PHY682 Special Topics in Solid-State Physics: Quantum Information Science

Lecture time: 2:40-4:00PM Monday & Wednesday

Today 10/14:

1. More on topological quantum computation:
   Magic state distillation and surface code quantum computation

Week 8: More topological please: Topological quantum computation continued, magic state distillation, and surface code

Arbitrary phase gate from `phase' state

- ❑ With the state available $|A_\theta\rangle = (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}$

  we can implement a general phase gate $U_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$

want to apply U_θ ?

assume Clifford gates are of high fidelity

- ➢ Suppose we have $|\psi\rangle \otimes |A_\theta\rangle = (a|0\rangle + b|1\rangle) \otimes (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}$

  we want to measure stabilizer $Z \otimes Z$ (+1 eigenspace spanned by 00, 11; -1 eigenspace by 01,10)

$$|\psi\rangle \otimes |A_\theta\rangle = \frac{1}{\sqrt{2}}[\underbrace{(a|00\rangle + e^{i\theta}b|11\rangle)}_{+1 \text{ eigenspace}} + e^{i\theta}\underbrace{(a|01\rangle + e^{-i\theta}b|10\rangle)}_{-1 \text{ eigenspace}}]$$

- ➢ Each with probability 1/2

+1 (i) $a|00\rangle + e^{i\theta}b|11\rangle$ CNOT $\Rightarrow$ $(a|0\rangle + e^{i\theta}b|1\rangle) \otimes |0\rangle = U_\theta|\psi\rangle \otimes |0\rangle$ → apply $U_{-\theta}$ gate

-1 (ii) $a|01\rangle + e^{-i\theta}b|10\rangle$ $(a|0\rangle + e^{-i\theta}b|1\rangle) \otimes |1\rangle = U_{-\theta}|\psi\rangle \otimes |1\rangle$

random (+1), (-1) +1 -1

[Bravyi & Kitaev, PRA71, 022316 (2005)]

Magic states |H> and |T>

Clifford gate g_c

g_c Pauli g_c^{-1} = Pauli

$H|H\rangle = |H\rangle$

- If we have $|H\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle$ $\quad U_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ $\quad S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$

Clifford

$\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}$ $\quad \cos\frac{\pi}{8}-i\sin\frac{\pi}{8}$

then $HS|H\rangle = (e^{i\pi/8}|0\rangle + e^{-i\pi/8}|1\rangle)/\sqrt{2} = e^{i\pi/8}|A_{-\pi/4}\rangle$ ⇒ non-Clifford

➔ Can implement non-Clifford gate $T^\dagger = U_{-\pi/4}$

$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$

use T to denote the "T" gate in the paper by Bravyi & Kitaev

- If we have $|T\rangle = \cos\beta|0\rangle + e^{i\pi/4}\sin\beta|1\rangle,\ \cos(2\beta) = 1/\sqrt{3}$

can we also generate a useful phase state? ⇒ from that I can get a phase gate

non-Clifford

- Start with $|\Psi_0\rangle = |T\rangle \otimes |T\rangle$ and measure stabilizer $Z \otimes Z$

$(\cos\beta|0\rangle + e^{i\pi/4}\sin\beta|1\rangle)(\cos\beta|0\rangle + e^{i\pi/4}\sin\beta|1\rangle)$

Only keep +1 outcome (probability 2/3)

When it succeeds ⇒ $\cos^2\beta|00\rangle + e^{i\pi/2}\sin^2\beta|11\rangle \sim |\Psi_1\rangle = \cos\left(\frac{\pi}{12}\right)|00\rangle + i\sin\left(\frac{\pi}{12}\right)|11\rangle$

$U_{-\pi/6}$

- Apply CNOT and then Hadamard: first qubit becomes $(e^{i\pi/12}|0\rangle + e^{-i\pi/12}|1\rangle)/\sqrt{2} = e^{i\pi/12}|A_{-\pi/6}\rangle$

[Bravyi & Kitaev, PRA71, 022316 (2005)]

Magic state distillation

applied to Ising anyons

- How do we generate high-fidelity H and T states?

$$|H\rangle = \cos\left(\frac{\pi}{8}\right)|0\rangle + \sin\left(\frac{\pi}{8}\right)|1\rangle \qquad H|H\rangle = |H\rangle$$

$$|T\rangle = \cos\beta|0\rangle + e^{i\pi/4}\sin\beta|1\rangle,\ \cos(2\beta) = 1/\sqrt{3}$$

$$\tilde{T} = e^{i\pi/4}SH \qquad \tilde{T}|T\rangle = e^{i\pi/3}|T\rangle \qquad |T^{\perp}\rangle \equiv \sigma_y H|T\rangle \quad \tilde{T}|T^{\perp}\rangle = e^{-i\pi/3}|T^{\perp}\rangle$$

- Bravyi & Kitaev exploited the 5-qubit error correcting code for T state and CSS codes for H state.

(complicated procedure)

➔ An iteratively procedure that takes many noisy copies to distill a single high-quality T and H states, provided, respectively

infidelity ≡ 1 − fidelity

$$\epsilon_T \equiv 1 - \langle T|\rho_{\text{noisy}}|T\rangle < \epsilon_0 \approx 0.173$$

$$\epsilon_H \equiv 1 - \langle H|\rho_{\text{noisy}}|H\rangle < \epsilon_0 \approx 0.141$$

close

Summary: use noisy gates to create states close to $|T\rangle$ & $|S\rangle$

⇒ can distill clean $|T\rangle$ & $|S\rangle$

⇓

can implement high-fidelity non-Clifford gate

+ "protected" Clifford gate

⇒ protected QC

[Bravyi & Kitaev, PRA71, 022316 (2005)]

T and H type states on Bloch sphere

useless inside

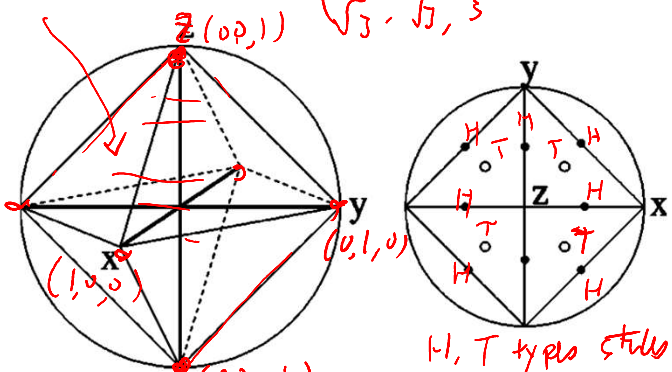
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{3}\right)$

they are mixture of

$|0/1\rangle$ $|+/-\rangle$

$|i/-i\rangle$

states



H, T type states are useful

FIG. 1. Left: the Bloch sphere and the octahedron O. Right: the octahedron O projected on the $x-y$ plane. The magic states correspond to the intersections of the symmetry axes of O with the Bloch sphere. The empty and filled circles represent T-type and H-type magic states, respectively.

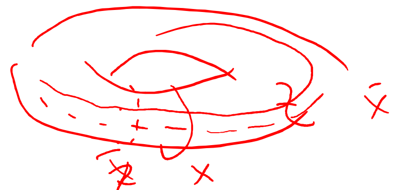
$$|T\rangle\langle T| = \frac{1}{2}\left[I + \frac{1}{\sqrt{3}}(\sigma^x + \sigma^y + \sigma^z)\right]$$

$$|H\rangle\langle H| = \frac{1}{2}\left[I + \frac{1}{\sqrt{2}}(\sigma^x + \sigma^z)\right]$$

$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

Clifford gates simply ~~move~~ rotate $|T\rangle\langle T|$ & $|H\rangle\langle H|$

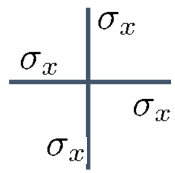
Surface code



Bravyi & Kitaev, quant-ph/9811052
Raussendorf, Harrington & Goyal, NJP 9, 199 (2007)
Folwer et al., Phys. Rev. A 86, 032324 (2012)
Fujii, arXiv:1504.01444

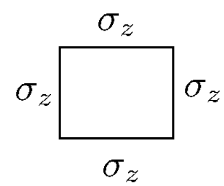
- It is essentially the toric code on a planar geometry
  - Star operators:

$$A_s = \prod_{j\in s} \sigma_x^{[j]}$$



  - Plaquette operators:

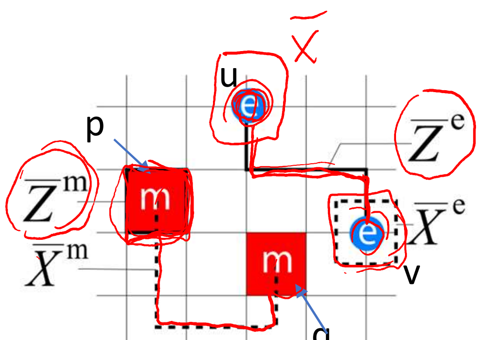
$$B_p = \prod_{j\in\partial(p)} \sigma_z^{[j]}$$



B_p
$\|$
$Z(\partial P)$

$\bar{D}$
$X(\partial \bar{D})$
A_s

- But a logical qubit is encoded in a pair of defects (or holes)



$Z(\partial P) \cdot Z(\partial q) = 1$

- Two kinds:
  (1) a pair of plaquette defects <p,q> → B_p and B_q no constraint but $B_p B_q$ is in stabilizer
  (2) a pair of star (vertex) defects <u,v> → A_u and A_v no constraint but $A_u A_v$ is in stabilizer

remove the two stabilizer
$-2+1 = -1$
⇒ one qubit

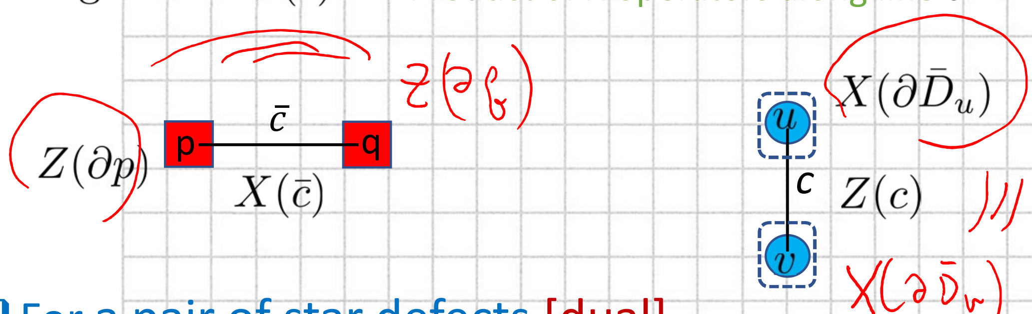
⇒ one qubit
remove constraints

Logical operators

- For a pair of plaquette defects [primal]

Logical Z: $Z(\partial p) \equiv Z(\partial q)$ → Product of Z operators around plaquette *p* or *q*

Logical X: $X(\bar{c})$ → Product of X operators along line $\bar{c}$



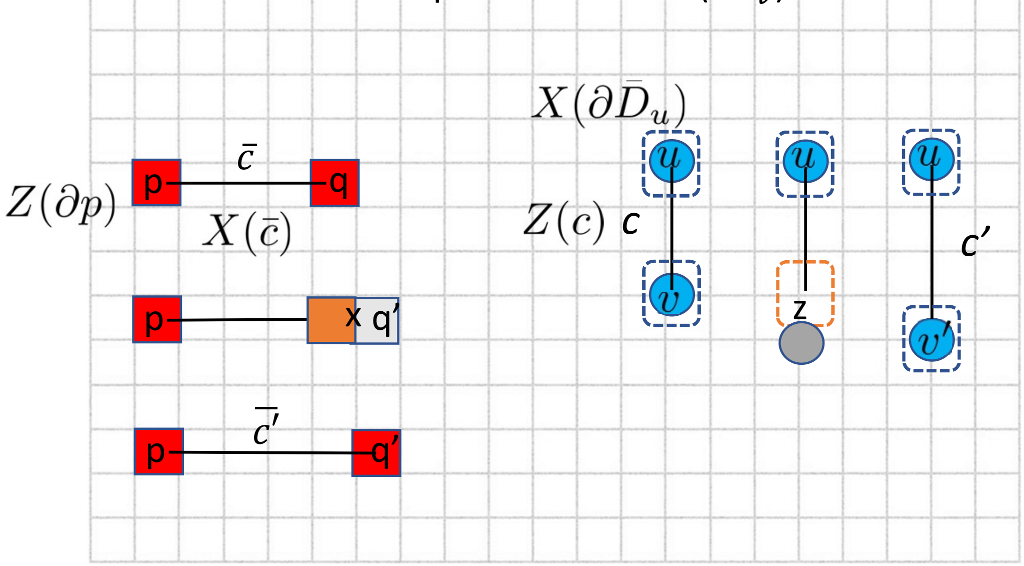
- For a pair of star defects [dual]

Logical Z: $Z(c)$ → Product of Z operators along line *c*

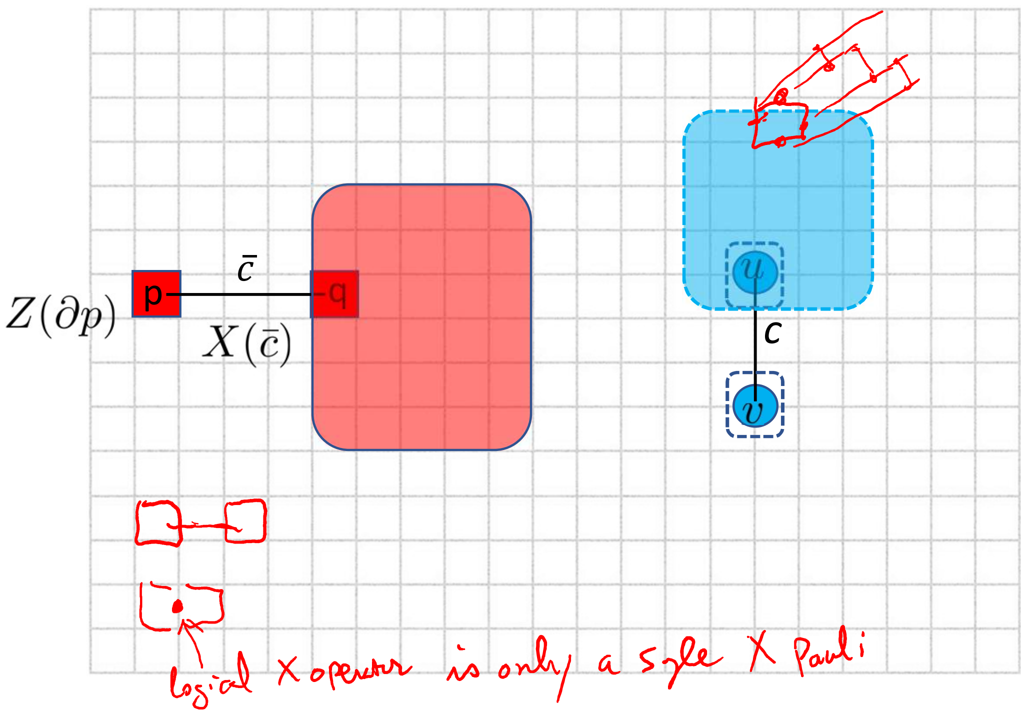
Logical X: $X(\partial \bar{D}_u) \equiv X(\partial \bar{D}_v)$ → Product of X operators around vertex *u* or vertex *v*

Logical operators: deformable

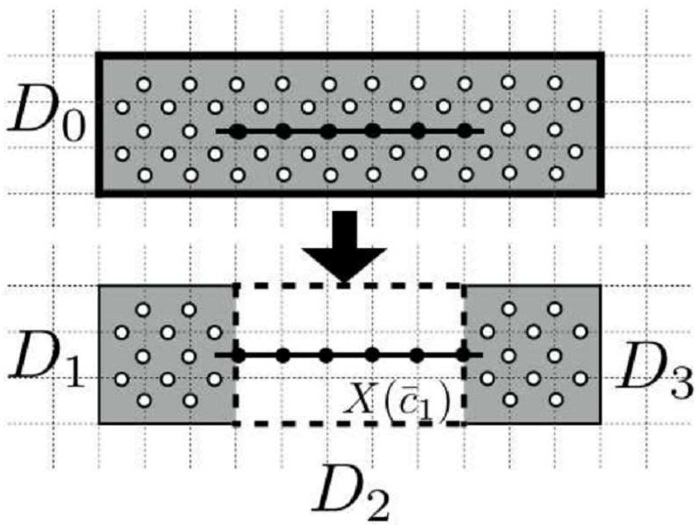
- Plaquette type: in toric code, we apply a string of x operators; here we measure X to expand and then Z(∂q)
- Star type: in toric code, we apply a string of z operators; here we measure Z to expand and then X(∂$\overline{D}_v$)



Logical operators: expandable/shrinkable



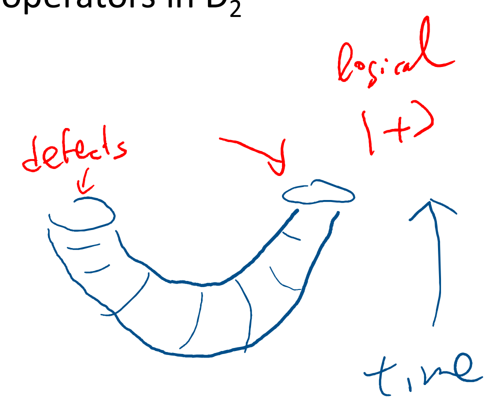
Logical X-basis state preparation



prepare logical $|+\rangle$

break stabilizer operators in that region

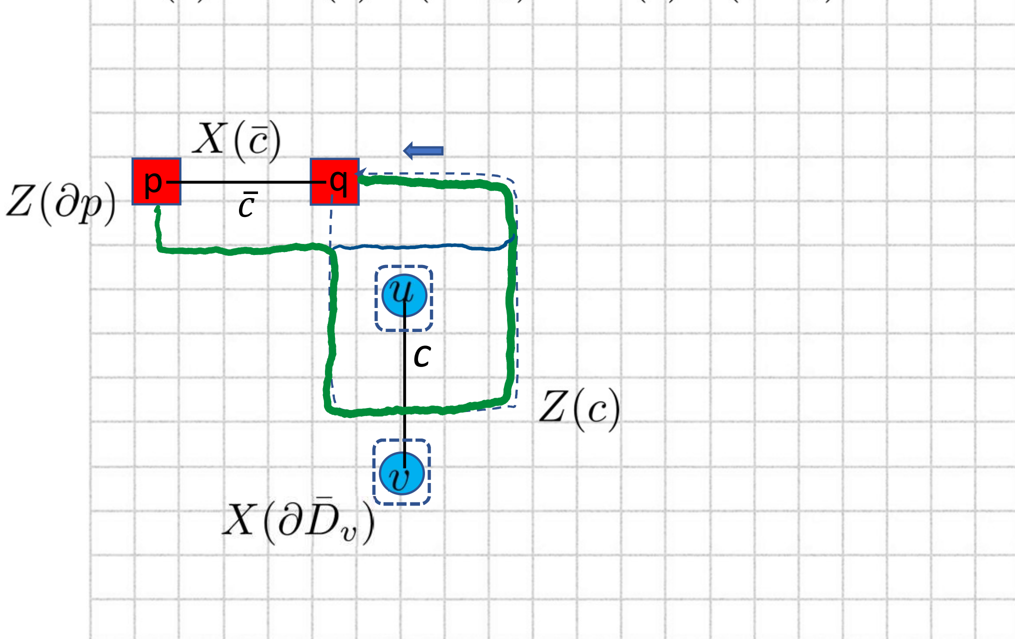
1. Create a defect region D_0 by measuring all qubits (circles) in X basis $|+\rangle$ or $|-\rangle \Rightarrow |+\rangle$
2. Annihilate defect region D_2 by measuring star operators in D_2

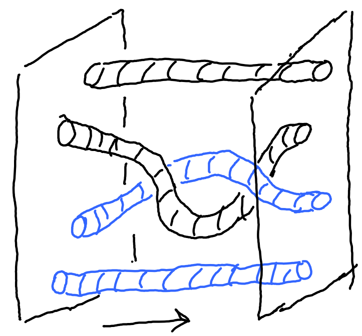


Protected CNOT gate by braiding: m around e

$$Z(\partial p) \to Z(\partial p) \qquad X(\partial \bar{D}_u) \to X(\partial \bar{D}_u)$$

$$X(\bar{c}) \to X(\bar{c})X(\partial \bar{D}_u) \equiv X(\bar{c})X(\partial \bar{D}_v)$$





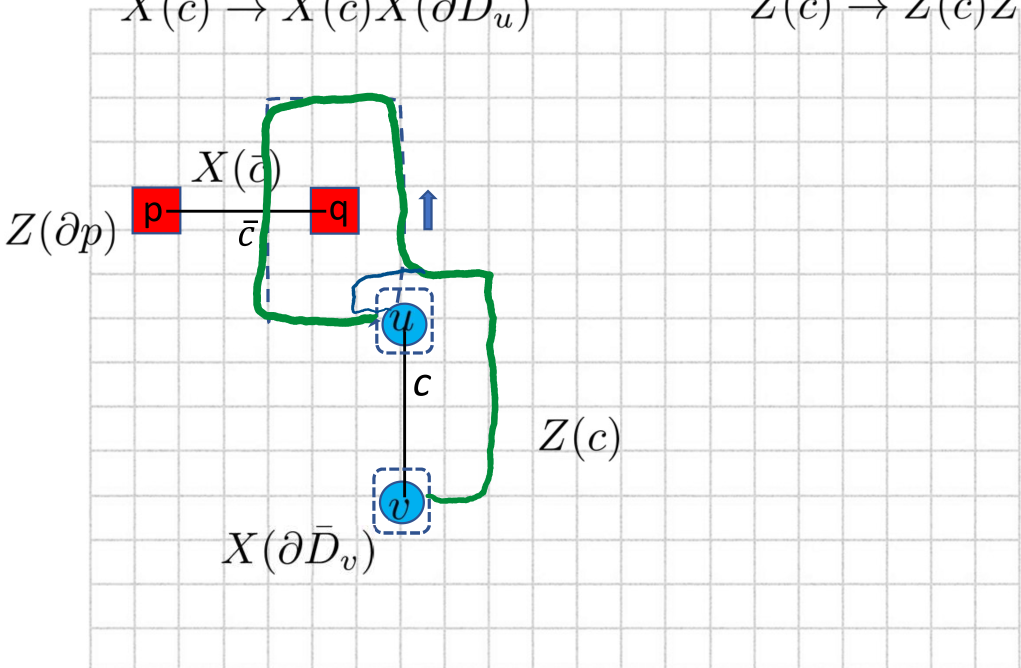
Equivalent view: braid e around m

$$Z(\partial p) \to Z(\partial p)$$

$$X(\bar{c}) \to X(\bar{c})X(\partial \bar{D}_u)$$

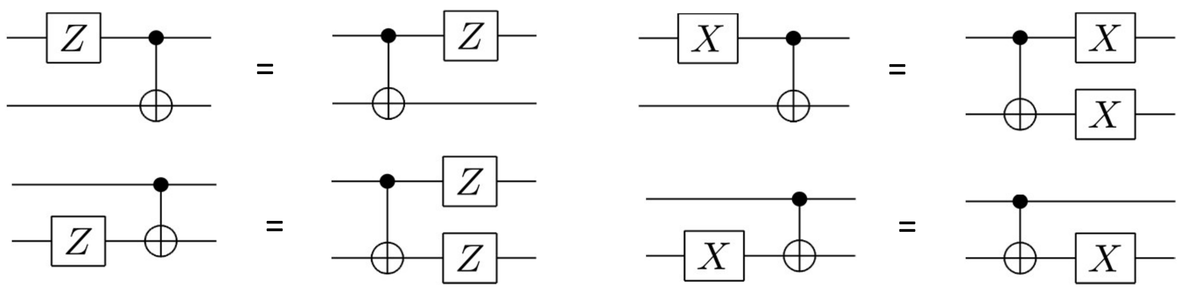
$$X(\partial \bar{D}_u) \to X(\partial \bar{D}_u)$$

$$Z(c) \to Z(c)Z(\partial q) = Z(c)Z(\partial p)$$

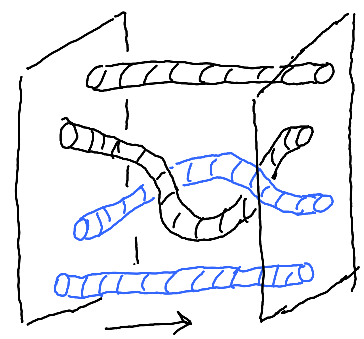


Verifying CNOT

- Recall:



- We have:

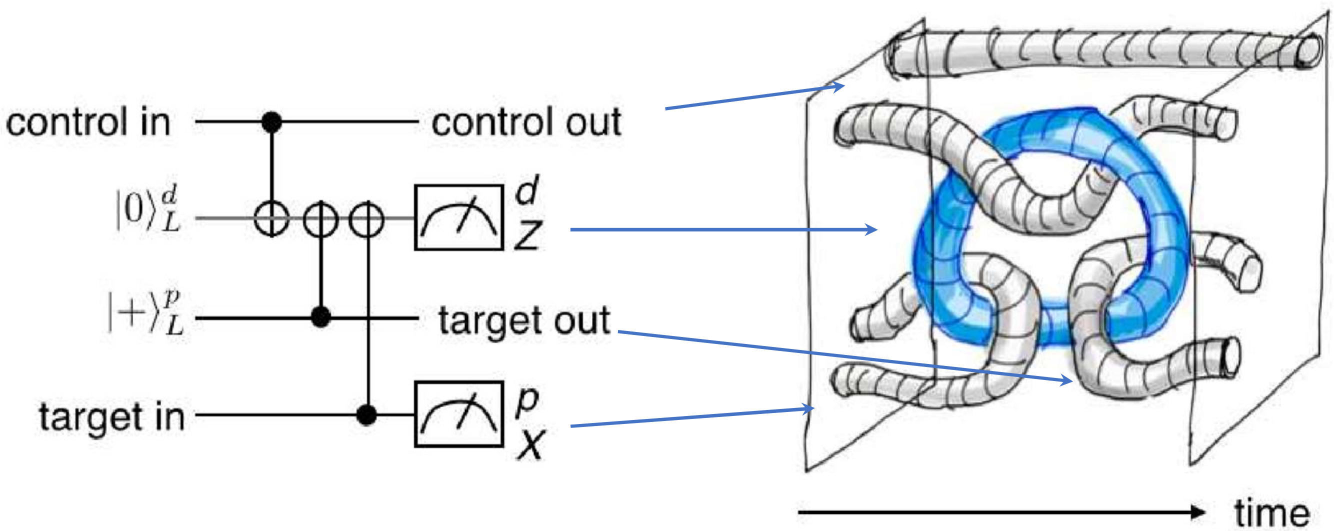


$$Z(\partial p) \to Z(\partial p)$$
$$X(\bar{c}) \to X(\bar{c})X(\partial \bar{D}_u)$$

$$X(\partial \bar{D}_u) \to X(\partial \bar{D}_u)$$
$$Z(c) \to Z(c)Z(\partial q) = Z(c)Z(\partial p)$$

➔ Primal qubit is the *control* and dual qubit is the *target*

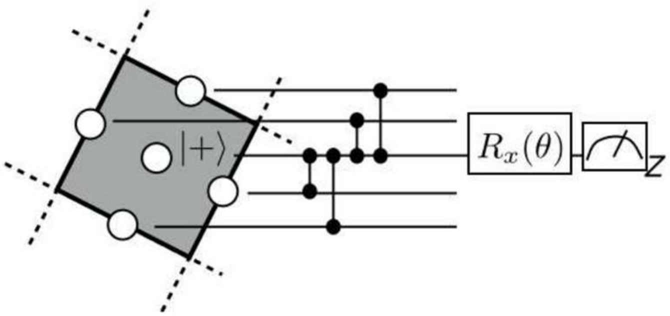
CNOT gate between two "primal" pairs



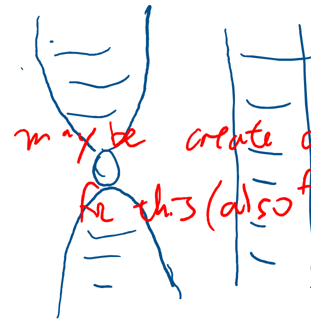
Fujii arXiv 15 . . .

Logical Z and X rotations (unprotected)

- **Logical Z rotation**: by adding an ancilla in + state, perform CZ gates (below), rotate the ancilla before Z measurement

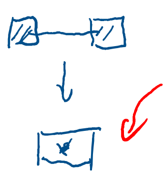


$$e^{-i(-1)^m(\theta/2)Z(f_l)}$$

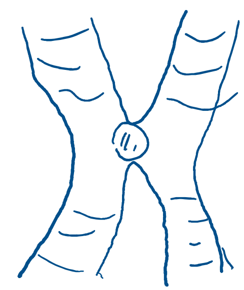


maybe create a homework for this (also for CNOT)

- **Logical X rotation**: bring two defects next to each other and rotate directly on the single qubit adjacent to both defects



simply rotate this single qubit by $R_x(\theta)$

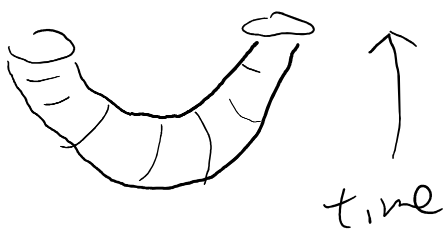


Even though they are not protected, they can be used to inject noisy magic states ➔ then distill to a higher-fidelity magic state

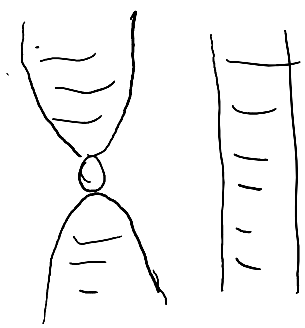
Logical arbitrary phase state

$$\frac{1}{\sqrt{2}}(e^{-i\theta/2}|0\rangle + e^{i\theta/2}|1\rangle) = e^{-i\theta Z/2}|+\rangle$$

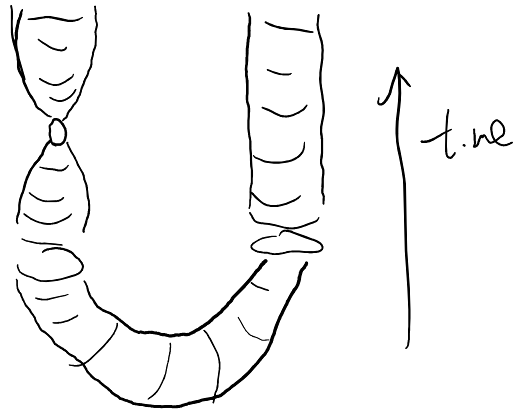
- First prepare logical + state



- Then perform logical Z rotation



Combined picture

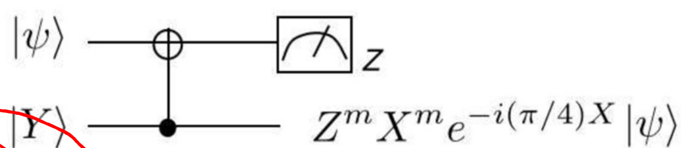


S gate and T gate

- So far, we have a protected CNOT, but no protected single-qubit gates

  - Need the eigenstate of the Pauli-Y operator, which is used to implement the S gate via gate teleportation

verify the circuit



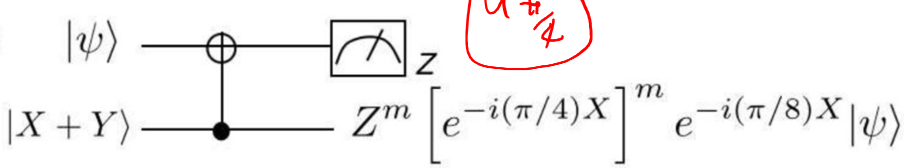
X-basis

$|H\rangle$ state

Z basis $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

  - Need the eigenstate of the (X + Y)/ √ 2 operator, which is used to implement the π/8 gate, a non-Clifford gate necessary for universal quantum computation.

$U_{\pi/4}$

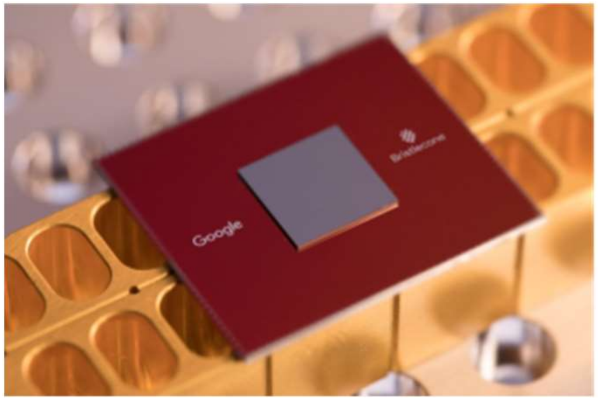


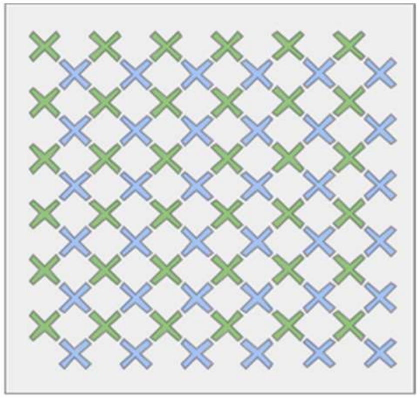
Surface code quantum computation

- Naturally protected CNOTs, but other gates from magic state distillation can be made of low error rates ➔ Universal
- Google is heavily invested on surface code and built devices suitable for its implementation

(Can also apply magic state distillation to Ising anyons to make universal gates)

Microsoft





[from Google]