PHY682 Special Topics in Solid-State Physics: Quantum Information Science

Lecture time: 2:40-4:00PM Monday & Wednesday

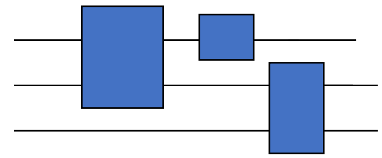
Today 10/26:

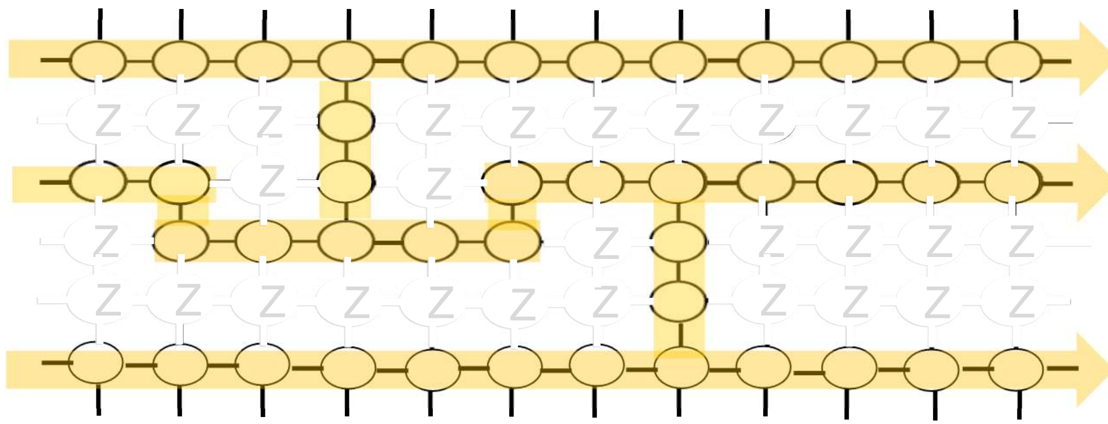
1. Brief review of MBQC; discuss Blind QC
2. Week 10: 'Quantum entangles'

QC by Local Measurement---an overview picture

[Raussendorf & Brigel '01]

- There is a highly entangled state on a 2D array of qubits. First carve out entanglement structure on cluster state by local Pauli Z measurement





One-way q. computer

Entanglement is the resource

- Then:

  (1) Measurement along each wire simulates one-qubit evolution (gates)

  (2) Measurement near & on each bridge simulates two-qubit gate (CNOT)

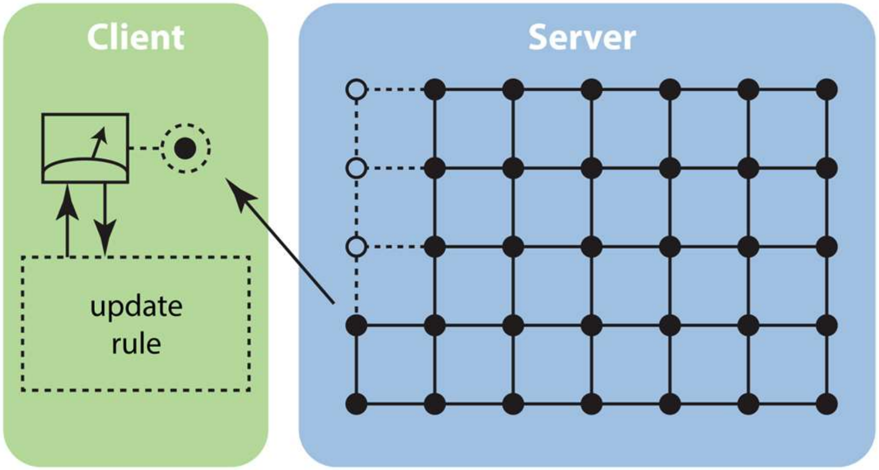
→ 2D or higher dimensions are needed for universal QC

One application of measurement-based QC

Suppose we have a cloud quantum computer server.

Q: Is it possible to run on this cloud quantum computer without the server figuring out what the client is actually running?

A: Blind quantum computation



Fitzsimons, npj Quantum Information (2017) 23

Universal blind quantum computation*

[Broadbent, Fitzsimons & Kashefi '09]

❑ Using the following cluster state (called brickwork state)

- Alice prepares

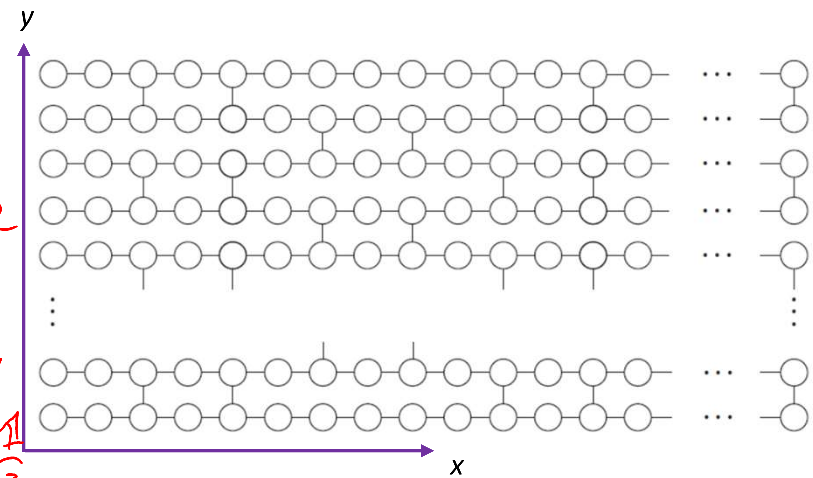
$$|\Psi\rangle = \bigotimes_{x,y} \left(|0\rangle_{x,y} + e^{i\theta_{x,y}}|1\rangle_{x,y}\right)$$

with random → On average

$$\theta_{x,y} = 0, \pi/4, \ldots 7\pi/4$$

qubits are in a completely mixed state $\frac{\mathbb{1}}{2}$

- Bob entangles all qubits according to the brickwork graph via CZ gates



- Alice tells Bob what measurement basis for Bob to perform and he returns the outcome (compute like one-way computer)

➔ Alice can achieve her quantum computation without Bob knowing what she computed!!

1 Alice computes $\phi'_{x,y}$ where $s^X_{0,y} = s^Z_{0,y} = 0$. $\quad \phi'_{x,y} = (-1)^{s^X_{x,y}}\phi_{x,y} + s^Z_{x,y}\pi$

2 Alice chooses $r_{x,y} \in_R \{0,1\}$ and computes $\delta_{x,y} = \phi'_{x,y} + \theta_{x,y} + \pi r_{x,y}$.

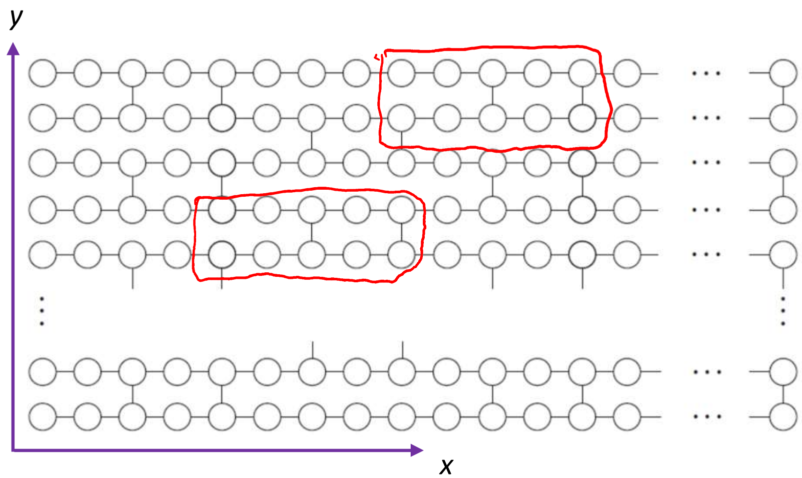
3 Alice transmits $\delta_{x,y}$ to Bob. Bob measures in the basis $\{|+_{\delta_{x,y}}\rangle, |-_{\delta_{x,y}}\rangle\}$.

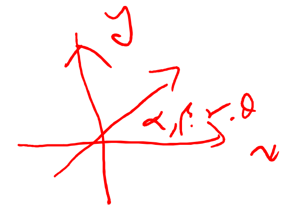
4 Bob transmits the result $s_{x,y} \in \{0,1\}$ to Alice.

5 If $r_{x,y} = 1$ above, Alice flips $s_{x,y}$; otherwise she does nothing.

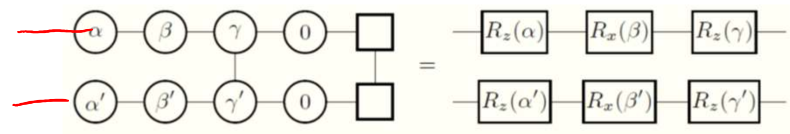
➔ Realized in an experiment Barz et al. 2012

Brickwork cluster state is universal

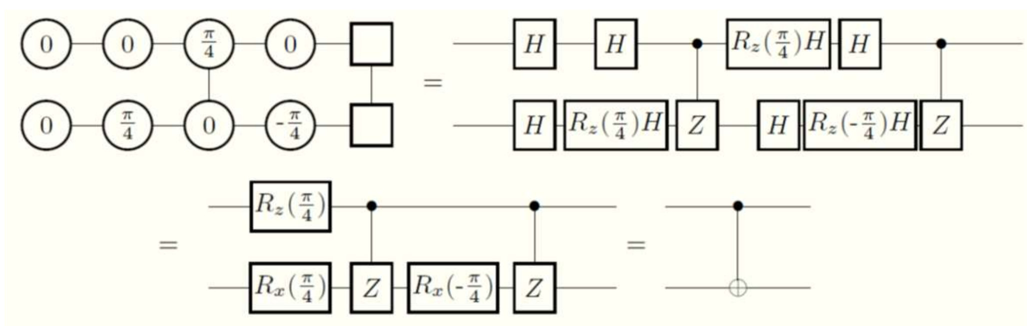




➢ Single-qubit gates:

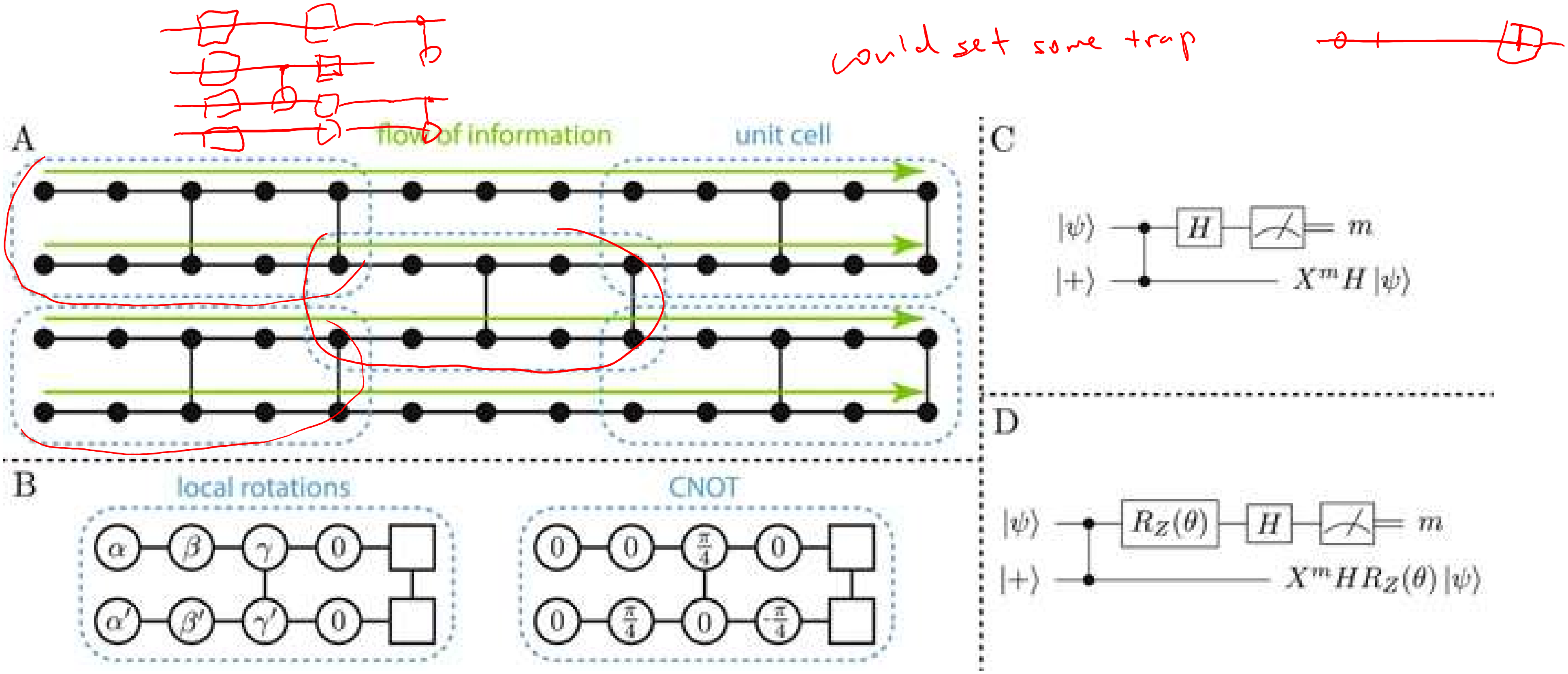


➢ CNOT gate:



Composing gates for a quantum circuit

could set some trap



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Week 10: Quantum entangles: Entanglement of quantum states, entanglement of formation and distillation, entanglement entropy, Schmidt decomposition, majorization, quantum Shannon theory

Entangled states

- We have seen the four Bell states (which allows several quantum information processing tasks, e.g. dense coding, teleportation, etc):

$$|\Phi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(\underset{AB}{|00\rangle} \pm \underset{AB}{|11\rangle}), \ |\Psi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$$

They cannot be factorized: $|\psi\rangle \neq |\phi^A\rangle \otimes |\phi^B\rangle$

- In contrast, examples such as $|00\rangle$, $|0+\rangle$ are factorizable in product form (we will call "separable")

- Question: which of the following is entangled? Separable? (ignore normalization)

$$|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle \;\; = (|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle + |\downarrow\rangle)$$ [separable]

$$|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle \;\; = |\uparrow\rangle(|\uparrow\rangle + |\downarrow\rangle) + |\downarrow\rangle(|\uparrow\rangle - |\downarrow\rangle)$$ [entangled]

Two-qubit pure states

- How am I sure that this state is really inseparable?

$\frac{1}{2}(-1-1) = -\frac{1}{2}$

$$|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle \quad = |\uparrow\rangle(|\uparrow\rangle + |\downarrow\rangle) + |\downarrow\rangle(|\uparrow\rangle - |\downarrow\rangle)$$

Let's consider a general two-qubit pure state:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

- If it is separable, we can write it as

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (\delta|0\rangle + \gamma|1\rangle) = \alpha\delta|00\rangle + \alpha\gamma|01\rangle + \beta\delta|10\rangle + \beta\gamma|11\rangle$$

Thus we must have

$$a = e^{i\theta}\alpha\delta,\ b = e^{i\theta}\alpha\gamma,\ c = e^{i\theta}\beta\delta,\ d = e^{i\theta}\beta\gamma$$

➔ a necessary consequence (for being separable):

$$ad - bc = e^{2i\theta}(\alpha\delta\beta\gamma - \alpha\gamma\beta\delta) = 0$$

$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = |+\rangle|+\rangle$

✓ Could define an entanglement quantity (concurrence) $\quad C = 2|ad - bc|$

e.g. $\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)$

$\Downarrow$

$C = 2\cdot\frac{1}{2} = 1$

Concurrence

$$C = 2|ad - bc| \qquad |\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

(spin degrees of freedom)

σ_y acts on

K : acts on everywhere

$$-\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Physical meaning: time reversal operation for a spin-1/2 (qubit)

$$-i\sigma_y K \quad (K: \text{ complex conjugation})$$

➔ Time reversal for ψ:

$$|\tilde{\psi}\rangle \equiv (-\sigma_y \otimes \sigma_y)K|\psi\rangle = d^*|00\rangle - c^*|01\rangle - b^*|10\rangle + a^*|11\rangle$$

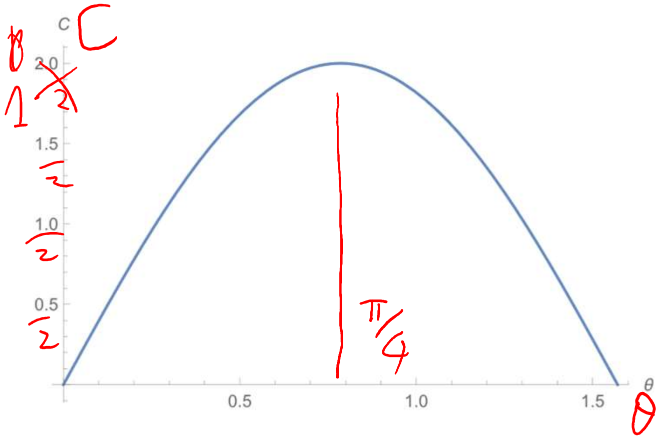
$$C = |\langle\tilde{\psi}|\psi\rangle| = 2|ad - bc|$$

❑ Example: concurrence for the following state

$$|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$$

$$C = 2|\sin\theta\cos\theta| = |\sin(2\theta)|$$

➔ C=1 for Bell states (at θ = π/4)



Schmidt decomposition

For any bipartite state (bipartite could arise by grouping subsystems into two parts)

$$|\psi\rangle = \sum_{i=1}^{N_A}\sum_{j=1}^{N_B} \psi_{ij} |i, j\rangle$$

A B

The coefficients is an $N_A x N_B$ matrix and can be "singular value" decomposed

≥0 singular values

$$\psi_{ij} = \sum_{k=1}^{\max(N_A,N_B)} U_{ik}\sigma_k V^*_{jk}$$

unitary unitary

$\max(N_A, N_B)$

$|i\rangle_A \xrightarrow{U} |\tilde{k}\rangle_A$

$|j\rangle_B \xrightarrow{V} |\tilde{k}\rangle_B$

$$|\psi\rangle = \sum_k \sigma_k \Big(\underbrace{\sum_i U_{ik}|i\rangle}\Big) \otimes \Big(\underbrace{\sum_j V^*_{jk}|j\rangle}\Big) = \boxed{\sum_{k=1} \sigma_k |\tilde{k}\rangle_A \otimes |\tilde{k}\rangle_B}$$

(Schmidt decomposition)

new basis

uses V for transform

- U and V define local unitary transformation so $|k\rangle$'s form an orthonormal basis
- The Schmidt coefficients σ_k (from singular values) quantify the entanglement of the system

not entangled

If only one $(\sigma_k = 1)$

$|\tilde{k}\rangle_A \otimes |\tilde{k}\rangle_B$

$$\text{they satisfy } \boxed{\sum_k \sigma_k^2 = 1}$$

Example of Schmidt decomposition

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) \quad C = 2/3$$

$2\left|\frac{-1}{3}\right|$

$\Rightarrow \{ \sigma_0 |\tilde{0}\rangle|\tilde{0}\rangle + \sigma_1 |\tilde{1}\rangle|\tilde{1}\rangle \}$

$\Downarrow$

$C = 2\sigma_0\sigma_1$

$$\psi = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{pmatrix} = U\, w\, V^{\dagger}$$

$\{U, w, V\}$ = SingularValueDecomposition[ψ]

diagonal matrix

$$U = \begin{pmatrix} 0.8507 & 0.5257 \\ 0.5257 & -0.8507 \end{pmatrix}$$

$\Rightarrow$ local transformation

$$w = \left\{\left\{\sqrt{\frac{1}{6}\left(3+\sqrt{5}\right)}, 0\right\}, \left\{0, \sqrt{\frac{1}{6}\left(3-\sqrt{5}\right)}\right\}\right\}$$

σ_1, σ_2

$$V = \begin{pmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{pmatrix}$$

→ Entangled if there are more than one nonzero Schmidt coefficients

$$C = 2\sigma_1\sigma_2 = 2/3$$

- Entanglement entropy from Schmidt coefficients

$\sum_k \sigma_k^2 = 1 \longrightarrow$ define a probability distribution

$\Downarrow$

define an entropy

Von Neumann entropy

$$S_V(\rho_A) = -\sum_k \sigma_k^2 \log_2(\sigma_k^2) \approx 0.55 \text{ (for above example)}$$

$$\boxed{\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)}$$

$A \,|\, \cancel{B}$ $\Rightarrow \rho_A \Rightarrow S_V(\rho_A) = -\mathrm{tr}\,\rho\log\rho$

Comments: partial trace and Sv

$$|\psi\rangle = \sum_k \sigma_k |\tilde{k}\rangle_A \otimes |\tilde{k}\rangle_B$$

$tr_B (A \otimes B)$

$$\begin{array}{cc} tr & tr \\ \begin{pmatrix} a_{11} B & a_{12} B \\ a_{21} B & a_{22} B \end{pmatrix} \\ tr & tr \end{array}$$

- Partial trace over second party:

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \sum_k \langle\tilde{k}|_B \cdot |\psi\rangle\langle\psi| \cdot |\tilde{k}\rangle_B \quad = \sum_k \sigma_k^2 |\tilde{k}\rangle_A \langle\tilde{k}|_A =$$

$$\begin{pmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_N^2 \end{pmatrix}$$

- Von Neumann entropy

$$S_V(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A) = -\sum_k \sigma_k^2 \log(\sigma_k^2)$$

$$\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

➔ Also known as the entanglement entropy of ψ

only one $\sigma_1 = 1$

➢ Unentangled pure state remains pure after partial tracing

Unentangled (separable) states can be prepared locally

- A pure state is called separable if it can be written as a product state

$$|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$$

E.g. $|HV\rangle \equiv \underbrace{|H\rangle}_{A} \otimes \underbrace{|V\rangle}_{B}$

$$|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle = \underbrace{(|\uparrow\rangle + |\downarrow\rangle)}_{|+\rangle}\underbrace{(|\uparrow\rangle + |\downarrow\rangle)}_{|+\rangle}$$

- A mixed state is called separable if it can be written as a mixture of separable pure states

$$\rho = \sum_i p_i |\phi_i^A\rangle\langle\phi_i^A| \otimes |\phi_i^B\rangle\langle\phi_i^B|$$

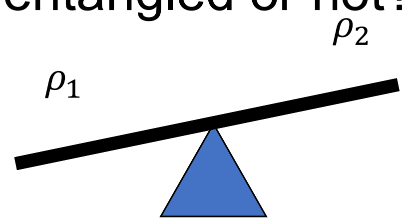
E.g. $\frac{1}{2}|HH\rangle\langle HH| + \frac{1}{2}|VV\rangle\langle VV|$ Can be created locally & communication

A: dice ⇒ $\frac{1}{2}$ prepares H & B prepares H [communication] ✓

- Separable states can be created locally by local operations and classical communication

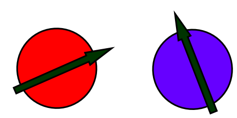
Entangled or not

1. How to determine a state is entangled or not?
2. Quantify how entangled?



➔ Generally, these are difficult problems. For special cases we can solve them.

❑ Example-- two interacting qubits :



(Heisenberg interaction) $H = J\vec{\sigma}^1 \cdot \vec{\sigma}^2$, with $J > 0$ antiferromagnetic coupling

➢ Eigenstates: $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$, $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$

➢ Eigenvalues: $J, J, J, -3J$

$\dfrac{e^{-\beta H}}{Tr(e^{-\beta H})}$

thermal density matrix

➢ $\boxed{\rho = \frac{1}{Z}\sum_n e^{-\beta E_n}|n\rangle\langle n|} = \frac{1-r}{4} I_{4\times 4} + r|\Psi^-\rangle\langle\Psi^-|$

$r = (e^{3\beta J} - e^{-\beta J})/(e^{3\beta J} + 3e^{-\beta J})$

➔ When is this entangled?

A separability criterion

How do we actually check if a state is entangled or not?
One useful tool the is Peres-Horodecki criterion for separability.
(PPT) [Peres '96, Horodecki et al. '96]

density matrix → Hermitian eigenvalues ≥ 0

under a transpose ⇒ these remain

$$\rho = \sum_i p_i \, \rho_i^A \otimes \rho_i^B$$

$$\rho^{T_B} = \sum_i p_i \, \rho_i^A \otimes \boxed{\left(\rho_i^B\right)}^T$$

Still a valid density matrix
➔ eigenvalues non-negative

This is called positive partial transpose criterion (PPT)

not entangled ⟹ positive (PPT)

If partial transpose of a density matrix is NOT positive ➔ must be entangled!)

Positive Partial Transpose

- ρ separable ➔ PPT
- PPT turns out to be sufficient for qubit-qubit and qubit-qutrit systems; for these PPT ➔ separable (both necessary and sufficient)

Ex. Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \left(\frac{1}{\sqrt{2}} \;\; 0 \;\; 0 \;\; \frac{1}{\sqrt{2}}\right)$$

$$\rho_{\Phi^+} = |\Phi^+\rangle\langle\Phi^+| = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \end{array} \xrightarrow{T_B} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{eigenvalues:} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \boxed{-\frac{1}{2}}$$

➔ Entangled ➔ Negativity= 2 x |sum of negative eigenvalues| = 1 (above e.g.)

PPT criterion: another example

Ex. Werner state

$$\rho_{Werner}(\gamma) \equiv \gamma|\Phi^+\rangle\langle\Phi^+| + (1-\gamma)I/4$$

$$= \begin{pmatrix} \frac{1+\gamma}{4} & 0 & 0 & \frac{\gamma}{2} \\ 0 & \frac{1-\gamma}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\gamma}{4} & 0 \\ \frac{\gamma}{2} & 0 & 0 & \frac{1+\gamma}{4} \end{pmatrix} \xrightarrow{T_B} \begin{pmatrix} \frac{1+\gamma}{4} & 0 & 0 & 0 \\ 0 & \frac{1-\gamma}{4} & \frac{\gamma}{2} & 0 \\ 0 & \frac{\gamma}{2} & \frac{1-\gamma}{4} & 0 \\ 0 & 0 & 0 & \frac{1+\gamma}{4} \end{pmatrix}$$

eigenvalues $\frac{1+\gamma}{4}, \frac{1+\gamma}{4}, \frac{1+\gamma}{4}, \frac{1-3\gamma}{4}$ ➔ certainly entangled when $\boxed{\gamma > \frac{1}{3}}$

Quantifying entanglement

- We will be discussing a several other ways to quantify entanglement:

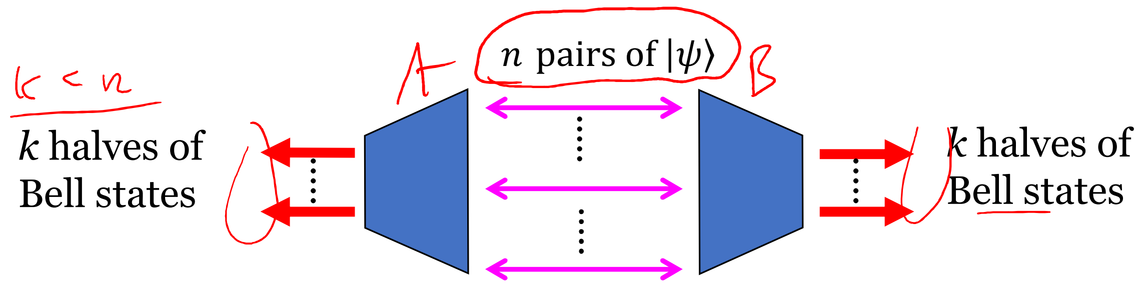
A. Entanglement of distillation [Bennett et al. ’96]

B. Entanglement of dilution or Entanglement cost
and Entanglement of formation [Bennett et al. ’97]

C. A geometric measure for multipartite entanglement
[Shimony ’95, Barnum & Linden ’01
Wei & Goldbart ’03]

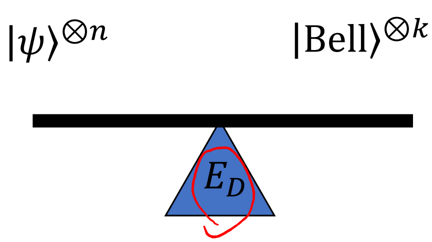
Entanglement of distillation/concentration

[Bennett et al. '96]



[Only local operations and classical communication are allowed]

- $E_D(\psi) \equiv \lim_{n\to\infty} (k/n)$ (idea also applies to mixed states)



For pure state ψ, $E_D(\psi) = S_V(Tr_B|\psi\rangle\langle\psi|)$

where S_V is the von Neumann entropy

$$S_V(\rho) \equiv -Tr\rho\log\rho \quad = -\sum_k \lambda_k \log(\lambda_k)$$

Comments: partial trace and Sv

$$|\psi\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$$

- ❑ Partial trace over second party (blue):

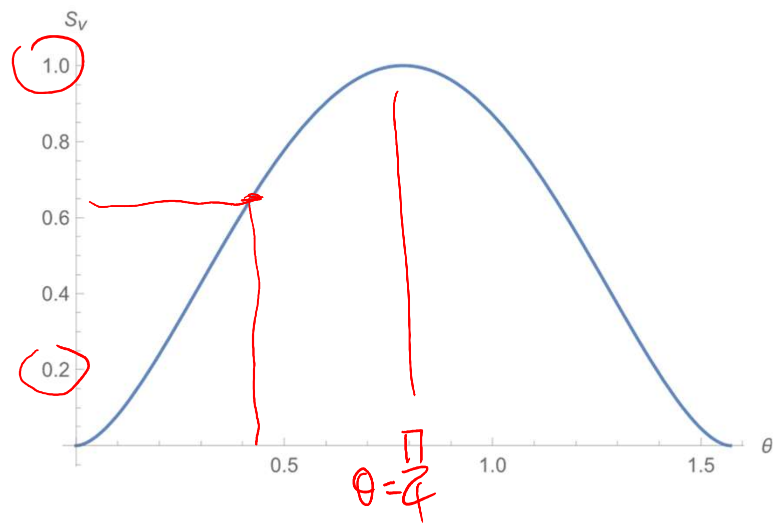
$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \sum_{k=0}^{1} \langle k|_B \cdot |\psi\rangle\langle\psi| \cdot |k\rangle_B \quad = \cos^2(\theta)|0\rangle\langle 0| + \sin^2(\theta)|1\rangle\langle 1|$$

- ❑ Von Neumann entropy

$$S_V(\rho_A) = -\sum_k \lambda_k \log(\lambda_k)$$

$$= -(\cos^2\theta)\log(\cos^2\theta) - (\sin^2\theta)\log(\sin^2\theta)$$

➔ Also known as the **entanglement entropy** of ψ



Entanglement of distillation: example

- Can Alice and Bob distill a Bell state from **two** pairs of $|\psi\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$?

Alice $\longleftarrow$ $|\psi_{12}\rangle = \cos\theta\,|0_1 0_2\rangle + \sin\theta\,|1_1 1_2\rangle$ $\longrightarrow$ Bob

$\longleftarrow$ $|\psi_{34}\rangle = \cos\theta\,|0_3 0_4\rangle + \sin\theta\,|1_3 1_4\rangle$ $\longrightarrow$

$$|\psi_{12}\rangle|\psi_{34}\rangle = \cos^2\theta\,|\overbrace{0_1 0_2 0_3}^{0} 0_4\rangle + \sqrt{2}\sin\theta\cos\theta\,\frac{1}{\sqrt{2}}\left(|\overbrace{0_1 0_2 1_3}^{1} 1_4\rangle + |\overbrace{1_1 1_2 0_3}^{1} 0_4\rangle\right) + \sin^2\theta\,|\overbrace{1_1 1_2 1_3}^{2} 1_4\rangle$$

~~the~~ a source for getting a Bell state

1. Alice measures k (total number of *1*'s on her side)

2. If k=0 or 2, repeat step 1 0000 1111 not useful

   If k=1, state collapses to $|\psi'\rangle = \frac{1}{\sqrt{2}}\left(|0_1 0_2 1_3 1_4\rangle + |1_1 1_2 0_3 0_4\rangle\right)$

   with probability $2\sin^2\theta\cos^2\theta$

   (Can obtain a Bell state from this?)

Entanglement of distillation

The state is now $|\psi'\rangle = \frac{1}{\sqrt{2}}(|0_1 0_2 1_3 1_4\rangle + |1_1 1_2 0_3 0_4\rangle)$, the goal is to get: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

3. Alice and Bob each perform a local unitary U

$$U_A^{(13)} = U_B^{(24)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad U|01\rangle = |00\rangle, \quad U|10\rangle = |10\rangle, \ etc.$$

➔ $$U_A^{(13)} \otimes U_B^{(24)} |\psi'\rangle = \frac{1}{\sqrt{2}}(|0_1 0_2\rangle + |1_1 1_2\rangle)|0_3 0_4\rangle$$

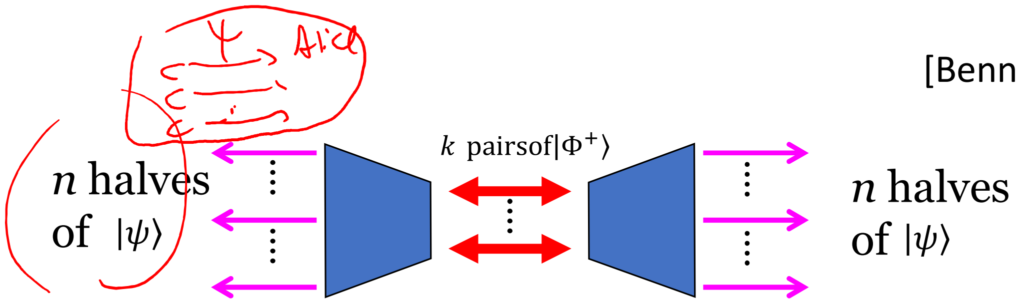
4. Throw away particles *3* and *4*, we get $\frac{1}{\sqrt{2}}(|0_1 0_2\rangle + |1_1 1_2\rangle)$

- Can show that given sufficient many copies

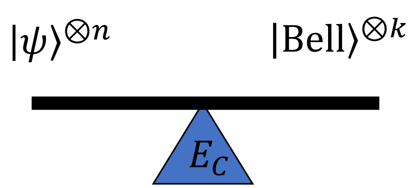
$$E_D(\psi) \equiv \lim_{n\to\infty} (k/n) = S_V(Tr_B|\psi\rangle\langle\psi|)$$

Entanglement cost and entanglement of formation

[Bennett et al. ’97]



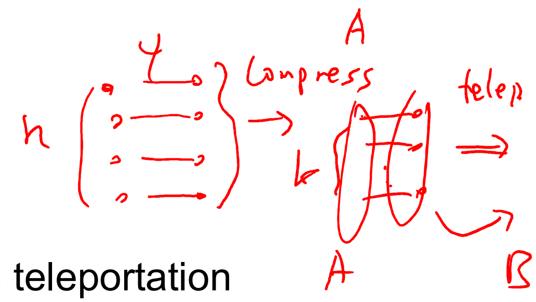
- $E_C(|\psi\rangle) \equiv \lim_{k\to\infty}(k/n)$ (idea also applies to mixed states)



For pure states,

$$E_C = E_D = S_V(Tr_B|\psi\rangle\langle\psi|)$$

Entanglement cost and formation



- How can we achieve the optimal "dilution" process?
- Ans. 'Quantum data compression' (explained later)+ quantum teleportation
  [Schumacher '96] [Bennett et al. '92]

1. Alice prepares ***n*** copies of the states locally, and compresses the part that will be shared by Bob

k Bell pairs to teleport

$$
\begin{aligned}
|\psi_{12}\rangle &= \cos\theta\,|0_1 0_2\rangle + \sin\theta\,|1_1 1_2\rangle \\
|\psi_{34}\rangle &= \cos\theta\,|0_3 0_4\rangle + \sin\theta\,|1_3 1_4\rangle \\
&\vdots \\
|\psi_{2n-1,2n}\rangle &= \cos\theta\,|0_{2n-1} 0_{2n}\rangle + \sin\theta\,|1_{2n-1} 1_{2n}\rangle
\end{aligned}
$$

Compressed into k qubits

$k \approx n\, S_V(Tr_B|\psi\rangle\langle\psi|)$

Teleport

Using k Bell pairs

Bob Decompresses

distillation $\Longleftrightarrow$ dilution

2. Alice and Bob share n copies of ψ by consuming ***k = n E*** copies of Bell states

Entanglement of formation

- Bennett et al. [‘96] constructed an average quantity for mixed states called entanglement of formation

$$E_F(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_C(|\psi_i\rangle)\,, \quad \text{with } \rho = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|$$

- Wootters [’98] has provided an analytic formula of E_F for two qubit states (exact form discussed later)

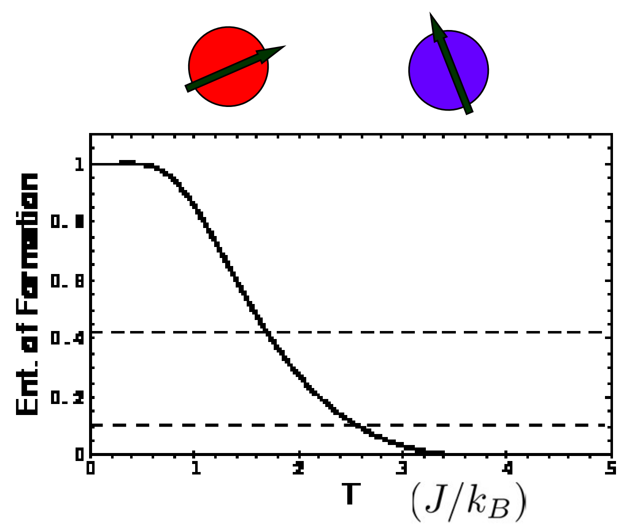
  - Applying it to our two-spin problem:

$$H = J\vec{\sigma}^1 \cdot \vec{\sigma}^2$$

$$\rho = \frac{1}{Z}\sum_n e^{-\beta E_n}|n\rangle\langle n| =$$

$$= \frac{1-r}{4} I_{4\times 4} + r|\Psi^-\rangle\langle\Psi^-|$$

$$r = (e^{3\beta J} - e^{-\beta J})/(e^{3\beta J} + 3e^{-\beta J})$$



Wootters' formula*

- Entangement of formation for mixed states is defined via

$$E_F(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_{\mathrm{F}}(|\psi_i\rangle)\,, \quad \text{with } \rho = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|$$

$$E_F(|\psi\rangle) = -S_V(\rho_A) = -\mathrm{Tr}\rho_A \log \rho_A\,, \quad \text{where } \rho_A = \mathrm{Tr_B}|\psi\rangle\langle\psi|$$

- For two-qubit mixed states, Wootters has found a closed form

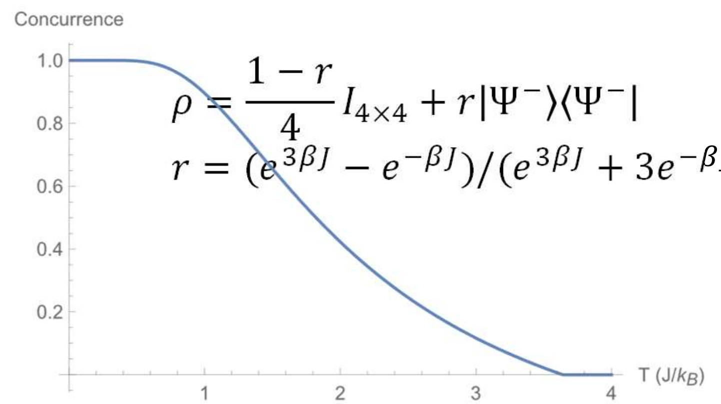
bath

$$H = J\,\vec{\sigma}_1 \cdot \vec{\sigma}_2 \qquad \Phi\ \Phi$$

$$E_F(\rho) = H\left(\frac{1+\sqrt{1-C^2(\rho)}}{2}\right), \quad \text{where } H(x)$$
$$= -x\log x - (1-x)\log(1-x),$$

$C(\rho) = \min\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$ is the **concurrence**
where λ_i's, in nonincreasing order, are eigenvalues of

$$\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$



$$\rho = \frac{1-r}{4} I_{4\times 4} + r|\Psi^-\rangle\langle\Psi^-|$$
$$r = (e^{3\beta J} - e^{-\beta J})/(e^{3\beta J} + 3e^{-\beta J})$$