dagger_i E_j$ needs to wrap around the torus)

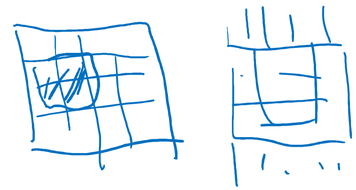
- Correctable condition from stabilizer formalism:

  $E^\dagger_j E_k$ not in N(S)−S for all j and k.

  - ✓ Errors can be corrected as long as $E^\dagger_j E_k$ do not wrap around the torus)

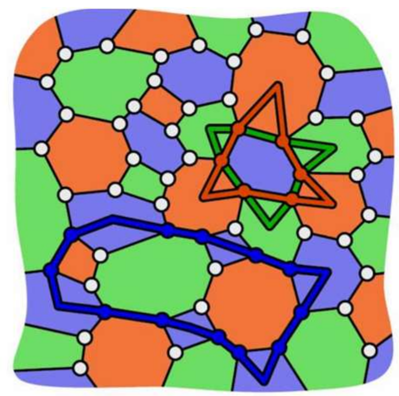
Other topological error codes

A_s B_p



logical qubits ⇒ surface code

google is aiming to implement surface code for Q.C.

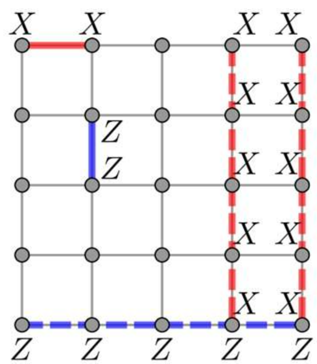


Surface code: “Toric code” on a plane with boundary

punch holes ⇒ a pair of holes

Color code: can be regarded as multiple copies of toric codes
[qubits are on vertices]

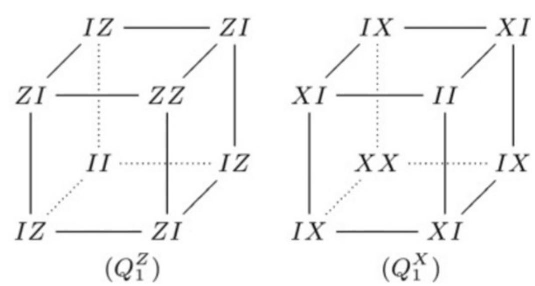
Bacon-Shor code:



⇒ recent 9-qubit implementation of Bacon-Shor

trapped ions C. Monroe

→ Fracton code:
logical operators have fractal structure



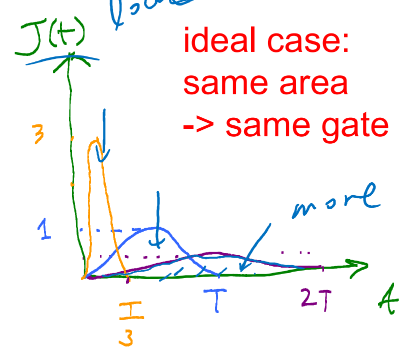
Error mitigation (not active correction)

- Basic idea:

[Temme, Bravyi & Gambetta, PRL 119, 180509 (2017); Li & Benjamin, PRX7, 021050(2017)]

  - A gate is achieved by some evolution operator via external field or coupling strength J(t)
  - Same area ➔ same ideal gate
  - Reality: longer pulse experience "larger" noise (effective larger λ)

lower error



more error

zero

$$\rightarrow \mathrm{Tr}(A\rho(t=T)) = E_K(\lambda) = E^* + \sum_{k=1}^{n} a_k \lambda^k + R_{n+1}(\lambda, \mathcal{L}, \mathcal{T})$$

tune strength of error

  - If we use different pulses to mimic different noise strengths $c_j\lambda$ can extract ideal E* up to small correction

⇒ extrapolate to zero error

$$E^* \approx \sum_{j=0}^{n} \gamma_j E_K(c_j \lambda) + \mathcal{O}(\lambda^{n+1})$$

$$\sum_j \gamma_j = 1, \; \sum_{j=0}^{n} \gamma_j c_j^k = 0$$

Measurement Error mitigation

[e.g Chen, Farahzad, Yoo & Wei, PRA 100, 052135 (2019); also standard in Qiskit]

- Basic idea: prepare computational states n and measure in computational basis. Gather enough statistics ➔ matrix M

- Mitigate readout errors [***M*** from detector tomography or above procedure]

ideal distribution

distribution

$$\underbrace{\tilde{P}_{(n_0,n_1,\ldots,n_{N-1})}[\text{measured}]}_{} = \sum_{\vec{m}} \mathbf{M}_{\vec{n};\vec{m}} \, P_{\vec{m}}[\text{ideal}]$$

↳ by experiment

  - Invert to find P with constraint P ≥ 0:

$$P_{\text{ideal}} = M^{-1} \tilde{P}_{\text{measure}}$$