PHY682 Special Topics in Solid-State Physics: Quantum Information Science

Lecture time: 2:40-4:00PM Monday & Wednesday

Today 10/7:

1. Reminder: Midterm report due Sunday 11:59pm 10/11
2. Today: Topological Quantum Computation (will review some aspects of Toric Code)

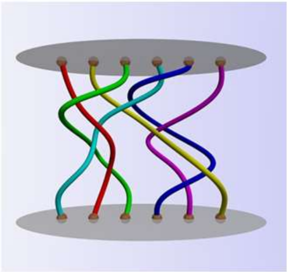
Week 7: Quantum computing by braiding: Anyons and topological quantum computation, Majorana fermions, Kitaev's chain

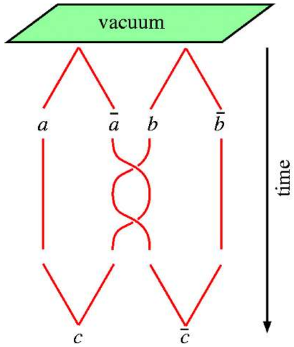
Topological Quantum Computation

[Kitaev, Freedman et al.]

- Use “topology” to passively protect against errors

- Braiding of anyons gives rise to certain set of quantum gates





- Fusing anyons to read out results

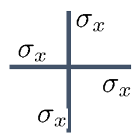
Review: Toric code [[2N^2, 2, N]]

- Hamiltonian: $\hat{H} = -\sum_s A_s - \sum_p B_p$

This defines the stabilizer group

- Star operators:

$$A_s = \prod_{j \in s} \sigma_x^{[j]}$$

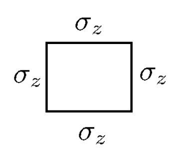


$n_s = N^2$

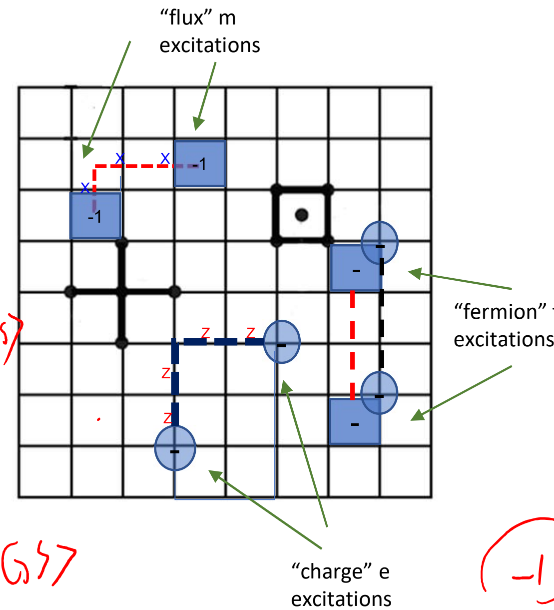
$A_s|GS\rangle = +1|GS\rangle$

- Plaquette operators:

$$B_p = \prod_{j \in \partial(p)} \sigma_z^{[j]}$$



$B_p|GS\rangle = +1|GS\rangle$



- Anyons: vacuum 1, charge e, flux m, and fermion f (= e x m)

e m -1

in same sense $\sqrt{-1}$

e m

- e: self-boson, m: self-boson, full rotation of e around m → -1 (mutual semion)
  f: fermion

f e +1

Review: Ground state: explicit construction

- Configuration 00…0 satisfies $B_p = \prod_{j\in\partial(p)} \sigma_z^{[j]}$ $\quad \sigma_z|0/1\rangle = \pm|0/1\rangle$

- *Action of A_s*: locally flipping 0000 to 1111 $\quad A_s = \prod_{j\in s} \sigma_x^{[j]}$

  ➔ apply all possible flipping $\quad$ 0110 → 1001

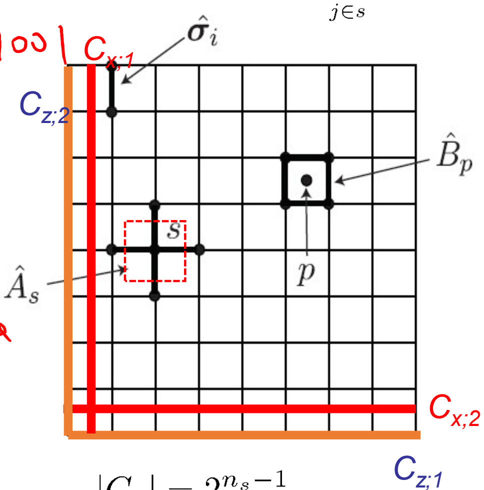
- Consider group G_s generating by all A_s

$$|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}} \sum_{g\in G_s} g|00...0\rangle$$

sum over all possible flips

$[B_p, A_s] = 0$

➔ Satisfies all A_s and B_p



$|G_s| = 2^{n_s - 1}$

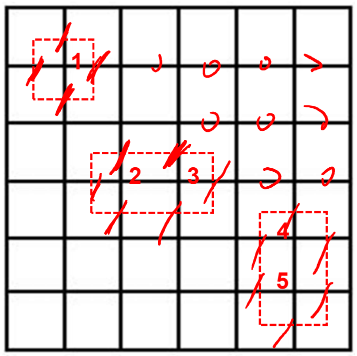
➔ Ground state G_{00} is a equal superposition of all possible (contractible) loop configurations [of 1…1]

Review: Pictorial understanding

$$|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}} \sum_{g \in G_s} g|00...0\rangle$$

e.g.

$$g = A_1 A_2 A_3 A_4 A_5 I \cdots I$$



➔ Ground state G_{00} is a equal superposition of all Possible (contractible) loop configurations [of 1…1]

Review: Other ground states

[4 degenerate states]

$$|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}} \sum_{g \in G_s} g|00\ldots0\rangle \qquad |G_s| = 2^{n_s - 1}$$

$g\,|G_{00}\rangle = |G_{00}\rangle$

$\uparrow$ S

- Ground space: effective two qubits

  ➔ Use logical Pauli X operators to flip to get other ground states:

$$|G_{\alpha,\beta}\rangle \equiv (X_1)^{\alpha}(X_2)^{\beta}|G_{0,0}\rangle$$

$|G_{10}\rangle = X_1|G_{00}\rangle$

$|G_{01}\rangle = X_2|G_{00}\rangle$

$|G_{11}\rangle = X_1 X_2|G_{00}\rangle$

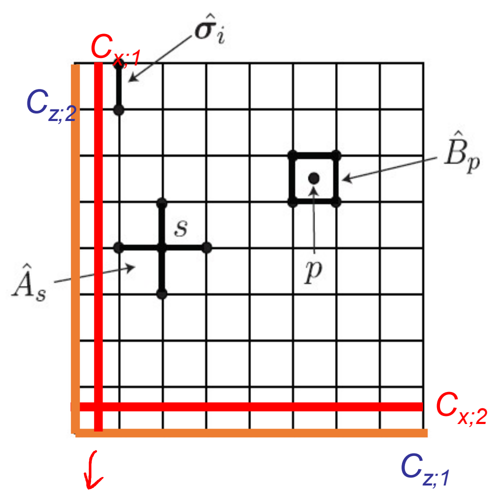
$$X_{1/2} \equiv \prod_{j \in C_{x;1/2}} \sigma_x^{[j]}$$

commute w. S

$\{X_1, Z_1\}$

$\{X_2, Z_2\}$

- Degenerate ground states cannot be distinguished locally
  (as *X* strings can be deformed)



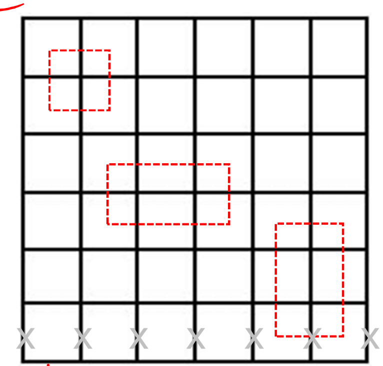
$C_{x;2} \Rightarrow X X \cdots X$

$C_{z;1} \Rightarrow Z \cdots Z$

$XZ = -ZX$

Review: Pictorial understanding

G_{00}



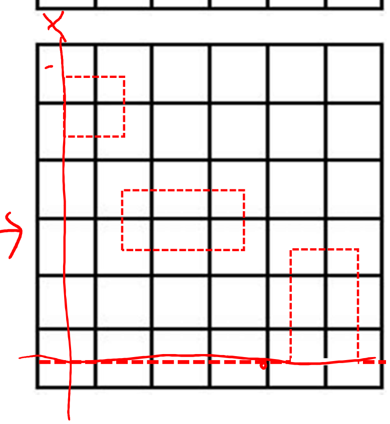
$$|G_{0,0}\rangle \equiv \frac{1}{\sqrt{|G_s|}} \sum_{g\in G_s} g|00...0\rangle$$

- Equal superposition of all possible (contractible) loop configurations

Another ground state:

$$|G_{0,1}\rangle = \big(\underline{X_2 \equiv \prod_{j\in C_{x;2}} \sigma_x^{[j]}}\big)|G_{0,0}\rangle$$

- Non-contractible loops with winding number=1 (odd) in horizontal direction

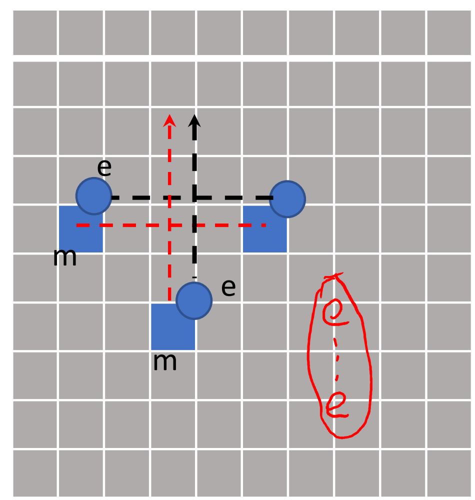


$X_2 \equiv X_2 \prod (A_s B_p)$

you can deform the loop

Review--Anyons: I, e, m and f (fermions)

- We have now seen three anyons: vacuum (I) charge (e): A_s=-1, and flux (m) B_p= -1
  - One more type of anyons is the usual "fermion" which is a "bound state" of e and m
    - Fusion: *e and m fuse to f (e x m = f);*
      e x f = m, m x f = e
    - Vacuum I is identity: *I x e = e, I x m = m,*
      I x f = f
    - Same anyons fuse to vacuum:
      e x e = I = m x m = f x f



Note: rotation of (e,m) around (e,m) gives +1

→ is non-negative integer

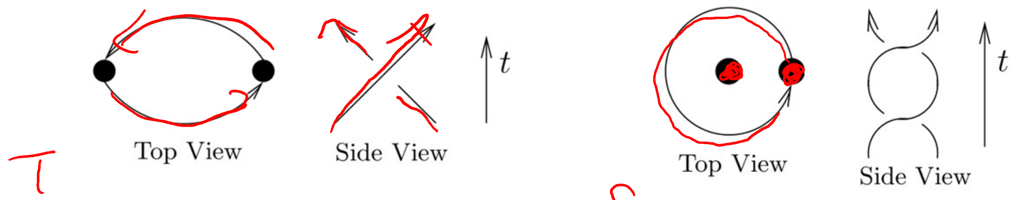
$$\text{Fusion: } a \times b = \sum_{c\in M} N^c_{ab}\, c \qquad M = \{1, a, b, c, ..\}\text{: anyon model}$$

product anyons

❖ If some N >1 → non-Abelian; Toric code hosts only Abelian anyons

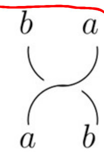
N^c_{ab} only one nonzero

Exchange (braiding) and full rotation

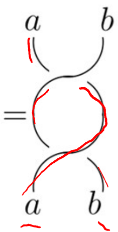


⇒ mutual statistics

- Useful notation:

$R_{ab} =$ 

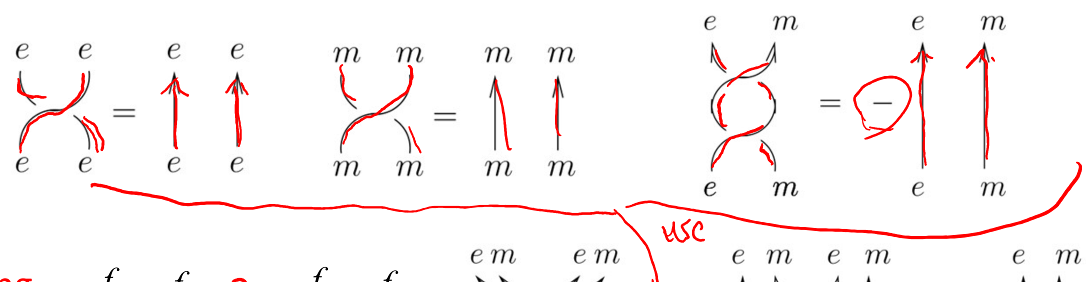
[arbitrary phase, unless a & b are same type of anyons]

$R_{ba} \cdot R_{ab} =$ 

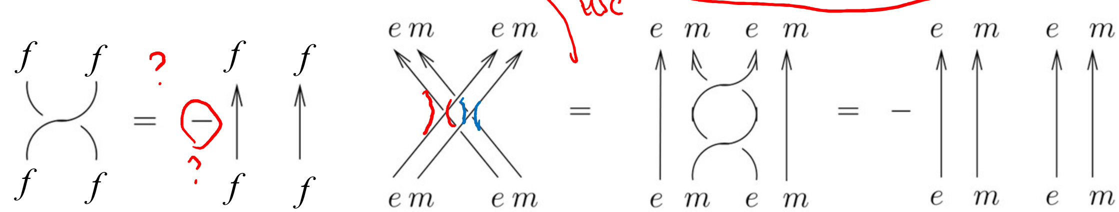
[topological invariant]

- ✓ From last lecture:

Toric Code



- Understanding fermion exchange?



Refs: Kitaev & Laumann, arXiv:0904.2771, Lahtinen & Pachos, arxiv:1705.04103, Nayak et al. Rev. Mod. Phys. **80**, 1083 (2008)

More about fusion

$a \to \quad \leftarrow b \Rightarrow (ab) \Rightarrow ?$

$a \times b \longrightarrow c$

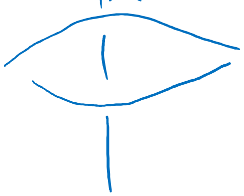
$e \to f \Rightarrow$

- If *a* and *b* fuse to several types of anyons *c*, *d*, …; define orthonormal states $|a,b;c\rangle$ (in fusion space) that satisfy $\langle a,b;d|a,b;c\rangle = \delta_{cd}$

can we use this $\sum_c \beta_c |a b; c\rangle$? to encode Q info? ↳ can not be useful in a coherent way because different anyons are the so-called superselection sectors

- If only two anyons: *a* and *b* , different *c* fused outcomes *cannot be superposed*
  ➔ need at least three anyons to be useful to encode quantum info:

$|X$ no peeky



  - (1) *a* and *b* fuse to *e*, then *e* fuses with *c* to *d* $\quad |(ab)c;ec;d\rangle$
  - (2) *b* and *c* fuse to *f*, then *a* and *f* fuse to *d* $\quad |a(bc);af;d\rangle$

  ➔ Different *e* can be used to encode information $\quad \sum q_e|(ab)c;ec;d\rangle$

different paths to fuse

fix final fusion product

e.g. $\nu = 12/5$

- Two different ways of fusion end up the same outcome *d*
  ➔ *Two bases are related by a unitary F*

Fibonacci anyons τ

$\tau \times \tau = 1 + \tau$ (from Fractional Hall effect)

$$|a(bc);af;d\rangle = \sum_e (F^d_{abc})_{fe} |(ab)c;ec;d\rangle$$

associativity $\quad a(bc) \quad (ab)c$

Anyon model: Pictorial representation

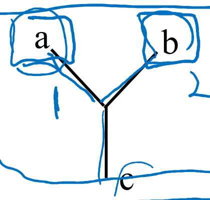
a .... b

⇓

(a b)

↓

c



Fusion: $a \times b = \sum_{c \in M} N_{ab}^{c} c$

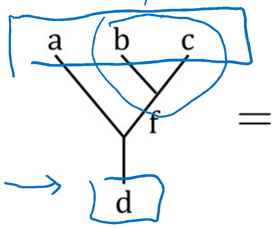
$a \times b = \sum N_{ab}^{c} c \Longrightarrow N_{ba}^{c}$

$b \times a = \sum N_{ba}^{c} c$

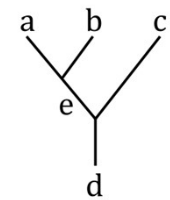
$= N_{ab}^{c}$

basis transform ⟹ unitary

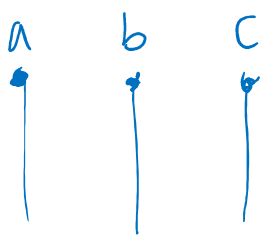
fix →



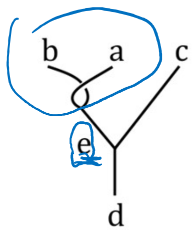
$= \sum_{e} (F_{abc}^{d})_{fe}$



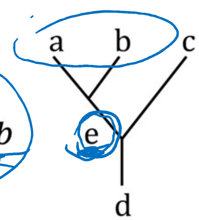
$|a(bc); af; d\rangle = \sum_{e} (F_{abc}^{d})_{fe} |(ab)c; ec; d\rangle$



different from



$= R_{ab}^{e}$

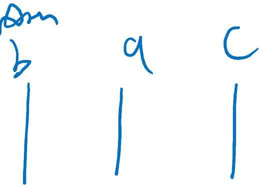


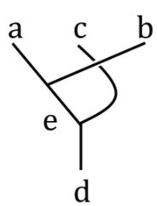
$|(ab)c; ec; d\rangle$

Kronecker delta fcn

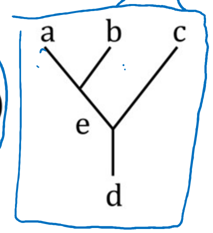
$|(ba)c; ec; d\rangle = \sum_{f} R_{ab}^{f} \delta_{e,f} |(ab)c; fc; d\rangle$

$\boxed{R_{ab}^{e} = e^{i\theta_{ab}^{e}}}$

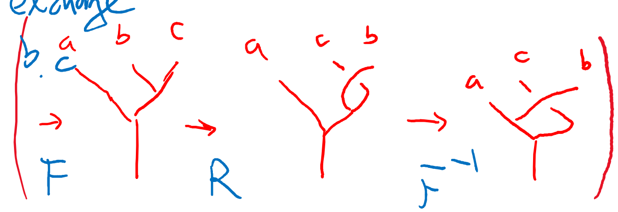




$= \boxed{(F_{abc}^{d})^{-1} R \, (F_{abc}^{d})}$

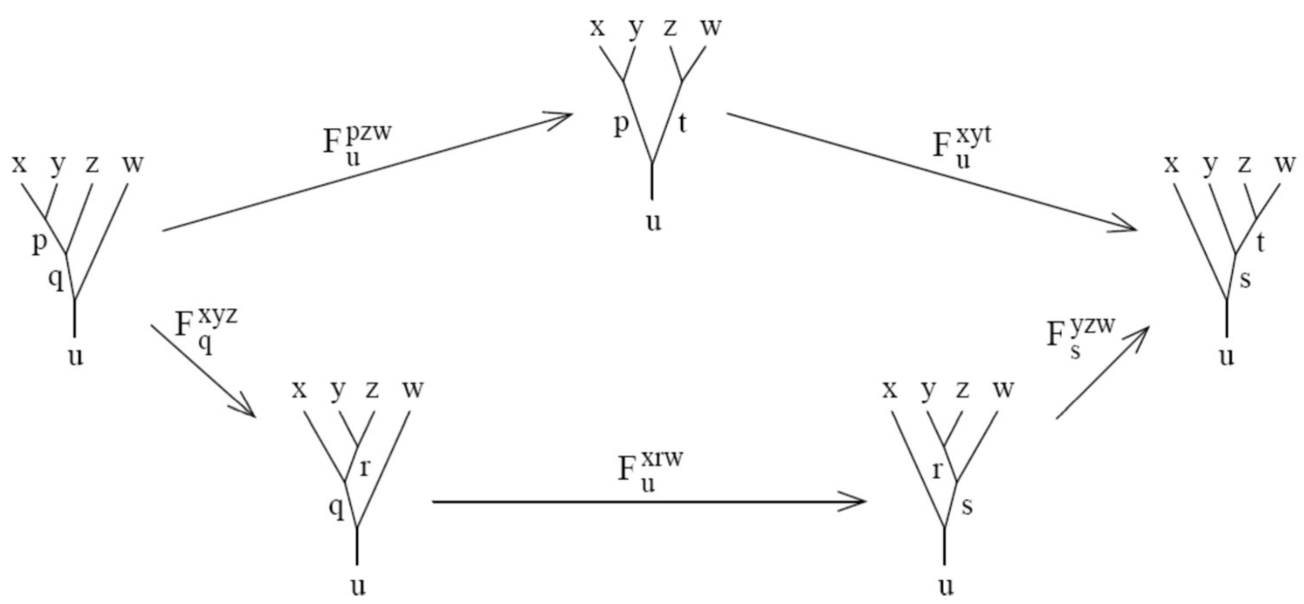


what if I want to exchange



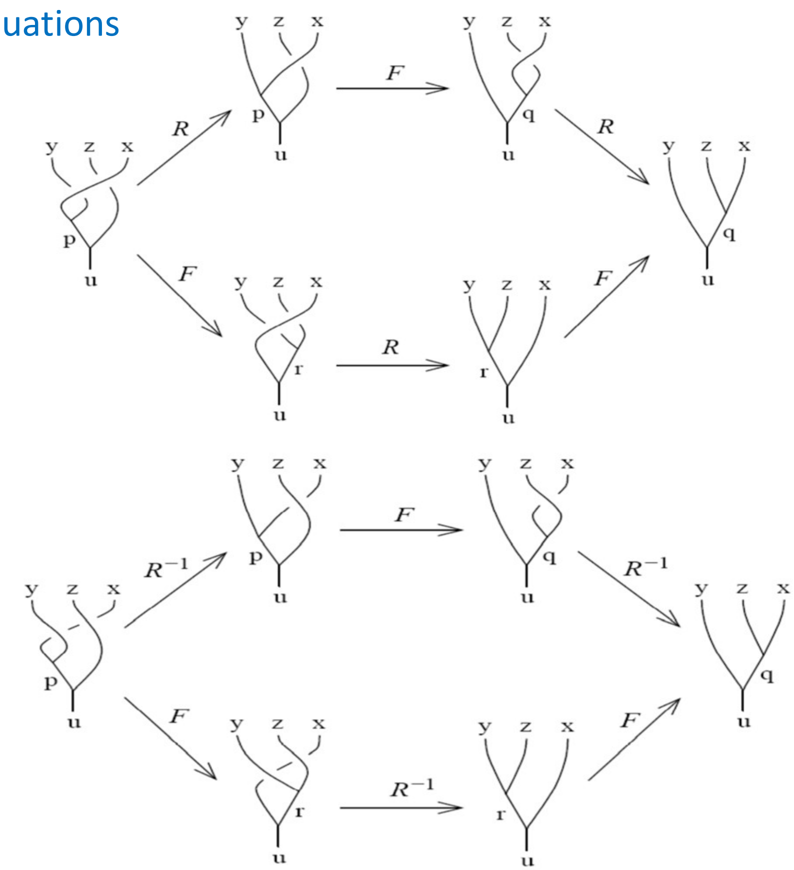
Consistency of Fusion*

- Pentagon equation



Consistency of fusion and braiding*

- hexagon equations



Example: Fibonacci anyons

- Only one nontrivial anyon: τ
- Fusion: $\tau \times \tau = 1 + \tau$ $\quad (\bullet,\bullet)_0 \quad (\bullet,\bullet)_1$ $\quad 1+\tau+\tau$

$$(\tau \times \tau) \times \tau = (1+\tau) \times \tau = \tau \times \tau + \tau = 1 + 2\tau$$

could be used as a qubit

$$\tau \times \tau \times \tau \times \tau = 2 \cdot 1 + 3\tau$$

$$\tau \times \tau \times \tau \times \tau \times \tau = 3 \cdot 1 + 5\tau$$

$$\tau \times \tau \times \tau \times \tau \times \tau \times \tau = 5 \cdot 1 + 8\tau$$

1, 1, 2, 3, 5, . . .

➔ Dimensions for a Fibonacci series: not a tensor-product structure

# of ways to fuse into this anyon "1"

- A qubit? E.g. fix product of 3 τ's fuse to τ
  → two ways to do that

fix ~~final~~ fusion outcome

$$|(\tau\tau)\tau; 1\tau; \tau\rangle, \ |(\tau\tau)\tau; \tau\tau; \tau\rangle$$ form basis states for a qubit

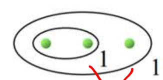
Exchange: $R = \begin{pmatrix} R^1_{\tau\tau} & 0 \\ 0 & R^\tau_{\tau\tau} \end{pmatrix} = \begin{pmatrix} e^{\frac{4\pi i}{5}} & 0 \\ 0 & e^{\frac{-3\pi i}{5}} \end{pmatrix}$

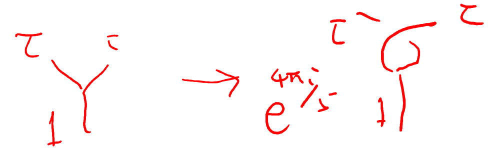
$R^\dagger = \begin{pmatrix} e^{-\frac{4\pi i}{5}} & \\ & e^{\frac{i3\pi}{5}} \end{pmatrix}$ this may be R in other convention

Gates from Fibonacci anyons

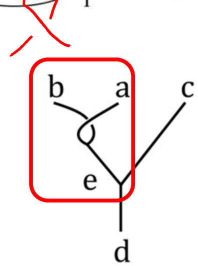
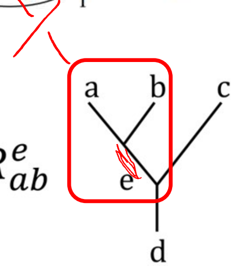
$$\tau \times \tau \times \tau = (1+\tau) \times \tau = \tau \times \tau + \tau = 1 + 2\tau$$

$$|(\tau\tau)\tau; 1\tau; \tau\rangle, \ |(\tau\tau)\tau; \tau\tau; \tau\rangle$$ form basis states for a qubit

 $= |0_L\rangle$  $= |1_L\rangle$



- Exchange:

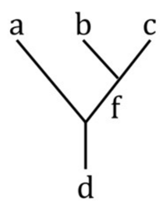
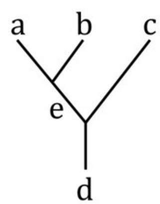
 $= R^e_{ab}$ 

$$R = \begin{pmatrix} R^1_{\tau\tau} & 0 \\ 0 & R^\tau_{\tau\tau} \end{pmatrix} = \begin{pmatrix} e^{\frac{4\pi i}{5}} & 0 \\ 0 & e^{\frac{-3\pi i}{5}} \end{pmatrix}$$

diagonal gate

$\boxed{R^{10} = 1}$

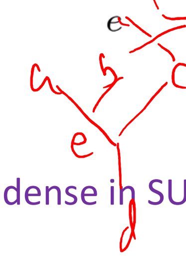
- Basis change:

 $= \sum_e \left(F^d_{abc}\right)_{fe}$ 

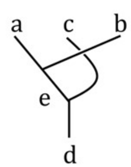
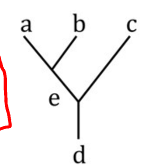
$$F = F^\tau_{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$$

$$\phi = (1+\sqrt{5})/2$$

iterate would be F^+



- Exchange (of b & c):
  F[-1]RF forms a group dense in SU(2)

 $= \underbrace{\left(F^d_{abc}\right)^{-1} R \left(F^d_{abc}\right)}_{R'}$ 

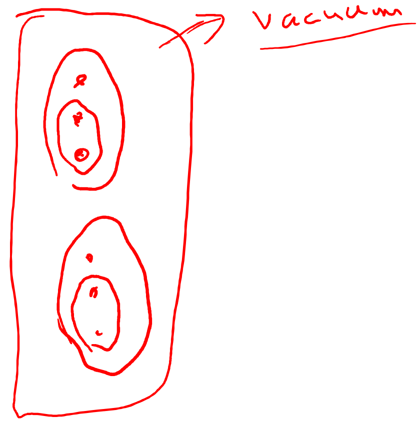
Two qubits from Fibonacci anyons?

$$\tau\times\tau\times\tau=(1+\tau)\times\tau=\tau\times\tau+\tau=1+2\tau$$

$$\tau\times\tau\times\tau\times\tau=2\cdot1+3\tau$$

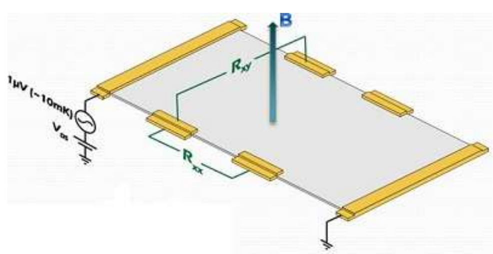
$$\tau\times\tau\times\tau\times\tau\times\tau=3\cdot1+5\tau$$

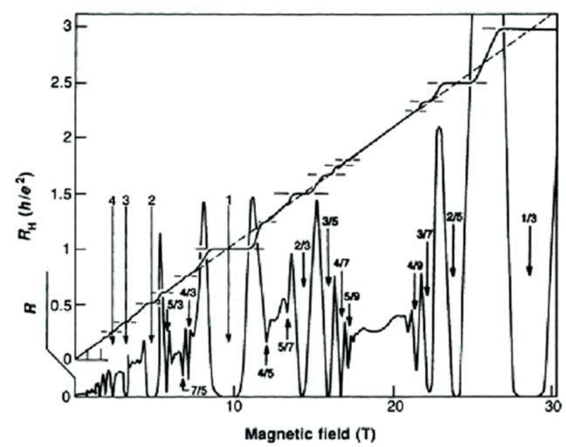
$$\tau\times\tau\times\tau\times\tau\times\tau\times\tau=\boxed{5}\cdot1+8\tau$$



- Naively, use two group of three anyons ➔ 6 anyons!
- 6 anyons fuse to vacuum: 5 different ways (slightly more than two-qubit dimension)
- Where do we get Fibonacci anyons? Fractional Quantum Hall state with ν=12/5

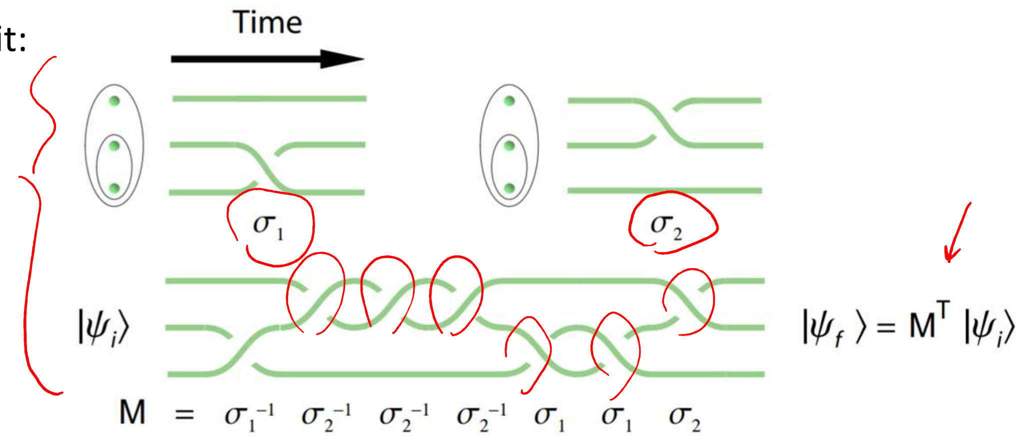
$\tau\times\tau=1+\tau$

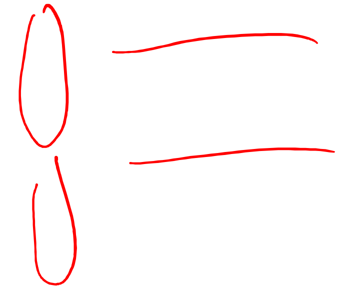




Braiding for gates

- One-qubit:



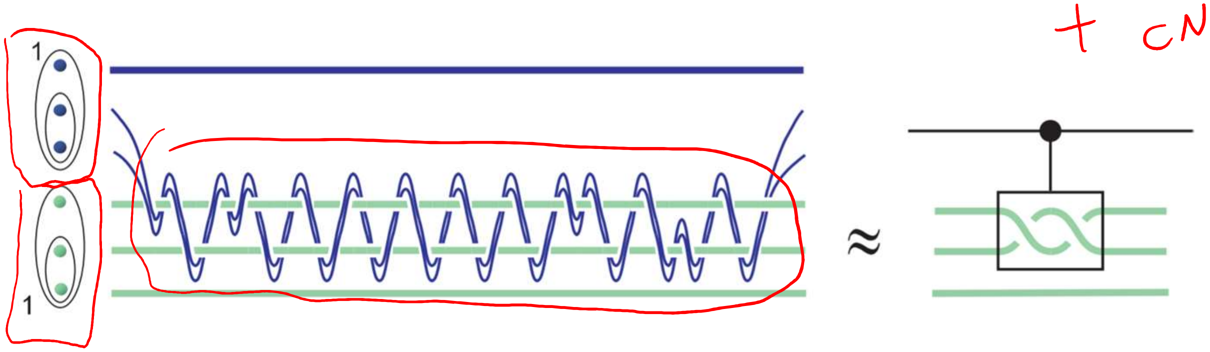


SU(2) dense

$\Rightarrow$ universal

+ CNOT

- Two-qubit:



approximate a CNOT gate

Ref: Bonesteel, Hormozi, Zikos & Simon, PRL95, 140503 (2005); Hormozi, Zikos, Bonesteel & Simon, PRB 75, 165310 (2007)

Example: Ising anyons

fermion

- Anyons: 1, ψ, σ

- Fusion: $1\times 1=1,\ 1\times\psi=\psi,\ 1\times\sigma=\sigma$

  $\psi\times\psi=1,\ \psi\times\sigma=\sigma,\ \sigma\times\sigma=1+\psi$

- Physical picture: 1 is condensate of Cooper pairs, ψ Bogoliubov fermion, σ Majorana zero mode bound to a vortex

e^-

e^-

$\gamma_1 = c_1 + i c_2$

$\gamma_2 = c_1 - i c_2$

- Qubits?

$$\sigma\times\sigma\times\sigma=(1+\psi)\times\sigma=2\cdot\sigma$$

$$\sigma\times\sigma\times\sigma\times\sigma=2\cdot 1+2\cdot\psi$$

$$\sigma\times\sigma\times\sigma\times\sigma\times\sigma=4\cdot\sigma$$

➔ one qubit

fixed

$$\{|(\sigma\sigma)\sigma;1\sigma;\sigma\rangle,|(\sigma\sigma)\sigma;\psi\sigma;\sigma\rangle\}$$

fixed

➔ 2n σ can encode n-1 qubits (assume fused to vacuum)